Seventy Years Using Fixed Points

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My First Research Experience

Jan Kalicki

Born: January 28, 1922, Warsaw, Poland
Died: November 25, 1953, Berkeley, California
MA: 1945, Underground University of Warsaw
Ph.D.: 1948, University of London
1951, Visiting Assistant Professor of Mathematics
1953, Assistant Professor of Philosophy, UC Berkeley

Definition. An equational theory is ***equationally complete*** if, and only if, it is non-trivial, but the addition of any non-provable equation renders it trivial.

Jan Kalicki and Dana Scott. "Equational completeness of abstract algebras." *Indagationes Mathematicae,* vol. 17 (1955), pp. 650-659.
This joint work was done in 1952.

Jan Kalicki. "The number of equationally complete classes of equations." *Indagationes Mathematicae*, vol. 17 (1955), pp. 660-662.

Dana Scott. "Equationally complete extensions of finite algebras." *Indagationes Mathematicae*, vol. 18 (1956), pp. 35-38

Alfred Tarski. "Equationally complete rings and relation algebras." I*ndagationes Mathematicae*, vol. 18 (1956), pp. 39-46.

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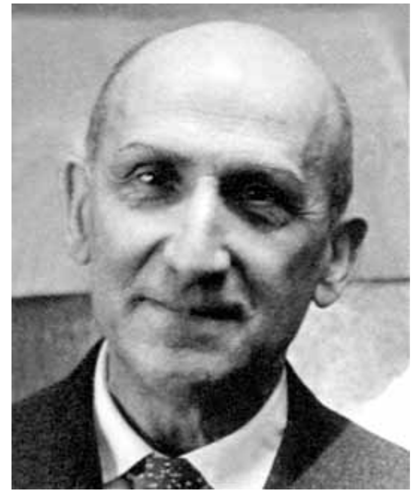
Jan Zygmunt. "The logical investigations of Jan Kalicki." *History and Philosophy of Logic*, vol. 2 (1981), pp. 41-53.

Jan Zygmunt. "Alfred Tarski: Auxiliary Notes on His Legacy." In: Ángel Garrido and Urszula Wybraniec-Skardowska, eds. *The Lvov-Warsaw School. Past and Present.* Springer International Publishing:, 2018, pp. 425-455.

Knaster-Tarski Fixed Point Theorem

Alfred (Teitelbaum) Tarski
Born: January 14, 1901, Warsaw
Died: October 26, 1983, Berkeley, California
Ph.D.: 1924, University of Warsaw



Bronisław Knaster
Born: 22 May 1893, Warsaw
Died: 3 November 1980, Wrocław
Ph.D.: 1923, University of Warsaw

B. Knaster. "Uή theoreme sur les fonctions d'ensembles." *Ann. Soc. Polon. Math.*, vol. 6 (1928), pp. 133-134. **Their joint work was done in 1927.**

A. Tarski. "A lattice-theoretical fixpoint theorem and its applications." *Pacific J. Math.*, vol. 5 (1955), pp. 285 - 309. **Most of the results contained in this paper were obtained in 1939.**

Anne C. Davis. "A characterization of complete lattices." *Pacific J. Math.*, vol. 5 (1955), pp. 311-319. **This result was found in 1950.**

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Tarski's Generalized Lattice-Theoretric Fixed-Point Theorem

Theorem. In a complete lattice the common fixed points of a commutative family of monotone functions form a complete lattice.

B. Banaschewski and G.C.L. Brümmer. “Thoughts on the Cantor-Bernstein Theorem.” *Quaestiones Mathematicae*, vol. 9 (1986), pp. 1-27.

Outline. The usual proofs of the well-known set-theoretical theorem "Given one-one maps $f : A \rightarrow B$ and $g : B \rightarrow A$, there exists a one-one onto map $h : A \rightarrow B$" actually produce a map $h : A \rightarrow B$ contained in the relation $f \cup g^{-1}$.

Considering Tarski's Fixed-point Theorem as the implicit basic ingredient of such proofs, the authors examine several classical proofs starting with Dedekind (1887), and illuminate their common feature by means of the categorical notion of a ***natural fixed point***. The authors consider a categorical form of the theorem in a variety of contexts, obtaining some examples of categories where the Cantor-Bernstein Theorem holds and others where it fails.

Among other results it is proved for a topos E, that the Cantor-Bernstein Theorem holds if E is Boolean, and conversely if E has a natural number object; moreover, The Axiom of Choice in E implies a dual version of Cantor-Bernstein Theorem, and conversely if E has splitting supports and a natural number object.

The Easy Fixed-Point Theorem

The ***powerset*** of the integers, $\mathcal{P}(\mathbb{N}) = \{ X \mid X \subseteq \mathbb{N} \}$, is not only a ***complete lattice***, but it is also a ***T_0-topological space*** with the sets of the form $\{ X \subseteq \mathbb{N} \mid E \subseteq X \}$ as a ***neighborhood base***, where E is taken as a ***finite*** set.

Exercise: What are the ***open*** subsets of $\mathcal{P}(\mathbb{N})$ in this topology?

Note: The ***continuous*** functions $F : \mathcal{P}(\mathbb{N}) \longrightarrow \mathcal{P}(\mathbb{N})$ for this topology are those where, for all $X \in \mathcal{P}(\mathbb{N})$ and all finite $E \in \mathcal{P}(\mathbb{N})$, we have:

$E \subseteq F(X)$ iff there is a finite $D \subseteq X$ with $E \subseteq F(D)$.

Such continuous functions are of course ***monotone*** for ***set inclusion*****.**

Exercise: The ***least fixed point*** of a continuous $F : \mathcal{P}(\mathbb{N}) \longrightarrow \mathcal{P}(\mathbb{N})$ is

$$\mathbf{Y}(F) = \bigcup_{n=0}^{\infty} F^n(\varnothing).$$

Enumeration Operators

Definitions. (1) ***Pairing:*** $(n,m) = 2^n(2m+1)-1$.

(2) ***Sequence numbers:*** $\langle\rangle = 0$ and

$\langle n_0, n_1, \ldots, n_{k-1}, n_k\rangle = (\langle n_0, n_1, \ldots, n_{k-1}\rangle, n_k)+1$.

(3) ***Sets:*** $\mathbf{E}(0) = \varnothing$ and $\mathbf{E}((n,m)+1) = \mathbf{E}(n) \cup \{m\}$.

(4) ***Kleene star:*** $X^* = \{n \mid \mathbf{E}(n) \subseteq X\}$, for sets $X \subseteq \mathbb{N}$.

Note: X^* consists of ***all*** the sequence numbers representing ***all*** the finite subsets of the set X.

Definition. An ***enumeration operator*** $\boldsymbol{F}: \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a mapping ***determined by*** a given subset $F \subseteq \mathbb{N}$ by the formula:

$$\boldsymbol{F}(X) = \{m \mid \exists n \in X^*. (n,m) \in F\}.$$

The operator is ***computable*** iff this set is ***recursively enumerable***.

Note: The enumeration operators on $\mathcal{P}(\mathbb{N})$ are ***exactly*** the continuous functions.

Hartley Rogers, Jr. *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, 1967, xix + 482 pp.

A λ-Calculus Model

Application: $F(X) = \{ m \mid \exists n \in X^*. (n,m) \in F \}$

Abstraction: $\lambda X.[..X..] = \{ (n,m) \mid m \in [..\mathbf{E}(n)..] \}$,

where $X \mapsto [..X..]$ is ***continuous***.

- Application is a continuous function of ***two*** variables.
- If $\boldsymbol{F}(X)$ is continuous, then $\lambda X.\boldsymbol{F}(X)$ is the ***largest*** *set* F where for all sets T, we have $F(T) = \boldsymbol{F}(T)$, but note that generally we only have $F \subseteq \lambda X.F(X)$.
- If the function $\boldsymbol{F}(X,Y)$ is continuous, then the abstraction term $\lambda X.\boldsymbol{F}(X,Y)$ is continuous in the ***other variable***.
- The computable enumeration operators are ***closed*** under application and abstraction.

Church's λ-Calculus

Definition. The **λ**-calculus — as a formal ***equational*** theory — has rules for the ***explicit definition*** of functions ***via*** these well known equational axioms:

α-conversion

$$\lambda X.[\ldots X\ldots] = \lambda Y.[\ldots Y\ldots]$$

β-conversion

$$(\lambda X.[\ldots X\ldots])(T) = [\ldots T\ldots]$$

η-conversion

$$\lambda X.F(X) = F$$

Note: The third axiom can be dropped in favor of a theory employing properties of a ***partial ordering***, as shown by our model.

Note: Church defined ***numerals*** in the pure **λ**-calculus, and Kleene originally developed ***partial recursive functions*** on this basis.

How to do Recursion in our Model?

The Basic Theorems.

- All pure λ-terms define ***computable*** operators in the model.
- If $\Phi(X)$ is continuous and if we let $\nabla = \lambda X.\Phi(X(X))$, then the set $P = \nabla(\nabla)$ is in fact the ***least fixed point*** of Φ.
- So, the least fixed-point operator is **Y** $= \lambda F.(\lambda X.F(X(X))(\lambda X.F(X(X))))$.
- The least fixed point of a ***computable*** operator is computable.

The Recursion Theorem. These computable operators:

$$\textbf{Succ}(X)=\{n+1 \mid n \in X\},$$

$$\textbf{Pred}(X)=\{n \mid n+1 \in X\}, \textbf{ and}$$

$$\textbf{Test}(Z)(X)(Y)= \{n \in X \mid 0 \in Z\} \cup \{m \in Y \mid \exists k.k+1 \in Z\},$$

together with **λ-calculus**, suffice for defining ***all RE sets***.

Dana Scott. "Data types as lattices." *SIAM Journal on Computing*, vol. 5 (1976), pp. 522-587.

The Category of Closure Operators

Definition. A set $C = \lambda X.C(X)$ represents a ***closure operator*** iff for all $X \subseteq \mathbb{N}$ we have $X \subseteq C(X) = C(C(X))$.

Note. The set of ***fixed points*** of a closure operator form a ***lattice*** and uniquely determine the operator. They give examples (up to isomorphism) of all ***countably based algebraic lattices***.

Theorem. We have ***function spaces*** and thus a ***category*** for closure operators via these definitions:

$$F: C \to D \text{ iff } F = D \circ F \circ C$$

and

$$(C \to D) = \lambda F.\, D \circ F \circ C, \text{ where } F \circ G = \lambda X.\, F(G(X)).$$

Products of Closures

Definition. ***Pairing functions*** for sets in $\mathcal{P}(\mathbb{N})$ can be defined by these enumeration operators:

$$\mathbf{Pair}(X)(Y) = \{2n \mid n \in X\} \cup \{2m+1 \mid m \in Y\}$$

$$\mathbf{Fst}(Z) = \{n \mid 2n \in Z\} \text{ and } \mathbf{Snd}(Z) = \{m \mid 2m+1 \in Z\}.$$

Theorem. In the category of closure operators, ***products*** of closures can be defined as: $(C \times D) = \lambda Z.\mathbf{Pair}(C(\mathbf{Fst}(Z)))(D(\mathbf{Snd}(Z)))$.

Theorem. The closure operators as a ***cartesian closed category*** is equivalent to the category of ***countably based algebraic lattices***.

Exercise: Show that $\mathbf{I} = (\mathbf{I} \times \mathbf{I})$. Are there ***other*** such closures?

Note. We may now regard $\mathcal{P}(\mathbb{N}) = \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$, and for $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ we write

$$X \,\mathcal{A}\, Y \text{ iff } \mathbf{Pair}(X)(Y) \in \mathcal{A}.$$

Note. Every closure operator C determines an ***equivalence relation*** over $\mathcal{P}(\mathbb{N})$:

$$X\,[C]\,Y \text{ iff } C(X) = C(Y)$$

A Universal Closure Operator

Theorem. Every enumeration operator $\mathtt{F}$ ***generates*** a closure operator $\mathtt{C}$ with range $\{\mathtt{X} \subseteq \mathbb{N} \mid \mathtt{F}(\mathtt{X}) \subseteq \mathtt{X}\}$,
where we can define

$$\mathtt{C}(\mathtt{X}) = \mathbf{Y}(\boldsymbol{\lambda}\mathtt{Y}.(\mathtt{X} \cup \mathtt{F}(\mathtt{Y}))).$$

Hint. The set $\{\mathtt{X} \subseteq \mathbb{N} \mid \mathtt{F}(\mathtt{X}) \subseteq \mathtt{X}\}$ is closed under ***arbitrary intersections***, and we want $\mathtt{C}(\mathtt{X})$ to be defined as ***the least*** $\mathtt{Y}$ in that set with $\mathtt{X} \subseteq \mathtt{Y}$.
Thus we can make the theorem into an ***operator*** by the definition:

$$\mathbf{Clos} = \boldsymbol{\lambda}\mathtt{F}.\boldsymbol{\lambda}\mathtt{X}.\mathbf{Y}(\boldsymbol{\lambda}\mathtt{Y}.(\mathtt{X} \cup \mathtt{F}(\mathtt{Y}))).$$

Theorem. $\mathbf{Clos}(\mathbf{Clos}) = \mathbf{Clos}.$

Using Fixed Points of Closures

Definition. In the category of closure operators, define:

$$\mathbf{D} = \mathbf{Y}(\lambda D.\,\mathbf{Clos}(D \to \mathbf{I})).$$

Theorem. Because $\mathbf{D} = \mathbf{D} \to \mathbf{I}$ and $\mathbf{I} = \mathbf{I} \times \mathbf{I}$, we have

$$\mathbf{D} \cong \mathbf{D} \times \mathbf{D} \quad \text{and} \quad \mathbf{D} \cong \mathbf{D} \to \mathbf{D},$$

where we invoke the idea of ***isomorphism***
in the cartesian closed category of closure operators.

Conclusion: The range of D can be made into a new model of *all* the three λ-calculus axioms along with a *surjective pairing*.

Are There More General Types?

Definition. The ***types*** over $\mathcal{P}(\mathbb{N})$ are the ***(partial) equivalence relations (PERs):***

$\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ and where, for all $\mathtt{X}, \mathtt{Y}, \mathtt{Z} \in \mathcal{P}(\mathbb{N})$, we have

$\mathtt{X}\ \mathcal{A}\ \mathtt{Y}$ implies $\mathtt{Y}\ \mathcal{A}\ \mathtt{X}$, and

$\mathtt{X}\ \mathcal{A}\ \mathtt{Y}$ and $\mathtt{Y}\ \mathcal{A}\ \mathtt{Z}$ imply $\mathtt{X}\ \mathcal{A}\ \mathtt{Z}$.

Additionally we often write $\mathtt{X}:\mathcal{A}$ for $\mathtt{X}\ \mathcal{A}\ \mathtt{X}$.

And let $\mathcal{T}$ be the ***class*** of all such types, which forms a ***complete lattice***.

Note. There is much more opportunity now for forming ***fixed points!***

Definition. The ***exponentiation*** of types $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$ is defined by:

$\mathtt{F}(\mathcal{A} \to \mathcal{B})\mathtt{G}$ ***iff*** $\forall \mathtt{X}, \mathtt{Y}.\ \mathtt{X}\ \mathcal{A}\ \mathtt{Y}$ implies $\mathtt{F}(\mathtt{X})\ \mathcal{B}\ \mathtt{G}(\mathtt{Y})$.

Note. $\mathtt{F}:\mathcal{A} \to \mathcal{B}$ ***implies*** $\forall \mathtt{X}.\ \mathtt{X}:\mathcal{A}$ implies $\mathtt{F}(\mathtt{X}):\mathcal{B}$.

Note. More information on ***PER***s and categorical fixed points can be found in:

W.P. Stekelenburg. "A note on “Extensional PERs”." *Journal of Pure and Applied Algebra*, vol. 215 (2011), pp. 253–256.

The Category of Types

Definition. The ***product*** of types $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$ is defined as:

$X(\mathcal{A} \times \mathcal{B})Y$ iff **Fst**$(X)\mathcal{A}$ **Fst**(Y) and **Snd**$(X)\mathcal{B}$ **Snd**(Y).

Note. $X{:}(\mathcal{A} \times \mathcal{B})$ iff **Fst**$(X){:}\mathcal{A}$ and **Snd**$(X){:}\mathcal{B}$.

Theorem. The types give us a **cartesian closed category** extending the category of closure operations.

Note: Every $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$ can be considered as a ***topological subspace*** of the the powerset. The ***type*** corresponding to $\mathcal{X}$ is the equivalence relation:

$X[\mathcal{X}]Y$ iff $X = Y \in \mathcal{X}$.

Theorem. Every countably based T_0-space is homeomorphic to a ***subspace*** of $\mathcal{P}(\mathbb{N})$.

Note: This theorem is originally due to:

P. Alexandroff. “Zur Theorie der topologischen Raume.” *C.R. (Doklady) Acad. Sci. URSS*, vol. 11 (1936), pp, 55-58.

Polymorphic Types

Note: As the class $\mathcal{T}$ of all types is a ***complete lattice***, because it is closed under ***arbitrary intersections,*** this allows for many ***typings***:

$$\lambda \texttt{X}.\lambda \texttt{Y}.\texttt{Pair(X)(Y)}: \bigcap_{\mathcal{A},\mathcal{B}} (\mathcal{A} \to (\mathcal{B} \to (\mathcal{A} \times \mathcal{B})))$$

Definition. The ***Scott numerals*** (1963) in λ-calculus are:

$\underline{\mathbf{0}} = \lambda \texttt{X}.\lambda \texttt{F}.\texttt{X}$, $\underline{\mathbf{1}} = \lambda \texttt{X}.\lambda \texttt{F}.\texttt{F}(\underline{\mathbf{0}})$, $\underline{\mathbf{2}} = \lambda \texttt{X}.\lambda \texttt{F}.\texttt{F}(\underline{\mathbf{1}})$, etc., and

$\underline{\mathbf{succ}} = \lambda \texttt{Y}.\lambda \texttt{X}.\lambda \texttt{F}.\texttt{F}(\texttt{Y})$, and

$\underline{\mathbf{pred}} = \lambda \texttt{Y}.\texttt{Y}(\underline{\mathbf{0}})(\lambda \texttt{X}.\texttt{X})$.

Note: Since a ***monotone*** $\Phi : \mathcal{T} \to \mathcal{T}$ has a ***least fixed point***, we can define:

$$\mathcal{S}cott = \bigcap_{\mathcal{A}} (\mathcal{A} \to ((\mathcal{S}cott \to \mathcal{A}) \to \mathcal{A})),$$

giving types to these numerals.

Martın Abadi, Luca Cardelli, & Gordon Plotkin. "Types for the Scott numerals." *Unpublished note* (1993).

Some Conclusions

- Enumeration operators on $\mathcal{P}(\mathbb{N})$ model λ-calculus and have a simple topology.
- Hence, the category of types (PERs) over $\mathcal{P}(\mathbb{N})$ inherits much topology.
- λ-calculus over $\mathcal{P}(\mathbb{N})$ together with the arithmetic combinators provides a basic notion of computability.
- Hence, the category of types over $\mathcal{P}(\mathbb{N})$ inherits aspects of computability.
- Polymorphism for types (PERs) then gives an abstract foundation for defining inductive and co-inductive data structures.
- Also Propositions-as-types can enforce using constructive logic.
- This modeling can in this way function as a laboratory for exploring many ideas in a very concrete fashion.

Note. The basis for this modeling uses standard impredicative classical logic. A new constructive, predicative approach is fully presented in:

Tom de Jong. *Domain Theory in Constructive and Predicative Univalent Foundations.* Ph.D.Thesis, University of Birmingham, 2022, 189 pp.

BUT WAIT … THERE'S MORE …

- **Jonathan Sterling** found that browsing the works of Marcelo Fiore and his collaborators, especially Gordon Plotkin, from the late 1990s made him realize how rudimentary his knowledge of later domain theory was. He especially recommends Fiore's thesis:

Marcelo P. Fiore. *Axiomatic Domain Theory in Categories of Partial Maps.* Cambridge University Press, 1996, 254 pp.

- The papers by Birkedal and collaborators on guarded domain theory are the touchstone for a lot of very interesting generalizations of metric approaches to domain theory:

Lars Birkedal, Kristian Støvring, and Jacob Thamsborg. "The category-theoretic solution of recursive metric-space equations." *Theoretical Computer Science,* vol. 411 (2010), pp. 4102-4122.

Kristian Støvring, Jan Schwinghammer, Rasmus Ejlers Møgelberg, and Lars Birkedal. "First steps in synthetic guarded domain theory: step-indexing in the topos of trees." *Logical Methods in Computer Science,* vol. 8 (2012), pp. 1–45.

- And a recent extension of the guarded domain theory program in his paper with Palombi generalizes many of earlier results to relative topos theory (e.g. the theory of bounded geometric morphisms):

Daniele Palombi and Jonathan Sterling. "Classifying topoi in synthetic guarded domain theory: The Universal Property of Multi-clock Guarded Recursion." In: Proceedings of MFPS 2022, *Electronic Notes in Theoretical Informatics And Computer Science*, vol. 1 (2023), pp. 12-1 – 12-14.

- For other references, see Sterling's bibliography of ***guarded domain theory*** at:

  https://www.jonmsterling.com/gdt-bibliography.html