Appendix

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The Kan Extension Approach

As commented in the blog post, my attempts to compute the push-pull using left Kan extensions involve some ugly adaptations to $\mathbb{T}_{\text{LV}}$. Namely, I want the set of tight arrows of $\mathbb{T}''_{\text{LV}}$ to be exactly the set of those which appear in the boundaries of β_{Growth} and $\beta_{\text{Interaction}}$. In particular, this means that only one leg of the tabulator $\intercal E$ appears. The idea is sketched below.

- objects: V, Term, State, V^2, State^2, Term^2 and $\intercal E$
- tight arrows: the seven following components and all those freely generated by them

$$\begin{array}{ccccccccc} \intercal E & \xrightarrow{\langle s,t\rangle} & V^2 & \xrightarrow{b^2_{\text{State}}} & \text{State}^2 & \xrightarrow{\text{Term}^2_{\text{State}}} & \text{Term}^2 & \xrightarrow{\mu} & \text{Term} \\ & \searrow^{t} & & & & & & & \\ & & V & \xrightarrow{b_{\text{State}}} & \text{State} & \xrightarrow{\text{Term}_{\text{State}}} & \text{Term} & & \end{array}$$

- proarrows: the single proarrow Contrib: Term $\nrightarrow$ State
- cells: β_{Growth} and $\beta_{\text{Interaction}}$

Given these adaptations I also define an adapted double theory for graphs $\mathbb{T}''_{\text{Gph}}$ by the theory freely generated by a proarrow $E : V \nrightarrow V$, the leg $t : \intercal E \to V$ and the unique $\langle s, t\rangle : \intercal E \to V^2$.

Now given a model M of $\mathbb{T}''_{\text{Gph}}$, let $M_{\text{LV}} : \mathbb{T}''_{\text{LV}} \to \mathbb{S}\text{et}$ be the result of the push. To assess whether we have implemented Lotka-Volterra ODE semantics after pulling back to $\mathbb{T}_{\text{ODE}}$ it will be sufficient to know only $M_{\text{LV}}(\text{Contrib} : \text{Term} \nrightarrow \text{State})$. In the following I attempt to apply (the dual of) theorems 5.2 and 4.6 in Grandis and Paré's paper. It might be best to have their paper up alongside you at this point if you want to fully understand where my computations come from.

Assignment on objects

The approach is almost the same as in the 1-dimensional case. For example, for $M_{\text{LV}}(\text{Term})$ we compute the colimit of

$$MQ_{\text{Term}} : (I \downdownarrows \text{Term}) \to \mathbb{T}_{\text{Gph}} \to \mathbb{S}\text{et}$$

where I is the inclusion : $\mathbb{T}_{\text{Gph}} \to \mathbb{T}_{\text{LV}}$, $(I \downdownarrows \text{Term})$ is a double comma category and $Q_{\text{Term}} : (\text{Term} \downdownarrows I) \to \mathbb{T}_{\text{Gph}}$ is the canonical projection. Double comma categories are defined as the obvious generalization of 1-dimensional comma categories:

A *double comma category* $(F \downarrow\downarrow G)$ for $F : \mathbb{C} \to \mathbb{E}$ colax and $G : \mathbb{D} \to \mathbb{E}$ lax is defined as the double category with the following data

- objects are triples $(A, X, f : FA \to GX)$
- tight arrows $(A, X, f : FA \to GX) \to (A', X', f' : FA \to GX)$ are pairs $(g : A \to A', h : X \to X')$ such that $f'Fg = G(h)f$. Composition is inherited from $\mathbb{C}$ and $\mathbb{D}$
- proarrows $(A, X, f : FA \to GX) \nrightarrow (\bar{A}, \bar{X}, \bar{f} : FA \to GX)$ are triples $(u : A \nrightarrow \bar{A}, v : X \nrightarrow \bar{X}, \alpha : [Fu \,{}^{f}_{\bar{f}}\, Gv])$ with composition given by

$$F[u, u'] \mid \frac{\alpha}{\alpha'} \mid G[v, v']$$

  where $F[u, u']$ and $G[v, v']$ are the laxity and colaxity cells of F and G
- cells $[(u, v, \alpha) \,{}^{(g,h)}_{(\bar{g},\bar{h})}\, (u', v', \alpha')]$ are pairs $(\beta : [u \,{}^{g}_{\bar{g}}\, u'], \gamma : [v \,{}^{h}_{\bar{h}}\, v'])$ such that $\alpha' F\beta = G(\gamma)\alpha$.

There are canonical projections onto $\mathbb{C}$ and $\mathbb{D}$ (often denoted by Q and P respectively) by projecting onto the first or second coordinate of each triple. We will consider two special cases. The first is when $\mathbb{D}$ is the singleton category $\mathbf{1}$, in which case G just picks out an object X of $\mathbb{E}$ so we write $(F \downarrow\downarrow X)$ for the resulting double category. The second comes when $\mathbb{D}$ is the category $\mathbf{2}$ of a single proarrow $0 \nrightarrow 1$, in which case G just picks out a proarrow $v : X \nrightarrow Y$ of $\mathbb{E}$ so we write $(F \downarrow\downarrow v)$ for the resulting double category.

So $(I \downarrow\downarrow \text{Term})$ is the vertically discrete double category illustrated below, where "multiply" really means $\mu \circ \text{Term}^2_{\text{State}} \circ b^2_{\text{State}}$ and "multiply endpoints" is the composition of that with $\langle s, t \rangle$.

$$(I \downarrow\downarrow \text{Term}) = \quad \begin{array}{c} (\text{T}E, \text{“multiply endpoints”} : \text{T}E \to \text{Term}) \\ {\scriptstyle \langle s,t \rangle}\downarrow \\ (V^2, \text{“multiply”} : V^2 \to \text{Term}) \end{array} \qquad \begin{array}{c} (\text{T}E, \text{Term}_{\text{State}} \circ b_{\text{State}} \circ t : \text{T}E \to \text{Term}) \\ {\scriptstyle t}\downarrow \\ (V, \text{Term}_{\text{State}} \circ b_{\text{State}} : V \to \text{Term}) \end{array}$$

Then it is clear that the colimit of MQ_{Term} is

$$\text{colim}(MQ_{\text{Term}}) = M(V) \coprod M(V^2).$$

which is as intended (as obtained in the blog post). If we worked with $\mathbb{T}_{\text{Gph}}$ instead of $\mathbb{T}''_{\text{Gph}}$ we would compute $(I \downarrow\downarrow \text{Term})$ to be the double category consisting of the singleton object $(V, \text{Term}_{\text{State}} \circ b_{\text{State}})$, whose colimit $M(V)$ is not what we would hope for.

We apply a similar approach to compute $M_{\text{ODE}}(\text{State})$ and the double comma category is the category of a single tight arrow

$$(I \downarrow\downarrow \text{State}) = (\text{T}E, b_{\text{State}} \circ t : \text{T}E \to \text{State}) \xrightarrow{t} (V, b_{\text{State}} : V \to \text{State})$$

and hence by taking the colimit we obtain $M_{\text{LV}}(\text{State}) = M(V)$ which again is as intended. Had we included $s : \text{T}E \to \text{State}$ in $\mathbb{T}''_{\text{LV}}$ we would've had to coequalize s and t in the colimit which would result in the colimit being the singleton set which is not as intended.

Assignment on Contrib

In Grandis and Paré's work, right double Kan extensions seem to act on proarrows like a 1-dimensional Kan extension would act given that proarrows are objects of 1-categories, with cells as morphisms. I will give a quick run through of their recipe and then finish with applying it to our example.

In the 1-categorical setting, the right Kan extension $\mathrm{Ran}_R S$ of some functor $S : \mathcal{C} \to \mathcal{D}$ along $R : \mathcal{C} \to \mathcal{C}'$ satisfies

$$\mathrm{Ran}_R S(c') \cong \lim((c' \downarrow R) \xrightarrow{Q_{c'}} \mathcal{C} \xrightarrow{S} \mathcal{D})$$

where $Q_{c'}$ is the canonical projection functor. But this limit is exactly the data of $\mathrm{Ran}_{(c'\downarrow R)\xrightarrow{!}\mathbf{1}}(SQ_{c'})$ because the apex of the limit cone defines the functor, and the legs of the limit cone define the natural transformation satisfying the universal property for Kan extensions. The analogous perspective for proarrows is that

$$\mathrm{Ran}_R S(v) = \mathrm{Ran}_{P_v}(SQ_v)$$

where $P_v : (v \Downarrow R) \to \mathbf{2}$ and $Q_v : (v \Downarrow R) \to \mathcal{C}$ are the canonical projections. This equality indeed holds, so it is sufficient to understand how to compute right kan extensions along functors whose codomain is $\mathbf{2}$.

To compute the right Kan extension $G : \mathbf{2} \to \mathbb{A}$ of a lax double functor $S : \mathbb{I} \to \mathbb{A}$ along $R : \mathbb{I} \to \mathbf{2}$ we can first compute the assignment of G on objects as in the previous section. Now let $v : 0 \nrightarrow 1$ be the unique proarrow of $\mathbf{2}$. For $G(v)$, first consider embedding the proarrows and cells of $\mathbb{I}$ into the tight direction of a double category. From here, take the subcategory $\mathbb{I}_v$ of proarrows which R sends to v and cells which R sends to the identity. Then consider the functors Q, J_0 and J_1 as follows.

$$\begin{aligned} Q &: \mathbf{2} \times \mathbb{I}_u \to \mathbb{I} \\ Q(0,-) &= \mathrm{dom}(-) \\ Q(1,-) &= \mathrm{cod}(-) \end{aligned} \qquad\qquad \begin{aligned} J_0, J_1 &: \mathbb{I}_u \rightrightarrows \mathbf{2} \times \mathbb{I}_u \\ J_0(-) &= (0,-) \\ J_1(-) &= (1,-) \end{aligned}$$

It transpires that the data of SQ in fact defines a vertical transformation

$$SQJ_0 \Rightarrow SQJ_1 : \mathbb{I}_v \to \mathbb{A}$$

and hence induces a unique proarrow $w : A \nrightarrow B$ between the limits of each functor together with a universal cone consisting of, for each proarrow u in $\mathbb{I}$, a cell

$$\begin{array}{ccc} A & \xrightarrow[\shortmid]{w} & B \\ {\scriptstyle p_u}\downarrow & \pi_u & \downarrow{\scriptstyle q_u} \\ S(\mathrm{dom} u) & \xrightarrow[\shortmid]{Su} & S(\mathrm{cod} u) \end{array}$$

where p_u and q_u are the legs of the respective cones, such that there exist fixed tight arrows $f : G(0) \to A$ and $g : G(1) \to B$ such that

$$p_u \circ f = p_{\mathrm{dom}(u)} \quad \text{and} \quad q_u \circ g = q_{\mathrm{cod}(u)} \quad \forall u.$$

where $p_{\mathrm{dom}(u)}$ and $q_{\mathrm{cod}(u)}$ are legs of the limit cones obtained in computing $G(0)$ and $G(1)$ (like we did in the first section). Finally

$$G(v) := f_* \otimes (w \otimes g^*) : G(0) \nrightarrow G(1)$$

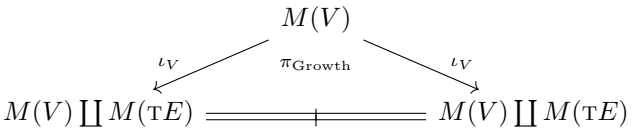
where f_* is the companion of f and g^* is the conjoint of g. There is some extra work to compute the universal cone to complete the requirements of a Kan extension but we will not need this.

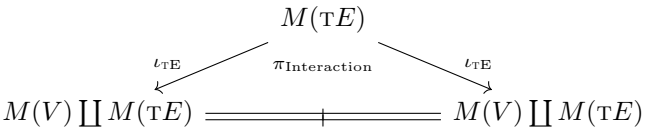
Applying the above to our example obviously requires some hand waving since Grandis and Paré's work was designed for double categories, not virtual ones, but here goes. By the dual of their results, for $M_{\mathrm{ODE}}(\mathrm{Contrib})$ we actually seek

$$\mathrm{Lan}_{P_{\mathrm{Contrib}}} MQ_{\mathrm{Contrib}} : \mathbf{2} \to \mathbb{T}''_{\mathrm{Gph}}$$

where $P_{\mathrm{Contrib}} : (I \Downarrow \mathrm{Contrib}) \to \mathbf{2}$ and $Q_{\mathrm{Contrib}} : (I \Downarrow \mathrm{Contrib}) \to \mathbb{T}''_{\mathrm{Gph}}$ are the canonical projections. In this case $\mathbb{I} = (I \Downarrow \mathrm{Contrib})$ and $\mathbb{I}_v$ is the discrete double category of two objects β_{growth} and $\beta_{\mathrm{interaction}}$. Where Grandis and Paré speak of SQ as a vertical transformation $SQJ_0 \Rightarrow SQJ_1$, in our case the vertical transformation is the identity transformation and both *colimits* (recall we are

dualizing the paper's results) are $M(V) \coprod M(\mathrm{T}E)$. Hence the unique proarrow $w : M(V) \coprod M(\mathrm{T}E) \nrightarrow M(V) \coprod M(\mathrm{T}E)$ is the identity proarrow in $\mathbb{S}\mathrm{et}$. As for the cells π_u, we have just two:





where ι_V and $\iota_{\mathrm{T}E}$ are the inclusions. We already computed $G(0)$ and $G(1)$ in the first section ie. $M_{\mathrm{LV}}(\mathrm{Term})$ and $M_{\mathrm{LV}}(\mathrm{State})$. By reviewing the legs of colimits we computed there and substituting into Grandis and Paré's method we obtain that there exist

$$f : M(V) \coprod M(\mathrm{T}E) \to M(V) \coprod M(V^2), \quad g : M(V) \coprod M(\mathrm{T}E) \to M(V)$$

such that

$$\begin{aligned} f(v) &= v & f(e) &= M\langle s, t\rangle(e) \\ g(v) &= v & g(e) &= Mt(e). \end{aligned}$$

These equations uniquely determine f and g. Then since the unique proarrow w was the identity, the composite with companions and conjoints just gives that $M_{\mathrm{LV}}(\mathrm{Contrib}) : M_{\mathrm{LV}}(\mathrm{Term}) \nrightarrow M_{\mathrm{LV}}(\mathrm{Term})$ is the set $M(V) \coprod M(\mathrm{T}E)$ with f and g providing the necessary projections, exactly as intended.

Clearly this approach isn't as pretty as the one in the blog post which is why it's been exiled to this appendix. In defense of the approach, maybe the final remark in the RFC about how to define the builder cells without tabulators would resolve some of the warts in the definition of $\mathbb{T}''_{\mathrm{LV}}$ such as where we only included one leg of the tabulator, but this is a consideration for future work.