Modeling decentralized markets using (mostly free) categories

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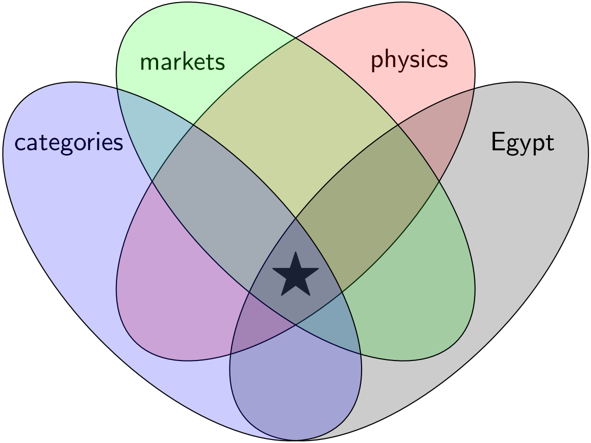
Abstract

Standard models of financial markets assume that transactions are mediated by a single unit of account. However, in some real-world markets, there is no distinguished unit of account, transactions consist of an exchange of two assets, and arbitrage is often possible. We attempt to define a categorical formalism for these "decentralized" markets, and a path integral description of the stochastic dynamics of exchange-values in such markets.

Overview

- Motivation
- Market structure: category theory
- Market dynamics: physics, or "physics"
- Numerical experiments: single period
- Numerical experiments: multiple periods
- Volatility and correlation structure: single period
- Next steps

Categorical finance: at the intersection of...



Examples of centralized markets

cker	Name	Last	Chg	Chg %	Volume	Rel
XOM	Exxon Mobil C...	72.96	1.09	1.52%	12.55M	1.9x
ABBV	AbbVie Inc.	137.32	1.45	1.07%	2.04M	1.1x
LLY	Eli Lilly and C...	245.19	1.07	0.44%	720.45K	0.9x
BA	The Boeing C...	226.91	0.95	0.42%	3.75M	1.2x
DIS	The Walt Disn...	152.55	0.61	0.40%	3.03M	1.1x
V	Visa Inc.	215.50	0.83	0.39%	2.72M	1.2x
MRK	Merck & Co.,...	81.67	0.29	0.35%	3.45M	1.2x
LMT	Lockheed Mar...	373.40	0.78	0.21%	457.23K	1.6x
COP	ConocoPhillips	86.91	0.17	0.20%	3.23M	1.7x

Figure 1: A single stock exchange

Centralized markets: the classical model

There is a single unit of account, call it \$.
At each instant, every asset has a price in terms of \$.
The vector X of log asset prices follows n-dim Brownian motion.
Covariance matrix bundles up the asset volatilities and correlations.

Path integral formulation (see Linetsky 1998): if $m^{-1} = \sigma^2$,

$$\text{density}(\gamma) \propto e^{-\int \frac{1}{2} m \left(\dot{\gamma(t)} - \mu\right)^2 dt}$$

The path integral approach goes back to Dash 1988, who applied it to option pricing, and argued that it was more fundamental than the well-known Black-Scholes PDE.

Examples of decentralized markets

(a) barter

(b) FX

(c) crypto

Figure 2: Some decentralized markets

See: Allen 2002

Stocks trade on multiple venues: exchanges, dark pools,. . .

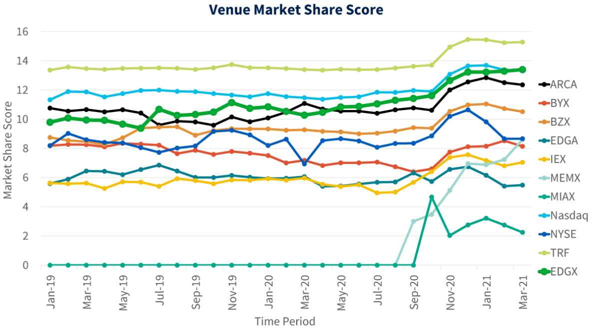


Figure 3: Market share of different equity trading venues

Source: "A deep dive into US equities trading venues" 2021

Cross-venue arbitrage in equities can persist for hundreds of milliseconds.

Previous efforts to analyze decentralized market structure

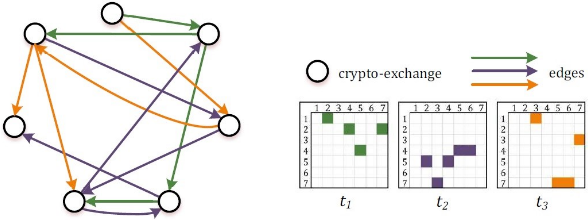


Figure 4: Graph structure of cryptocurrency exchanges

Source: Kabašinskas and Šutienė 2021

Previous efforts to model arbitrage in markets

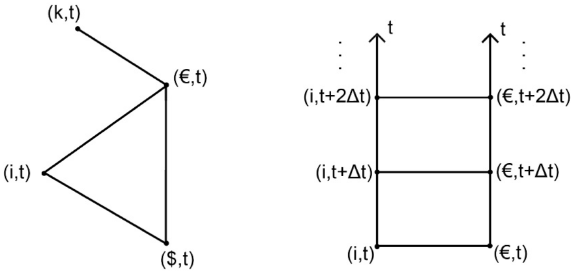


Figure 5: 4-asset market, and 2-asset market evolving in time

Source: Rodrigues 2019; the approach originates with Ilinski 2001

What would an interesting theory be able to do?

- Show that classical theory is a special case
- Understand comovement, generalizing classical correlation
- Understand how shocks can lead to transitory arbitrage
- Realistically model elimination of arbitrage
- Define fundamental value and convergence to fair value
- Understand how merging markets affects their dynamics
- Efficiently carry out multi-period simulation
- Efficiently calibrate to observed market data

The model in a nutshell

"Markets in which fungible assets are exchanged for each other"

- A *market* is defined by a weighted, directed (multi)graph **M**
- Form the free weighted category (Perrone 2021) $\mathbb{M}$ on **M**
- A *state* is a functor $X \colon \mathbb{M} \to \langle \mathbb{R}, + \rangle$
  - Intuitively, $f \mapsto \log(\text{exchange ratio})$
- A *path* γ is a sequence of states, with specified time step
- *Lagrangian* $L(\gamma)$ with kinetic and potential energy terms
  - Weiner measure term: each arrow f contributes *kinetic energy*
  - Arbitrage at parallel pairs, loops: *elastic potential energy*
  - $X(f)$ vs its fundamental value: *gravitational potential energy*
- The probability density of a path γ is $e^{-\int L(\gamma)\,dt}$

Interpretation

ob(**M**) is the set of assets. Each asset $A \in$ ob(**M**) is assumed to be a fungible, infinitely divisible asset.

An edge $f : A \to B$ in **M** is an offer to provide asset B in exchange for asset A: "owner of B sets the terms, then owner of A decides".

- The mass m_f represents how much the exchange ratio resists change, i.e. how "sticky" the exchange ratio is.

Composites $f_n \circ f_{n-1} \circ \cdots \circ f_1$ in $\mathbb{M}$ are sequences of exchanges.

When the market state is X, the arrow f lets an owner of asset A decide to exchange 1 unit of asset A for $e^{X(f)}$ units of asset B.

NB: Change-of-units for an individual asset defines a local gauge transformation; gauge invariance is just translation invariance.

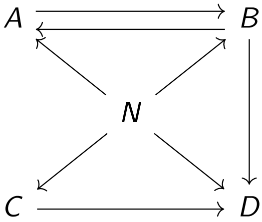
- e.g. imperial to metric units; redenomination of Turkish lira

Definitions: numéraire, medium of exchange

A numéraire is an asset that can serve as an accounting measure.

- e.g. dollars, sacks of grain, ounces of silver, Ethereum.

Formally, a *numéraire* in $\mathbb{M}$ is a spanning star. An arrow $n_A \colon N \to A$ in the star corresponds to buying the asset A with the numéraire asset N, at the *offer* price of A.



A numéraire N becomes a *medium of exchange* by specifying, for every A, an arrow $n'_A \colon A \to N$; this corresponds to a *bid* for A.

Centralized markets

We say a market is *weakly centralized* by a medium of exchange.

- e.g. suppose the asset N is "dollars": it is a numéraire if there's always a posted offer in dollars for every (other) asset, and a medium of exchange if there's also always a posted bid

A medium of exchange is a *perfect medium of exchange* if n_A and n'_A are inverses, for all assets A. Note that:

- This notion takes us out of the realm of free categories
- Any two perfect mediums of exchange are isomorphic
- The exchange ratio of two perfect mediums can fluctuate
- Perfect mediums of exchange do not exist in the real world

A market is *perfectly centralized* by a perfect medium of exchange. We expect such a market to have "nearly classical behavior".

Shocks; supply and demand shocks

A *shock* to a state $X\colon \mathbb{M} \to \langle \mathbb{R}, + \rangle$ perturbs it to a new state X'.
Given an asset A in $\mathbb{M}$, there are two special kinds of shock:

- A *supply shock* to A affects $X(f)$ for all $f\colon B \to A$
- A *demand shock* to A affects $X(f)$ for all $g\colon A \to B$

(*NB:* classical model can't distinguish supply and demand shocks.)

Interpretation when A is a numéraire N, e.g. fiat money:

- A positive demand shock for N means owners of other assets will offer them at lower prices in terms of N; e.g. if N is fiat money and a financial crisis triggers a flight to cash.
- If N is a medium of exchange, a positive supply shock for N means owners of N will bid higher for other assets; e.g. if N is fiat money and the central bank expands the money supply.

Merging markets

Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be two markets; without loss of generality, we can assume that $\mathrm{ob}(\mathbb{M}_1) = \mathrm{ob}(\mathbb{M}_2) = M$.

The *merger* of the two markets $\mathbb{M}_1$ and $\mathbb{M}_2$ is the pushout

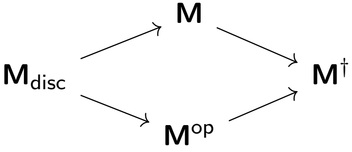
$$\begin{array}{ccccc} & & \mathbb{M}_1 & & \\ & \nearrow & & \searrow & \\ \mathbb{M}_{\mathsf{disc}} & & & & \mathbb{M}_1 \sqcup \mathbb{M}_2 \\ & \searrow & & \nearrow & \\ & & \mathbb{M}_2 & & \end{array}$$

where $\mathbb{M}_{\mathsf{disc}}$ is the discrete category on M; note that $\mathbb{M}_1 \sqcup \mathbb{M}_2$ is a free category.

Markets with refunds

Given a market $\mathbb{M}$, we would like to construct a market $\mathbb{M}^\dagger$ by "adjoining refunds", i.e. for each $f : A \to B$ in $\mathbb{M}$ there is an arrow $f^\dagger : B \to A$ in $\mathbb{M}^\dagger$. We don't impose that $f, f^\dagger$ are inverses.

Form the pushout in the category of weighted directed multigraphs:



Then we can let $\mathbb{M}^\dagger$ be the free weighted category on $\mathbf{M}^\dagger$.

- *Claim:* $\mathbb{M}^\dagger$ is the free weighted dagger category on $\mathbb{M}$
- Does the cofree dagger category (of pairs $A \rightleftarrows B$) have a role?

What terms should be in the Lagrangian?

The Lagrangian $L(\gamma)$ contains the following terms:

- *Kinetic energy*: when $\dot{X}(f) \neq 0$
- *Arbitrage potential energy*:
  - Parallel pairs $f, g\colon A \rightrightarrows B$ in $\mathbb{M}$, where $X(f) \neq X(g)$
  - Loops $h\colon A \to A$ in $\mathbb{M}$, where $X(h) \neq 0$
- *Fundamental potential energy*: when $X(f)$ deviates from a predetermined "fundamental value" (a.k.a. "fair value")

$\gamma = \langle X_0, X_1, \ldots X_n \rangle$ is discrete, so $L(\gamma) = \frac{T}{n} \sum_{i=1}^{n} L(X_{i-1}, X_i)$.

Also recall the requirement of gauge invariance: the Lagrangian should be invariant to local change-of-units (asset redenomination). This means $L(X_{i-1}, X_i)$ should be invariant to local translations.

Terms in the Lagrangian $L(X_{i-1}, X_i)$

Kinetic energy term

$$\sum_{f\colon A\to B} \frac{m_f}{2}(\Delta_i X(f))^2 \quad \text{where} \quad \Delta_i X(f) \coloneqq X_i(f) - X_{i-1}(f)$$

Arbitrage potential energy term (*NB:* uses m^{-1} rather than m)

$$\sum_{f,g\colon A\rightrightarrows B} \frac{(X_i(g) - X_i(f))^2}{2 m_f m_g} \quad + \quad \sum_{h\colon A\to A} \frac{X_i(h)^2}{2 m_h^2}$$

Fundamental potential energy term

$$\sum_{f\colon A\to B} \frac{m_f}{2} |X_i(f) - X^{\text{fair value}}(f)|$$

A coherent notion of fair value

There is a natural constraint on the specification $X^{\text{fair value}}$ of predetermined "fair value" exchange ratios: there should be no arbitrage in a fair value state.

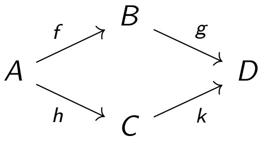
Call a state $X : \mathbb{M} \to \langle \mathbb{R}, + \rangle$ *arbitrage-free* if it factors through the quotient directed graph $q_{\mathbb{M}} : \mathbb{M} \twoheadrightarrow \tilde{\mathbb{M}}$.

If we specify an arbitrage-free fair value state $X^{\text{fair value}}$, then

$$L(X^{\text{fair value}}, X^{\text{fair value}}) = 0$$

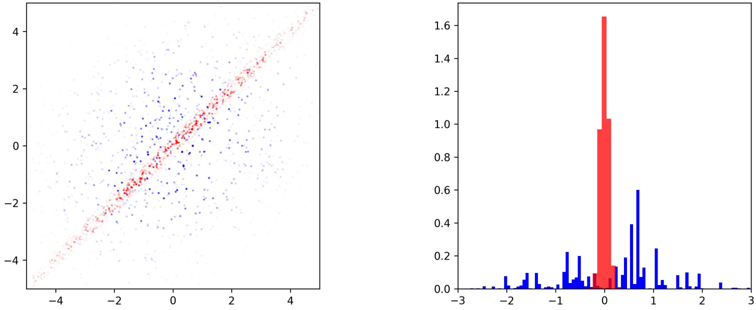
and the constant path $\gamma(t) \equiv X^{\text{fair value}}$ is, trivially, the most probable state evolution starting at $X^{\text{fair value}}$.

Single period simulations: a commutative square



$$m_f = m_g = m_h = m_k = 1$$

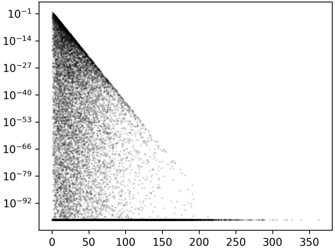
Nearly independent pair vs arbitrage constrained pair



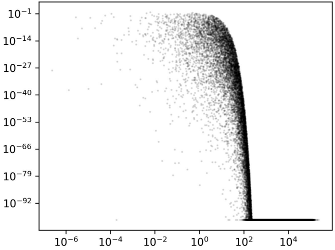
(a) Δ_f, Δ_h and $\Delta_{g \circ f}, \Delta_{k \circ h}$ (b) $\Delta_f - \Delta_h$ and $\Delta_{g \circ f} - \Delta_{k \circ h}$

Figure 6: Simulated distribution of f, h (blue) and $g \circ f, k \circ h$ (red)

How efficient is naive sampling?



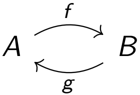
(a) weight vs kinetic energy

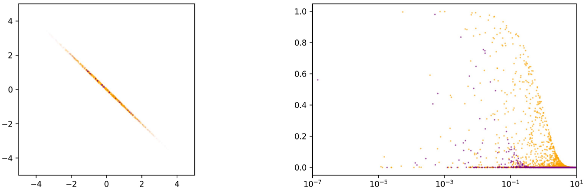


(b) weight vs arbitrage potential

Figure 7: Distribution of sample weights – 100,000 draws

Opposite pair: how many terms in expansion do you need?





(a) samples of f, g (b) weights vs. loop potential

Figure 8: Simulation results for $N = 2$ (orange) and $N = 4$ (purple)

Why efficient multi-period sampling is difficult

We would like to sample mostly from the region of path space where most of the probability density lies. Statisticians refer to this as the "typical set".

The typical set – i.e. the paths with modest variability *and* low arbitrage along the whole path – turns out to be a small and special region of path space.

When sampling naively, we end up sampling few or no paths with low arbitrage throughout the path, even though these are the ones with high probability density. That is, we sample very few paths from the typical set – often none at all, even from $\sim 10^5$ samples.

Instead, an excessively high weight is assigned to a handful of paths which randomly happen to have lower arbitrage potential than the others (but still not low, compared to paths in the typical set).

Partly mitigating weight degeneracy via "dissipation"

Include an *ad hoc* "dissipation" adjustment at each time step. Wherever there is parallel pair arbitrage or loop arbitrage, this adjustment should make it shrink.

This is meant to be an approximation to the true dynamics (as yet unidentified) of how arbitrage closes. The better the approximation, the more efficiently we can sample.

In practice, imposing exponential adjustment with half-life inversely proportional to mass works reasonably well in small examples. We conjecture that this simple *ansatz* describes the true "arbitrage dissipation" dynamics.

Even using this method, to sample even a few paths from the typical set, we have to either (a) draw many millions of samples, or (b) sample $\Delta X(f)$ from a suboptimally narrow distribution.

Single arrow: dependence on mass

$$A \xrightarrow{f} B$$

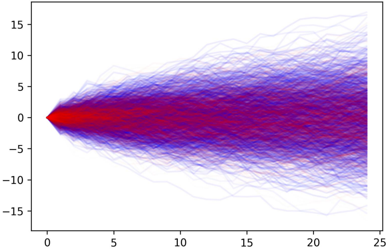


Figure 9: Simulated paths, $m_f = 0.5$ (blue) versus $m_f = 2$ (red)

Single arrow: impact of reversion to fair value

$$A \xrightarrow{f} B$$

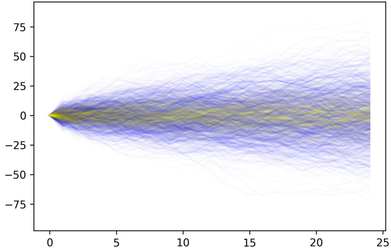


Figure 10: Free (blue) versus reverting to fair value $X_f = 0$ (yellow)

Single arrow: reversion to fair value after a shock

$$A \xrightarrow{f} B$$

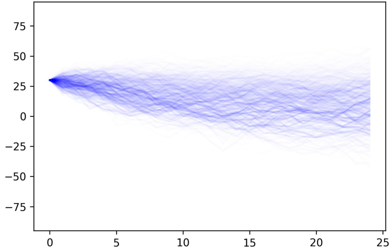
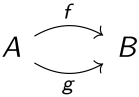


Figure 11: Reversion to fair value after an initial shock

Parallel pair: dynamics limited by arbitrage



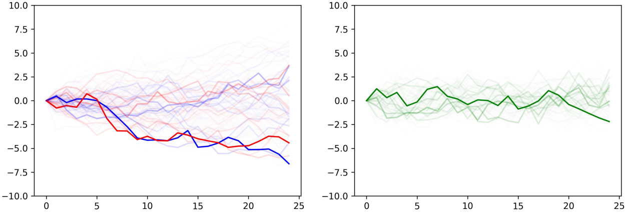
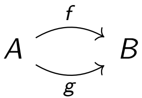


Figure 12: Simulated paths for f, g (blue, red) and their difference (green)

Parallel pair: with reversion to fair value



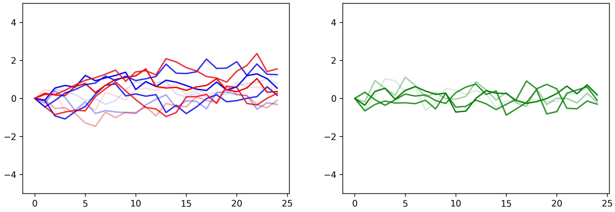
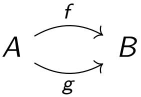


Figure 13: Simulated paths for f, g (blue, red) and their difference (green)

Parallel pair: reversion to fair value after a shock to f



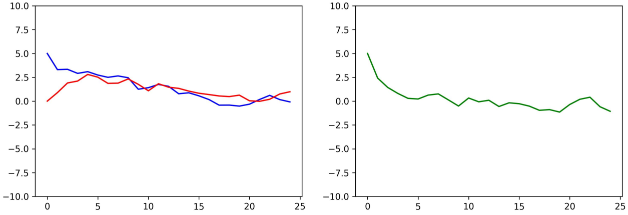
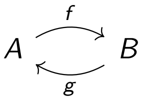


Figure 14: Simulated paths for f, g (blue, red) and their difference (green)

Opposite pair: dynamics limited by arbitrage



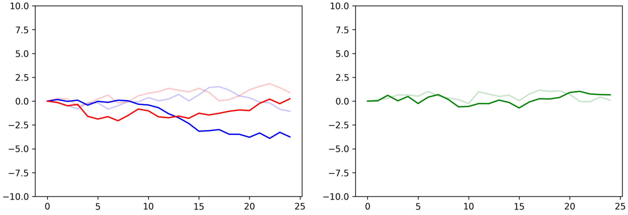


Figure 15: Simulated paths for f, g (blue, red) and $g \circ f$ (green); $N = 2$

Opposite pair: simulation has severe weight degeneracy

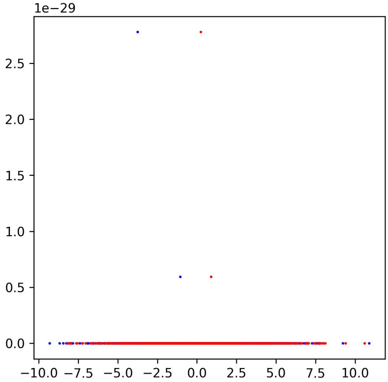
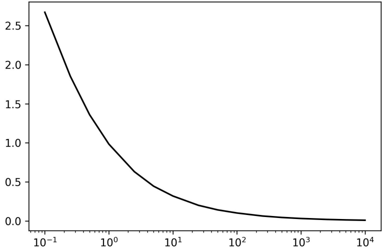


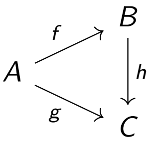
Figure 16: Path weight vs terminal value of $X(f)$ (blue), $X(g)$ (red)

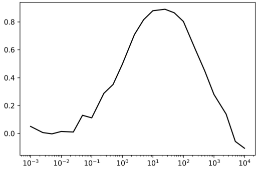
Single arrow: dependence of volatility on mass

$$A \xrightarrow{f} B$$

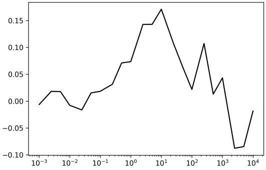


Two linked arrows: dependence of correlation on link mass



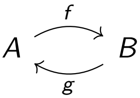


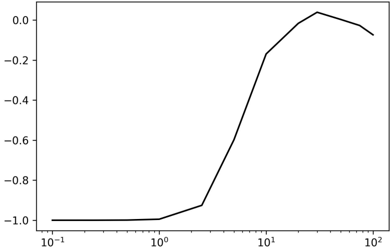
(a) $m_f = m_g = 1$



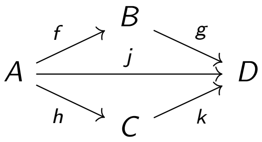
(b) $m_f = m_g = 10$

Opposite pair: dependence of correlation on mass





Correlation matrix generated by a diagram ($m = 1$)



	f	h	g	k	$g \circ f$	$k \circ h$	j
f	1.00	−0.40	−0.69	0.59	0.36	0.32	0.36
h	−0.40	1.00	0.58	−0.79	0.26	0.28	0.27
g	−0.69	0.58	1.00	−0.27	0.43	0.46	0.42
k	0.59	−0.79	−0.27	1.00	0.38	0.36	0.37
$g \circ f$	0.36	0.26	0.43	0.38	1.00	0.99	0.99
$k \circ h$	0.32	0.28	0.46	0.36	0.99	1.00	0.99
j	0.36	0.27	0.42	0.37	0.99	0.99	1.00

Better sampling for multi-period simulation

Most important open issue: how can we avoid weight degeneracy in multi-period simulation?

The dissipation heuristic only seems to help for very small markets. What about adopting the opposite approach, i.e. drawing arbitrage-free paths first, and then introducing random arbitrage?

Proposed procedure for drawing a random path:

1. Sample a path $\tilde{\gamma}$ of arbitrage-free states $\tilde{X}_i : \tilde{\mathbb{M}} \to \langle \mathbb{R}, + \rangle$
2. $X_i := \tilde{X}_i \circ q_{\mathbb{M}} : \mathbb{M} \twoheadrightarrow \tilde{\mathbb{M}} \to \langle \mathbb{R}, + \rangle$ is a path γ for $\mathbb{M}$
3. Compute the kinetic energy $E_K(\gamma)$ of γ
4. Modify the X_i by adding small arbitrages, such that the arbitrage potential energy of the new path is $\ll E_K(\gamma)$

In practice, we can probably repeat step 4. many times for each γ.

Coupling terms increase modeling flexibility

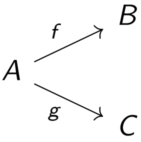
The above framework cannot model the following phenomena:

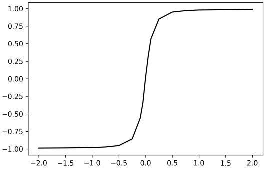
1. Negatively correlated assets – e.g. stocks vs. bonds
2. Short positions and leveraged positions

An *ad hoc* approach: add *coupling terms* to the Lagrangian:

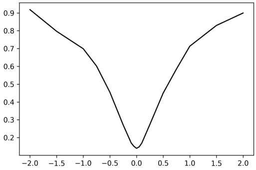
1. To force a correlation between $f : A \to B$ and $g : A \to C$, add a term $\propto (\Delta_i X(f) - \lambda \Delta_i X(g))^2$; e.g. $\lambda = -1$ forces $\rho < 1$
2. Given $f : A \to B$, adjoin a *derived* asset $B_{\lambda f}$, a new edge $f_\lambda : A \to B_{\lambda f}$ and a coupling term $\propto (\Delta_i X(f_\lambda) - \lambda \Delta_i X(f))^2$
   - One can think of $B_{\lambda f}$ as wrapping the notion of a "$\lambda \times$ leveraged position in B, financed by A" in a synthetic asset such as an ETF; the coupling strength controls tracking error.

Dependence of correlation and volatility on coupling λ





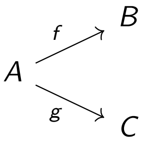
(c) correlation of $X(f), X(g)$

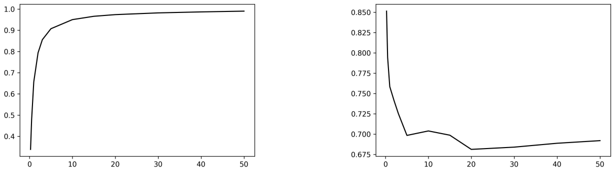


(d) volatility of $X(g)$

Figure 17: Dependence on λ ($m_f = m_g = 1$, coupling strength $= 25$)

Dependence on coupling strength





(a) correlation of $X(f), X(g)$ (b) volatility of $X(g)$

Figure 18: Dependence on coupling strength $(m_f = m_g = 1,\ \lambda = 1)$

Other open issues: theory, simulation, applications

- Describe stochastic evolution by a Fokker-Planck equation
- Find efficient ways to calibrate to real world market data
- Formulate a tractable notion of impulse-response function
- Model how shocks create arbitrages, and how arbitrage closes
- Define notions of borrowing, discounting, own rates of interest
- Develop financial math: option pricing, portfolio theory

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