Looking for a ruler to measure complexity

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Computability Theory

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(Every finite object can be coded by a natural number.)

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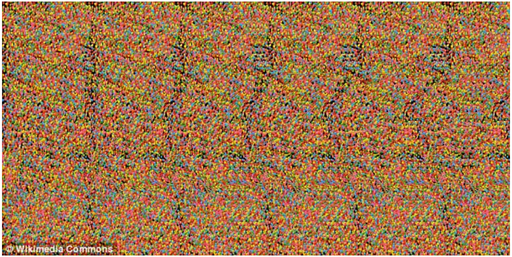

(3) The structure underneath the chaos: Martin's conjecture

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4. $\mathcal{P}(\mathbb{N})/\equiv_T$ is uncountable.

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There are many more results showing that $(\mathcal{P}(\mathbb{N}); \leq_T)$ is extremely complex.

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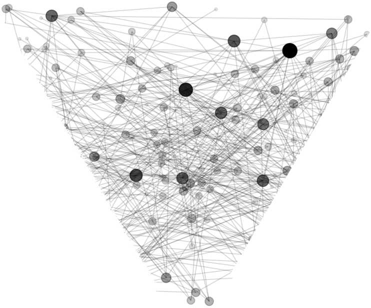

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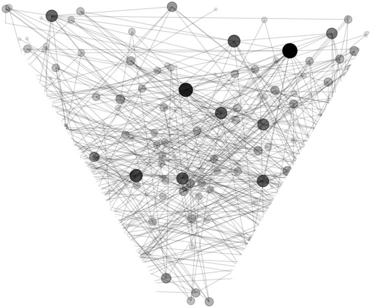

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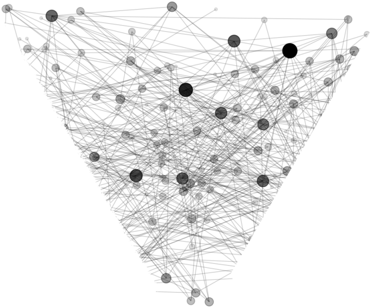

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We know that
all natural examples are:

- constructible
- relativizable

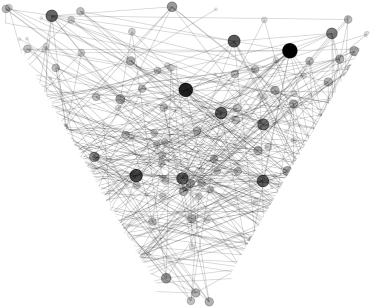
Natural examples

$0 <_T K \equiv_T WP \equiv_T HTP <_T TF <_T COF <_T TA <_T WF.$

$\leq_T$

$\Leftarrow$ Natural examples

All the sets $\Rightarrow$



Mathematically, we don't have a definition of "natural example".

We know that
all natural examples are:

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Plan: relativizable examples $\Longleftrightarrow$ functions $\mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ $\equiv_T$-invariants

$\Updownarrow$ $\Updownarrow$

ordered by $\leq_T$ $\Longleftrightarrow$ Ordered by $\leq_T^{\nabla}$

Relativization

Observación:
The functions that are computable in A behave "like" the computable functions.

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K $=$ The set of programas `p` that eventually halt.

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$\mathbf{K}^A$ = The set of programas $\mathtt{p}^A$ that use the funciton χ_A and eventually stop.

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Example of relativization in algebra

(all rings are countable, commutative, and with unity)

Theorem: Every ring $(D; 0, 1, +, \times)$ has a maximal ideal.

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Examples: $\mathbf{K}^{\odot}$, $\mathbf{FIN}^{\odot}$, $\mathbf{COF}^{\odot}$, $\mathbf{TA}^{\odot}$, $\mathbf{WF}^{\odot}$, $\mathbf{MI}(\odot)$, $\mathbf{JI}(\odot)$,....

Empirical observation: Natural Turing degrees induce $\equiv_T$ -invariant functions.

Definition: Given $F, G\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, we say that $F \leq_T G$ if,

for every B, $F(B) \leq_T G(B)$.

We know that for every B

$$0 <_T \mathbf{K}^B \equiv_T \mathbf{MI}(B) <_T \mathbf{FIN}^B \equiv_T \mathbf{JI}(B) <_T \mathbf{COF}^B <_T \mathbf{TA}^B <_T \mathbf{WF}^B$$

thus, if we look at the corresponding functions:

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Problem: There are $\equiv_T$ -invariant functions of all shapes and colors.

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Problem: There are $\equiv_T$ -invariant functions of all shapes and colors.

...but not if we compare them at the “limit”.

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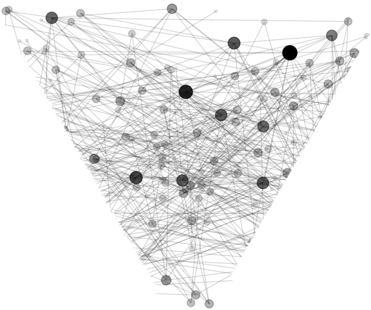

F is increasing if $F(A) \geq_T A$ for every A.

$\equiv_T$-invariants increasing
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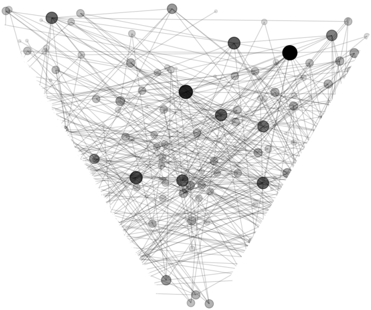

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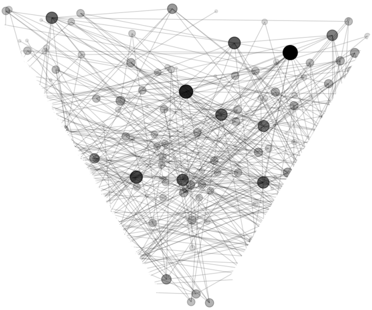

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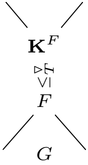

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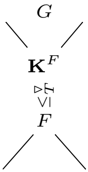

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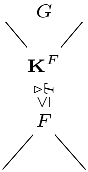

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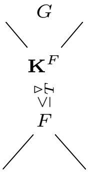

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Thm:[Kihara-Montalbán 18]: Connect natural many-one degrees and Wadge degrees.

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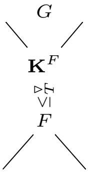

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The conjecture is still open for the general case.

Wadge degrees

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Theorem: [Wadge 83](AD) The Wadge degrees are almost linearly ordered:

- For every $A, B \subseteq \mathbb{N}^{\mathbb{N}}$, either $A \leq_w B$ or $B \leq_w A^c$.
- For every $A, B \subseteq \mathbb{N}^{\mathbb{N}}$, if $A <_w B$, then $A <_w B^c$.

Theorem: (AD) [Martin] The Wadge degrees are well founded.

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Theorem: [Kihara, M.] There is a one-to-one correspondence between $(\equiv_T, \equiv_m)$-UI functions ordered by $\leq_{\mathbf{m}}^{\nabla}$ and $\mathcal{P}(2^{\mathbb{N}})$ ordered by Wadge reducibility.