Università di Genova | DIMA DIPARTIMENTO DI MATEMATICA

Equality is coalgebraic

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joint work with Jacopo Emmenegger and Fabio Pasquali

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TOPOS INSTITUTE

Categories and Logic

Logic for categories	monoidal closed category cartesian closed category Grothendieck topos exact category elementary topos pretopos Heyting pretopos arithmetic universe logos…
Category theory for logic	hyperdoctrine existential elementary doctrine tripos
$D\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{Pos}$	indexed poset Grothendieck fibration elementary fibration…

F.W. Lawvere

Adjointness in foundations *Dialectica* 1969, available also in *Repr. TAC*

Categories and Logic

Logic for categories	monoidal closed category
	cartesian closed category
	Grothendieck topos
Sub: $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{P}os$	exact category
	elementary topos
	pretopos
	Heyting pretopos
	arithmetic universe
	logos...
Category theory for logic	hyperdoctrine
	existential elementary doctrine
	tripos
$D\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{P}os$	indexed poset
	Grothendieck fibration
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F.W. Lawvere
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Categories and Logic

Logic for categories

monoidal closed category
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$\mathrm{Sub}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{P}os$

Category theory for logic

hyperdoctrine
existential elementary doctrine
tripos
indexed poset
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$D\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{P}os$

F.W. Lawvere
Adjointness in foundations *Dialectica* 1969, available also in *Repr. TAC*

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is a *primary doctrine* when

- $\mathcal{B}$ has finite products
- each fibre $P(b)$ has finite meets $$\begin{tikzcd} P(b)\times P(b) \arrow[r, shift right, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, shift right, "\langle \mathrm{id}, \mathrm{id}\rangle"'] \arrow[r, shift left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, shift left, "\top_b"] \end{tikzcd}$$
- the families $(\wedge_b)_b\colon P\times P \overset{\cdot}{\longrightarrow} P$ and $(\top_b)_b\colon 1 \overset{\cdot}{\longrightarrow} P$ are natural

F.W. Lawvere
Equality in hyperdoctrines and the comprehension schema as an adjoint functor
Proc.AMS Symp.Pure Math. 1970

B. Jacobs
Categorical Logic and Type Theory 1999

A.M. Pitts
Categorical logic *Handbook of Logic in Computer Science, vol. 5, 2000*

M.E. Maietti, G. R.
Quotient completion for the foundation of constructive mathematics *Log.Univer.* 2013

$\wp\colon \mathcal{Set}^{\mathrm{op}} \longrightarrow \mathcal{Pos}$

The powerset functor

$$\begin{tikzcd}
\mathcal{Set}^{\mathrm{op}} \arrow[rrr, "\wp"] & & & \mathcal{Pos} \\
S \arrow[rrr, mapsto] \arrow[dd, red] & & & \wp(S) \arrow[dd] \\
& f \arrow[r, mapsto] & f^{-1} & \\
S' \arrow[rrr, mapsto] & & & \wp(S')
\end{tikzcd}$$

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \qquad f^{-1}(S) = S'$$

$\mathrm{LT}^{\mathcal{T}} : \mathcal{Ctx}_{\mathcal{L}}^{\mathrm{op}} \longrightarrow \mathcal{Pos}$

$\mathcal{T}$: a theory in the $\wedge$-fragment of first order logic

$$\begin{tikzcd}
\mathcal{Ctx}_{\mathcal{L}}\colon (x_{i_1}, \ldots, x_{i_n}) \arrow[r, "{(t_1, \ldots, t_m)}"'] \arrow[rr, bend right, "{(s_1[t_1/x_{j_1}, \ldots, t_m/x_{j_m}], \ldots, s_k[t_1/x_{j_1}, \ldots, t_m/x_{j_m}])}"'] & (x_{j_1}, \ldots, x_{j_m}) \arrow[r, "{(s_1, \ldots, s_k)}"'] & (x_{\ell_1}, \ldots, x_{\ell_k})
\end{tikzcd}$$

$\mathrm{LT}^{\mathcal{T}}(x_{i_1}, \ldots, x_{i_n})$: the Lindenbaum-Tarski algebra of formulas with free variables among $(x_{i_1}, \ldots, x_{i_n})$

$$\begin{tikzcd}
\mathcal{Ctx}_{\mathcal{L}}^{\mathrm{op}} \arrow[r, "\mathrm{LT}^{\mathcal{T}}"] & \mathcal{Pos} \\
\vec{x}_j \arrow[r, mapsto] \arrow[dd] & \mathrm{LT}^{\mathcal{T}}(\vec{x}_j) \arrow[dd] \\
\vec{t} \arrow[r, mapsto] & (-)[\vec{t}/\vec{x}_i] \\
\vec{x}_i \arrow[r, mapsto] & \mathrm{LT}^{\mathcal{T}}(\vec{x}_i)
\end{tikzcd}$$

$$(\varphi \wedge \psi)[\vec{t}/\vec{x}_i] = \varphi[\vec{t}/\vec{x}_i] \wedge \psi[\vec{t}/\vec{x}_i] \qquad \top[\vec{t}/\vec{x}_i] = \top$$

Sub: $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{P}os$

$\mathcal{C}$: a category with finite limits

Sub(c): the poset reflection of the preorder

$$\begin{tikzcd}
a \arrow[rr] \arrow[dr, tail, "m"'] & & b \arrow[dl, tail, "n"] \\
& c &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[rr, "\mathrm{Sub}"] & & \mathcal{P}os \\
c \arrow[rr, mapsto] \arrow[d, "f"] & & \mathrm{Sub}(c) \arrow[d, "f^*"] \\
c' \arrow[rr, mapsto] & & \mathrm{Sub}(c')
\end{tikzcd}$$

$f \longmapsto f^*$

$$\begin{tikzcd}
& p' \arrow[rrrr] \arrow[ddl] \arrow[dr] \arrow[ddd, tail, "f^*(m \wedge n)"] & & & & p \arrow[dr] \arrow[ddl] \arrow[ddd, tail, "m \wedge n"] & \\
& & b' \arrow[rrrr] \arrow[ddl, tail, "f^*n"] & & & & b \arrow[ddl, tail, "n"] \\
a' \arrow[rrrr] \arrow[dr, tail, "f^*m"] & & & & a \arrow[dr, tail, "m"] & & \\
& c' \arrow[rrrr, "f"] & & & & c &
\end{tikzcd}$$

$\mathrm{Sub}_{\mathcal{M}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{P}os$

$\mathcal{C}$: a category with finite limits

$\mathcal{M}$: a class of monos in $\mathcal{C}$ closed under identities and pullbacks

$\mathrm{Sub}_{\mathcal{M}}(c) \subseteq \mathrm{Sub}(c)$: the sub-poset of $\mathrm{Sub}(c)$ on those equivalence classes with representatives in $\mathcal{M}$

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[r, "\mathrm{Sub}_{\mathcal{M}}"] & \mathcal{P}os \\
c \arrow[r, mapsto] \arrow[dd, "f"] & \mathrm{Sub}_{\mathcal{M}}(c) \arrow[dd, "f^{*}"] \\
{} \arrow[r, mapsto] & {} \\
c' \arrow[r, mapsto] & \mathrm{Sub}_{\mathcal{M}}(c')
\end{tikzcd}$$

$\mathrm{Sub}_{\mathcal{M}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{P}os$ is a *primary sub-doctrine* of $\mathrm{Sub}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{P}os$

Homomorphisms of primary doctrines

Definition

A *homomorphism of primary doctrines* from $P\colon \mathcal{A} \longrightarrow \mathcal{P}os$ to $Q\colon \mathcal{B} \longrightarrow \mathcal{P}os$ is a pair (F, α) where

- $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ preserves finite products
- $$\begin{tikzcd} \mathcal{A}^{\mathrm{op}} \arrow[rrd, "P"] \arrow[dd, "F^{\mathrm{op}}"'] & & \\ & \alpha \downarrow\cdot & \mathcal{P}os \\ \mathcal{B}^{\mathrm{op}} \arrow[rru, "Q"'] & & \end{tikzcd}$$
- every $\alpha_a\colon P(a) \longrightarrow Q(F(a))$ preserves finite meets

They actually form a 2-category $\mathcal{PD}oct$

Homomorphisms of primary doctrines

Examples

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[rrd, "\mathrm{Sub}_{\mathcal{M}}"] \arrow[dd, "\mathrm{Id}_{\mathcal{C}}^{\mathrm{op}}"'] & & \\
& \subseteq \, \dot{\downarrow} & \mathcal{P}os \\
\mathcal{C}^{\mathrm{op}} \arrow[rru, "\mathrm{Sub}"'] & &
\end{tikzcd}$$

A homomorphism

$$\begin{tikzcd}
\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \arrow[rrd, "\mathrm{LT}^{\mathcal{T}}"] \arrow[dd, "F^{\mathrm{op}}"'] & & \\
& \alpha \, \dot{\downarrow} & \mathcal{P}os \\
\mathcal{S}et^{\mathrm{op}} \arrow[rru, "\wp"'] & &
\end{tikzcd}$$

is exactly a model of the Horn sentences provable in $\mathcal{T}$

$H^{(-)}\colon \mathcal{S}et^{\mathrm{op}} \longrightarrow \mathcal{P}os$

H: an inf-semilattice

$$\begin{tikzcd}
\mathcal{S}et^{\mathrm{op}} \arrow[r, "H^{(-)}"] & \mathcal{P}os \\
S \arrow[r, mapsto] \arrow[dd, "f"'] & H^S \arrow[dd, "- \circ f"] \\
{} \arrow[r, mapsto] & {} \\
S' \arrow[r, mapsto] & H^{S'}
\end{tikzcd}$$

with the pointwise order

$$\begin{aligned}
\big(H^f(\varphi \wedge \psi)\big)(s) &= (\varphi \wedge \psi)(f(s)) = \varphi(f(s)) \wedge \psi(f(s)) \\
&= H^f(\varphi)(s) \wedge H^f(\psi)(s) = \big(H^f(\varphi) \wedge H^f(\psi)\big)(s)
\end{aligned}$$

$\mathrm{Vrn}\colon \mathscr{A}^{\mathrm{op}} \longrightarrow \mathscr{P}os$

$\mathscr{A}$: a category with finite (strict) products and weak pullbacks

$\mathrm{Vrn}(a)$: the poset reflection of the category $\mathscr{A}/a$

$$\begin{tikzcd}
v \arrow[rr] \arrow[dr, "e"'] & & w \arrow[dl, "d"] \\
& a &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{A}^{\mathrm{op}} \arrow[rr, "\mathrm{Vrn}"] & & \mathscr{P}os & & \\
a \arrow[rr, mapsto] \arrow[dd] & & \mathrm{Vrn}(a) \arrow[dd] \arrow[r, phantom, "\ni"] & {[e]} \arrow[dd, mapsto] \\
f \arrow[rr, mapsto] & & \mathrm{Vrn}(f) & \\
a' \arrow[rr, mapsto] & & \mathrm{Vrn}(a') \arrow[r, phantom, "\ni"] & {[e']}
\end{tikzcd}$$

where

$$\begin{tikzcd}
v' \arrow[r] \arrow[d, "e'"'] \arrow[dr, phantom, "\text{w.p.}"] & v \arrow[d, "e"] \\
a' \arrow[r, "f"'] & a
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Vrn}(a) \arrow[r, bend right, "\mathrm{Vrn}(f)"'] \arrow[r, phantom, "\perp"] & \mathrm{Vrn}(a') \arrow[l, bend right, "f \circ -"']
\end{tikzcd}$$

M. Grandis
Weak subobjects and the epi-monic completion of a category *J.Pure Appl.Algebra* 2000

The internal logic of a primary doctrine

Given a primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$
consider the many-sorted language the objects of $\mathcal{B}$ as sorts
Write arrows in $\mathcal{B}$ as terms in context $x_1 : b_1, \ldots, x_n : b_n$ as follows:

- $x_i : b_i$ is the i-th projection $\quad \pi_i \colon b_1 \times \ldots \times b_n \longrightarrow b_i$
- for $t : b$ and $f \colon b \longrightarrow b'$ in $\mathcal{B}$, the term $f(t)$ is the composite

$$b_1 \times \ldots \times b_n \xrightarrow{t} b \xrightarrow{f} b'$$

- for $t_1 : b'_1 \ldots t_m : b'_m$ the term $\langle t_1, \ldots, t_n \rangle : b'_1 \times \ldots \times b'_m$ is the unique arrow determined into the product $b'_1 \times \ldots \times b'_n$ by the family $\quad t_j \colon b_1 \times \ldots \times b_n \longrightarrow b'_j \quad j = 1, \ldots, m$

Terms $\vec{x} \mid t_1 : b$ and $\vec{x} \mid t_2 : b$ are equal iff their equality follows from the axioms

$$\mathrm{id}_b(t) = t \qquad g(f(t)) = (g \circ f)(t) \qquad \pi_i\langle t_1, \ldots, t_n \rangle = t_i \qquad \langle \pi_1(t), \ldots, \pi_n(t) \rangle = t$$

The internal logic of a primary doctrine, II

Write objects in fibres $P(b)$ as formulas in context $x_1 : b_1, \ldots, x_n : b_n$ as follows:

- for $t : b$ and $\varphi \in P(b)$ in $\mathcal{B}$, the formula $\varphi(t)$ is $P(t)(\varphi) \in P(b_1 \times \ldots \times b_n)$

Write $\quad x_1 : b_1, \ldots, x_n : b_n \mid \Gamma \vdash_{\mathcal{T}_P} \varphi \quad$ when $\quad \bigwedge_{\gamma \in \Gamma} \gamma \leq \varphi \quad$ in $P(b_1 \times \ldots \times b_n)$

The following rules are derivable

$$\frac{\vec{x} \mid \Gamma, \varphi, \psi, \Delta \vdash_{\mathcal{T}_P} \theta}{\vec{x} \mid \Gamma, \psi, \varphi, \Delta \vdash_{\mathcal{T}_P} \theta} \qquad \frac{\vec{x} \mid \Gamma \vdash_{\mathcal{T}_P} \theta}{\vec{x} \mid \Gamma, \varphi \vdash_{\mathcal{T}_P} \theta}$$

$$\frac{}{\vec{x} \mid \Gamma, \varphi \vdash_{\mathcal{T}_P} \varphi} \qquad \frac{\vec{x} \mid \Gamma \vdash_{\mathcal{T}_P} \varphi \qquad \vec{x} \mid \Gamma, \varphi \vdash_{\mathcal{T}_P} \psi}{\vec{x} \mid \Gamma \vdash_{\mathcal{T}_P} \psi} \qquad \frac{\vec{x} \mid \Gamma, \varphi \vdash_{\mathcal{T}_P} \psi}{\vec{x}' \mid \Gamma, \varphi(t) \vdash_{\mathcal{T}_P} \psi(t)}$$

$$\frac{}{\vec{x} \mid \Gamma \vdash_{\mathcal{T}_P} \top} \qquad \frac{\vec{x} \mid \Gamma \vdash_{\mathcal{T}_P} \varphi \wedge \psi}{\vec{x} \mid \Gamma \vdash_{\mathcal{T}_P} \varphi} \qquad \frac{\vec{x} \mid \Gamma \vdash_{\mathcal{T}_P} \varphi \wedge \psi}{\vec{x} \mid \Gamma \vdash_{\mathcal{T}_P} \psi} \qquad \frac{\vec{x} \mid \Gamma \vdash_{\mathcal{T}_P} \varphi \qquad \vec{x} \mid \Gamma \vdash_{\mathcal{T}_P} \psi}{\vec{x} \mid \Gamma \vdash_{\mathcal{T}_P} \varphi \wedge \psi}$$

Elementary doctrines

Definition

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is *elementary* when
for each parametrized diagonal $d_{z,b} \overset{\mathrm{df}}{=} \langle \pi_1, \pi_2, \pi_2 \rangle : z \times b \longrightarrow z \times b \times b$ in $\mathcal{B}$

$$\begin{tikzcd} P(z \times b \times b) \arrow[r, "P(d_{z,b})"] \arrow[r, phantom, shift right=2, "\top"] & P(z \times b) \arrow[l, bend left, shift left=2, "\exists_{d_{z,b}}"] \end{tikzcd}$$

- the left adjoints are natural in z
- they satisfy Frobenius Reciprocity

$$\exists_{d_{z,b}}[P(d_{z,b})(\alpha) \wedge \beta] \xrightarrow{\sim} \alpha \wedge \exists_{d_{z,b}}(\beta)$$

F.W. Lawvere
Equality in hyperdoctrines and the comprehension schema as an adjoint functor
Proc.AMS Symp.Pure Math. 1970

Examples of elementary doctrines

$$\mathcal{S}et^{\mathrm{op}} \xrightarrow{\wp} \mathcal{P}os$$

$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Vrn}} \mathcal{P}os$ for $\mathcal{A}$ with finite products and weak pullbacks

$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathcal{P}os$ for $\mathcal{C}$ with finite products and pullbacks of monos

$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}_{\mathcal{M}}} \mathcal{P}os$ for c with finite products and
$\mathcal{M}$ a class of monos closed under pullbacks and diagonals

A characterization of elementary doctrines

Theorem

Let $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ *be a primary doctrine. TFAE*

(i) *P is elementary*

(ii) *for every b in $\mathcal{B}$ there is δ^P_b in $P(b \times b)$ such that*

  (a) $x : b \vdash_{\mathcal{T}_P} \delta^P_b(x, x)$

  (b) $x_1, y_1 : b_1, x_2, y_2 : b_2 \mid \delta^P_{b_1}(x_1, y_1), \delta^P_{b_2}(x_2, y_2) \vdash_{\mathcal{T}_P} \delta^P_{b_1 \times b_2}(\langle x_1, x_2 \rangle, \langle y, y_2 \rangle)$

  (c) $x, y : b \mid \beta(x), \delta^P_b(x, y) \vdash_{\mathcal{T}_P} \beta(y)$ *for every β in $P(b)$*

Corollary

Let $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ *be an elementary doctrine. Then*

- $x, y : b, z : b' \mid \alpha(x, z), \delta^P_b(x, y) \vdash_{\mathcal{T}_P} \alpha(y, z)$ *for every α in $P(b \times b')$*
- $x, y : b \mid \delta^P_b(x, y) \vdash_{\mathcal{T}_P} \delta^P_{b'}(f(x), f(y))$

More examples of elementary doctrines

$\mathcal{S}et^{\mathrm{op}} \xrightarrow{H^{(-)}} \mathcal{P}os$ for H an inf-semilattice with a least element

$$\begin{tikzcd} S \times S \arrow[r, "\delta^{P}_{S}"] & H \\ (s_1, s_2) \arrow[r, mapsto] & \begin{cases} \top & \text{if } s_1 = s_2 \\ \bot & \text{if } s_1 \neq s_2 \end{cases} \end{tikzcd}$$

$\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}^{\mathcal{T}}} \mathcal{P}os$ for T a theory in the $\wedge =$-fragment of first order logic

$\delta^{\mathrm{LT}^{\mathcal{T}}}_{(x_{i_1},\ldots,x_{i_n})}$ is the formula $x_{i_1} = x'_{i_1} \wedge \ldots \wedge x_{i_n} = x'_{i_n}$

Elementary doctrines are 2-coalgebras

$$\begin{tikzcd}
\mathcal{PD}oct \arrow[loop above, "C"] \arrow[rr, bend left, "R"] \arrow[rr, phantom, "\top"] \arrow[d, shift right=2] \arrow[d, phantom, "\vdash"] & & \mathcal{ED}oct \arrow[ll, hook'] \arrow[lld, "J", "\equiv"'] \\
\text{Ps-}C\text{-Coalg} \arrow[u, hook', shift right=2] & &
\end{tikzcd}$$

Theorem

The comonad on C is KZ and the comparison functor J is a 2-equivalence

J. Emmenegger, F. Pasquali, G. R.
Elementary doctrines as coalgebras *J.Pure Appl.Algebra* 2020

Elementary doctrines are 2-coalgebras

$$\begin{tikzcd}
\mathcal{PD}oct \arrow[loop above, "C"] \arrow[rr, shift left=2, "R"] \arrow[rr, phantom, "\top"] \arrow[d, shift right=2] \arrow[d, phantom, "\vdash"] & & \mathcal{ED}oct \arrow[ll, hook', shift left=2] \arrow[loop above, "M"] \arrow[rr, shift left=2, "(-)_{\mathrm{q}}"] \arrow[rr, phantom, "\perp"] \arrow[dll, "\equiv", "J"'] \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] & & \mathcal{QED}oct \arrow[ll, hook', shift left=2] \arrow[dll, "\equiv"] \\
\text{Ps-}C\text{-Coalg} \arrow[u, hook, shift right=2] & & \text{Ps-}M\text{-Alg} \arrow[u, hook, shift right=2] & &
\end{tikzcd}$$

M.E. Maietti, G. R.
Elementary quotient completion *Theory Appl.Categ.* 2013
F. Pasquali
A co-free construction for elementary doctrines *Appl.Categ. Structures* 2015
D. Trotta
Existential completion and pseudo-distributive laws:
an algebraic approach to the completion of doctrines *PhD Thesis* 2019

The functor of the comonad on $\mathcal{PD}oct$

Given $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ a primary doctrine

the primary doctrine $C(P)$ is $\mathcal{E}_P^{\mathrm{op}} \xrightarrow{\mathrm{Des}_P} \mathcal{P}os$ where

Objects of $\mathcal{E}_P$ are (b, ρ) where

- b is an object of $\mathcal{B}$
- ρ is an object of $P(b \times b)$ such that
  - $x : b \vdash_{\mathcal{T}_P} \rho(x, x)$
  - $x, y : b \mid \rho(x, y) \vdash_{\mathcal{T}_P} \rho(y, x)$
  - $x, y, z : b \mid \rho(x, y), \rho(y, z) \vdash_{\mathcal{T}_P} \rho(x, z)$

Arrows of $\mathcal{E}_P$ are $f\colon (b, \rho) \longrightarrow (b', \rho')$ where

- $f\colon b \longrightarrow b'$ is in $\mathcal{C}$ such that
  - $x, y : b \mid \rho(x, y) \vdash_{\mathcal{T}_P} \rho'(f(x), f(y))$

$\mathrm{Des}_P(b, \rho)$ is the sub-poset of $P(b)$ on the objects β in $P(b)$ such that

$$x, y : b \mid \beta(x), \rho(x, y) \vdash_{\mathcal{T}_P} \beta(y)$$

Examples

$$\begin{tikzcd}
\mathcal{P}\mathcal{D}oct \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] & \mathcal{E}\mathcal{D}oct \arrow[l, hook']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}tx_{T}^{\mathrm{op}} \arrow[r, "\mathrm{LT}_{T}"] & \mathcal{P}os & \mathcal{C}tx_{T^{\mathrm{eq}}}^{\mathrm{op}} \arrow[r, "\mathrm{LT}_{T^{\mathrm{eq}}}"] & \mathcal{P}os
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{op}} \arrow[r, "\mathrm{Vrn}"] & \mathcal{P}os & \mathcal{E}_{\mathrm{Vrn}}^{\mathrm{op}} \arrow[r, two heads] \arrow[rr, bend right, "R(\mathrm{Vrn})"'] & (\mathcal{A}_{\mathrm{ex}})^{\mathrm{op}} \arrow[r, "\mathrm{Sub}"] & \mathcal{P}os
\end{tikzcd}$$

B. Poizat
Une théorie de Galois imaginaire *J. Symb. Log.* 1983
A. Carboni, E. Vitale
Regular and exact completions *J.Pure Appl.Algebra* 1998

Examples

$$\begin{tikzcd} \mathcal{PD}oct \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] & \mathcal{ED}oct \arrow[l, hook'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Fzs}(H)^{\mathrm{op}} \arrow[r, "\mathrm{Gr}\left(H^{(-)}\right)"] & \mathcal{P}os \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}^{\mathrm{op}}_{\mathrm{Gr}\left(H^{(-)}\right)} \arrow[r, two heads] \arrow[rr, bend right, "R\left(\mathrm{Gr}\left(H^{(-)}\right)\right)"'] & \mathrm{sep}([\mathcal{H}^{\mathrm{op}}, \mathcal{S}et])^{\mathrm{op}} \arrow[r, "\mathrm{ClSub}"] & \mathcal{P}os \end{tikzcd}$$

M.P. Fourman, D.S. Scott
Sheaves and logic *Applications of sheaves* LNM#753, 1979
G.P. Monro
Quasitopoi, logic and Heyting valued-models *J.Pure Appl.Algebra* 1986
O. Wyler
Lecture notes on topoi and quasitopoi 1991

Examples

$$\begin{tikzcd}
\mathcal{PD}oct \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] & \mathcal{ED}oct \arrow[l, hook']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Fzs}(H)^{\mathrm{op}} \arrow[r, "\mathrm{Gr}\left(H^{(-)}\right)"] & \mathcal{P}os
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}^{\mathrm{op}}_{\mathrm{Gr}\left(H^{(-)}\right)} \arrow[r, two heads] \arrow[rr, bend right=10, "R\left(\mathrm{Gr}\left(H^{(-)}\right)\right)"'] & \mathrm{sep}([\mathcal{H}^{\mathrm{op}}, \mathcal{S}et])^{\mathrm{op}} \arrow[r, "\mathrm{ClSub}"] & \mathcal{P}os
\end{tikzcd}$$

Proposition

The primary doctrine $\mathrm{Fzs}(H)^{\mathrm{op}} \xrightarrow{\mathrm{Gr}\left(H^{(-)}\right)} \mathcal{P}os$ *is the comprehension completion of the primary doctrine* $\mathcal{S}et^{\mathrm{op}} \xrightarrow{H^{(-)}} \mathcal{P}os$

Examples

$$\begin{tikzcd} \mathcal{PD}oct \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] & \mathcal{ED}oct \arrow[l, hook'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Fzs}(H)^{\mathrm{op}} \arrow[r, "\mathrm{Gr}\left(H^{(-)}\right)"] & \mathcal{P}os \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}^{\mathrm{op}}_{\mathrm{Gr}\left(H^{(-)}\right)} \arrow[r, two heads] \arrow[rr, bend right, "R\left(\mathrm{Gr}\left(H^{(-)}\right)\right)"'] & \mathrm{sep}([\mathcal{H}^{\mathrm{op}}, \mathcal{S}et])^{\mathrm{op}} \arrow[r, "\mathrm{ClSub}"] & \mathcal{P}os \end{tikzcd}$$

$\mathrm{Fzs}(H)$:

$$\begin{tikzcd} (S, \alpha) \arrow[r, "f", "f : S \to S'"'] & (S', \alpha') \\ \text{s.t. } \alpha \in H^S & \text{s.t. } \alpha(s) \leq \alpha'(f(s)),\ s \in S \qquad \text{s.t. } \alpha' \in H^{S'} \end{tikzcd}$$

$\mathrm{Gr}\left(H^{(-)}\right)(S, \alpha)$: the sub-poset of H^S on the objects β such that $\beta(s) \leq \alpha(s)$ for all $s \in S$

Explaining the quotients

The quotients $\mathcal{E}_{\mathrm{Vrn}} \twoheadrightarrow \mathcal{A}_{\mathrm{ex}}$ and $\mathcal{E}_{\mathrm{Gr}(H^{(-)})} \twoheadrightarrow \mathrm{sep}([\mathcal{H}^{\mathrm{op}}, \mathcal{S}et])$ identify arrows exactly when they have the same actions on “parametrized propositions”

Proposition

Let $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ *be an elementary doctrine.*
For $f, g\colon b \longrightarrow c$ *the following are equivalent*

(i) $x : b, z : b' \mid \alpha(f(x), z) \dashv\vdash_{\mathcal{T}_P} \alpha(g(x), z)$ *for every* α *in* $P(c \times b')$

(ii) $x : b \vdash_{\mathcal{T}_P} \delta^P_c(f(x), g(x))$

M.E. Maietti, G. R.
Unifying Exact Completions *Appl.Categ. Structures* 2013
C.J. Cioffo
Homotopy setoids and generalized quotient completion *PhD thesis*, 2022

Elementary fibrations

Definition

A fibration $\mathcal{E} \xrightarrow{p} \mathcal{B}$ is *elementary* when

- $\mathcal{B}$ and $\mathcal{E}$ have finite products, and p preserves them
- for each parametrized diagonal $d_{z,b} \overset{\mathrm{df}}{=} \langle \pi_1, \pi_2, \pi_2 \rangle : z \times b \longrightarrow z \times b \times b$ in $\mathcal{B}$

$$\begin{tikzcd} p^{-1}(z \times b \times b) \arrow[r, "p^{-1}(d_{z,b})", shift left=2] \arrow[r, phantom, "\top"] & p^{-1}(z \times b) \arrow[l, "\exists_{d_{z,b}}", shift left=2, bend left=5] \end{tikzcd}$$

and the left adjoints are natural in z, and satisfy Frobenius Reciprocity

$$\exists_{d_{z,b}}[p^{-1}(d_{z,b})(\alpha) \wedge \beta] \xrightarrow{\sim} \alpha \wedge \exists_{d_{z,b}}(\beta)$$

Elementary fibrations are pseudo-2-coalgebras

$$\begin{tikzcd}
\mathcal{PF}ib \arrow[loop above, "T"] \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] \arrow[d, shift right] \arrow[d, phantom, "\vdash"] & \mathcal{EF}ib \arrow[l, hook', shift left] \arrow[dl, "K"] \\
\text{Ps-}T\text{-Coalg} \arrow[u, hook, shift right] &
\end{tikzcd}$$

Theorem

The comonad on T is KZ and the comparison functor K is a 2-equivalence

J. Emmenegger, F. Pasquali, G. R.
Equality is pseudo-coalgebraic *in preparation*

The functor of the comonad on $\mathcal{PFib}$

For a fibration $\mathcal{E} \xrightarrow{p} \mathcal{B}$, $\mathcal{B}$ and $\mathcal{E}$ with finite products, preserved by p
the fibration $T(p)$ is as follows:
$\mathcal{E}_p$ is a full subcategory of Gpd$(\mathcal{E})$:

$$\begin{tikzcd}
\mathcal{E} \arrow[d, "p"'] & g_0 \arrow[r, bend left, "i"] & g_1 \arrow[l, shift right, "d_0"'] \arrow[l, shift left, "d_1"] \arrow[loop above, "s"] & g_2 \arrow[l, shift right=2, "d'_0"'] \arrow[l, "c"] \arrow[l, shift left=2, "d'_1"] \\
\mathcal{B} & b = p(g_0) \arrow[r, bend left, "{\langle \mathrm{id}_b, \mathrm{id}_b \rangle}"] & b \times b \arrow[l, shift right, "\pi_1"'] \arrow[l, shift left, "\pi_2"] \arrow[loop above, "{\langle \pi_2, \pi_1 \rangle}"] & b \times b \times b \arrow[l, shift right=2, "{\langle \pi_1, \pi_2 \rangle}"'] \arrow[l, "{\langle \pi_1, \pi_3 \rangle}"] \arrow[l, shift left=2, "{\langle \pi_2, \pi_3 \rangle}"]
\end{tikzcd}$$

$\mathcal{D}_p$ is a full subcategory of Act$(\mathcal{E})$:

$$\begin{tikzcd}
e_0 \arrow[d, "a"] \arrow[dr, phantom, "p.b."] & e_1 \arrow[l, "d'_0"'] \arrow[r, "m"] \arrow[d, "a'"] & e_0 \arrow[d, "a"] \\
g_0 & g_1 \arrow[l, "d_0"'] \arrow[r, "d_1"] & g_0
\end{tikzcd}$$

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