My 70 Years with Heyting Algebras

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Early History of Lattice Theory

Richard Dedekind (1831 – 1916) was an influential German mathematician. However, hostility towards lattice theory began when Dedekind published the two fundamental papers that brought the theory to life well over one hundred years ago. ***Kronecker*** in one of his letters accused Dedekind of "losing his mind in abstractions," or something to that effect.

The concept of a lattice comes from two sources. The first source is usually cited as Dedekind's two classic papers, 1897 and 1900. But, by tracing back the references in these, one can see that he was thinking (modular) lattice-theoretically for at least twenty years prior to that. He took notes at ***Dirichlet's*** lectures on number theory and wrote them up as a book with eleven "Supplements" (1863, 1871,1879, and 1893). Section 169 in Supplement XI of the 1893 edition is about lattices, including the axioms, modular law, duality, and the free modular and distributive lattices on three generators—all developed as properties of modules and ideals.

In 1897 Dedekind notes that general lattices were treated by ***Ernst Schröder*** in his famous ***Algebra der Logik*** (1880) where he introduced—but did not name—lattices as orders. There was a well publicized debate in which ***C. S. Pierce*** claimed that all lattices were distributive, but counterexamples were later provided.

Gian-Carlo Rota. "The many lives of lattice theory." Notices of the AMS, vol. 44 (1997), pp. 1440 – 1445.

George Grätzer. "Two problems that shaped a century of lattice theory." Notices of the AMS, vol. 54 (2007), pp. 696 – 707.

More Recent Lattice-Theory Pioneers

Alfred Tarski
Born: 14 January 1901 in Warsaw, Russian Poland
Died: 26 October 1983 in Berkeley, California

J. C. C. McKinsey
Born: 30 April 1908 in Frankfort, Indiana
Died: 26 October 1953 in Palo Alto, California

Saunders Mac Lane
Born: 4 August 1909 in Taftville, Connecticut
Died: 14 April 2005 in San Francisco, California

Andrzej Mostowski
Born: 1 November 1913 in Lwów, Austria-Hungary
Died: 22 August 1975 in Vancouver, Canada

Roman Sikorski
Born: 11 July 1920 in Mszczonów, Poland
Died: 12 September 1983 in Warsaw, Poland

Marshall H. Stone
Born: 8 April 1903 in New York, New York
Died: 9 January 1989 in Madras, India

Garrett Birkhoff
Born: 19 January 1911 in Princeton, New Jersey
Died: 22 November 1996 in Water Mill, New York

Paul R. Halmos
Born: 3 March 1916 in Budapest, Hungary
Died: 2 Oct 2006 in Los Gatos, California

Helena Rasiowa
Born: 20 June 1917 in Vienna, Austria
Died: 9 Aug 1994 in Warsaw, Poland

Stanisław Jaśkowski
Born: 22 April 1906 in Warsaw, Congress Poland
Died:16 November 1965 in Warsaw, Poland

Rasiowa-Sikorski's Textbook Motto

Die Mathematiker sind eine Art Franzosen:
redet man zu ihnen, so übersetzen sie
es in ihre Sprache
und dann ist es alsobald ganz etwas Anderes.

—Johann Wolfgang von Goethe

Helena Rasiowa and Roman Sikorski.
"The Mathematics of Metamathematics."
Panstwowe Wydawnictwo Naukowe, Warszawa, 1963, 519 pp.

The Dutch Intuitionists

L. E. J. Brouwer

Born: 27 February 1881 in Overschie, Netherlands
Died: 2 December 1966 in Blaricum, Netherlands

Arend Heyting

Born: 9 May 1898 in Amsterdam, Netherlands
Died: 9 July 1980 in Lugano, Switzerland

Evert Willem Beth

Born: 7 July 1908 in Almelo, Netherlands
Died: 12 April 1964 in Amsterdam, Netherlands

Dirk van Dalen

Born: 20 December 1932, Amsterdam, Netherlands

Anne Sjerp Troelstra

Born: 10 August 1939 in Maartensdijk, Utrecht
Died: 7 March 2019 in Blaricum, Netherlands

1923. L. E. J. Brouwer: *"On the significance of the principle of excluded middle in mathematics, especially in function theory."*

1927. L. E. J. Brouwer: *"On the domains of definition of functions."*

1927. L. E. J. Brouwer: *"Intuitionistic reflections on formalism."*

Early Papers of Arend Heyting

Die formalen Regeln der intuitionistischen Logik. I, II, III. In: **Sitzungsberichte der preußischen Akademie der Wissenschaften,** 1930, pp. 42–56, pp. 57-71, pp. 158-169.

Mathematische Grundlagenforschung. Intuitionismus. Beweistheorie.
Berlin, Springer, 1934. iv+73 pp.

This pamphlet is one of the series Ergebnisse der Mathematik published by the editors of the Zentralblatt. It deals chiefly with the foundations of mathematics,and mathematical logic, from two points of view, the intuitionism of Brouwer and the formalism of Hilbert, and gives an able, clear, and concise account of the essentials of these two points of view and of the important results which have been obtained in connection with them. As explained in the introduction, no attempt is made to give an account of the logistic formulation of the foundations of mathematics, a subject which is to be treated in a later number of the series.

This work is recommended, not only to mathematical logicians, but also to mathematicians in general who desire an understandable survey of its field. The reviewer knows of no better such survey, indeed of none nearly so good. The first section begins with a notice of Poincaré as historical forerunner of intuitionism, describes the point of view of the French semi-intuitionists as they are here called (Borel, Lebesgue, Baire), the first theory of Weyl, and the point of view of F. Kaufmann, and then gives an account of the intuitionism of Brouwer and of the results in connection with it of Brouwer, Heyting, Kolmogoroff, Glivenko, Gödel, Gentzen, de Loor, Belinfante.

The second section discusses the classical axiomatic method and the concepts of consistency and categoricity, then proceeds to an account of Hubert's formal system and the Hilbert concept of a metamathematical proof of consistency. The consistency proofs of Ackermann and von Neumann are outlined; and brief mention is made of the consistency proof of Herbrand; also of the famous theorem of Gödel and its significance in this connection. And the section ends with a discussion of the relationship between formalism and intuitionism.

The third section gives a description of several other points of view on the foundations of mathematics, notably those of Mannoury and Pasch. The fourth section discusses the relation of mathematics to the natural sciences, comparing the formalistic and the intuitionistic accounts of this relation. At the end of the pamphlet is a five-page bibliography of publications in this field.

ALONZO CHURCH, Reviewer

Early Papers of Marshall H. Stone

Linear Transformations in Hilbert Space and their Applications to Analysis. American Mathematical Society, New York, 1932.

On the structure of Boolean algebras. **Bulletin of the American Mathematical Society**, vol. 39 (1933), p. 200.

Boolean algebras and their application to topology. **Proceedings of National Academy of Sciences**, vol. 20 (1934), pp. 197–202.

Postulates for Boolean algebras and generalized Boolean algebras. **American Journal of Mathematics**, vol. 57 (1935), pp. 703–732.

Subsumption of the theory of Boolean algebras under the theory of rings. **Proceedings of National Academy of Sciences**, vol. 21 (1935), pp.103–105.

The theory of representation for Boolean algebras. **Transactions of the American Mathematical Society**, vol. 40 (1936), pp. 37–111.

Algebraic characterizations of special Boolean algebras. **Fundamenta Mathematica**, vol. 29 (1937), pp. 223–303.

Applications of the theory of Boolean rings to general topology. **Transactions of the American Mathematical Society**, vol. 41 (1937), pp. 375–481.

Topological representations of distributive lattices and Brouwerian logics. **Časopis pro pěstování matematiky a fysiky**, vol. 67 (1938), pp. 1–25.

Early Papers of Alfred Tarski

Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften. **Monatshefte für Mathematik und Physik**, vol. 37 (1930), pp. 361–404.

Über einige fundamentale Begriffe der Metamathematik. **C. R. Soc. Sci. Let. Varsovie**, vol. 23 (1930), pp. 22–29.

Grundzüge des Systemenkalküls I. **Fundamenta Mathematicae**, vol. 25 (1935), pp. 503–526.

Zur Grundlegung der Booleschen Algebra I. **Fundamenta Mathematicae**, vol. 24 (1935), pp. 177–198.

Grundzüge des Systemenkalküls II. **Fundamenta Mathematicae**, vol. 26 (1936), pp. 283–301.

Der Aussagenkalkül und die Topologie. **Fundamenta Mathematicae**, vol. 31 (1938), pp. 103-134.

Ideale in vollständigen Mengenkörpern. I. **Fundamenta Mathematicae***,* vol. 32 (1939), pp. 45-63.

Ideale in vollständigen Mengenkörpern. II. **Fundamenta Mathematicae**, vol. 33 (1945), pp. 51-65.

Logic, Semantics, Metamathematics. Papers from 1923 to 1938. OUP, London, 1956. Translated by J.H. Woodger. Second edition 1983.

Some Very, Very Brief Later History

Gödel gave us two translations: **(1)** classical into intuitionistic using the not-not translation, and **(2)** intuitionistic into S4-modal logic.

Tarski and **McKinsey** reviewed all this algebraically in propositional logic, proving completeness of **(2)**.

Mostowski suggested the algebraic interpretation of quantifiers.

Rasiowa and **Sikorski** went further with first-order logic, giving many completeness proofs (*pace* **Kanger**, **Hintikka** and **Kripke**).

Montague applied higher-order modal logic to linguistics.

Solovay discovered and he and **Scott** then showed how **Cohen's** forcing for ZFC can be reconstructed under **(1)**.

Fitting and **Smullyan** worked it out under **(2)** (also using **(1)**).

And **Topos Theory** had much, much more to say about such relations under **(1)**. (But maybe not as much about **(2)**?)

Some further basic background on sheaf theory and toposes can be found in:

Dana Scott and Michael Fourman. *Sheaves and logic.*
In: **Applications of Sheaves**, Durham Proceedings (1977).
Springer LNM, vol. 753 (1979), pp. 302-401.

Lattices as Partially Ordered Sets

$0 \leq x \leq 1$	Bounded
$x \leq x$	Reflexive
$x \leq y \ \& \ y \leq z \Rightarrow x \leq z$	Transitive
$x \leq y \ \& \ y \leq x \Rightarrow x = y$	Anti-symmetric
$x \vee y \leq z \Leftrightarrow x \leq z \ \& \ y \leq z$	With sups &
$z \leq x \wedge y \Leftrightarrow z \leq x \ \& \ z \leq y$	With infs

An equivalent first-order axiomatization uses just the binary relation. The sups and infs are uniquely determined.

Lattices as Algebras

$x \wedge 1 = x$	Neutral
$x \vee 0 = x$	
$x \wedge x = x$	Idempotent
$x \vee x = x$	
$x \wedge y = y \wedge x$	Commutative
$x \vee y = y \vee x$	
$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	Associative
$x \vee (y \vee z) = (x \vee y) \vee z$	
$x \wedge (x \vee y) = x$	Absorptive
$x \vee (x \wedge y) = x$	

With the partial order being equationally definable:

$$x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y$$

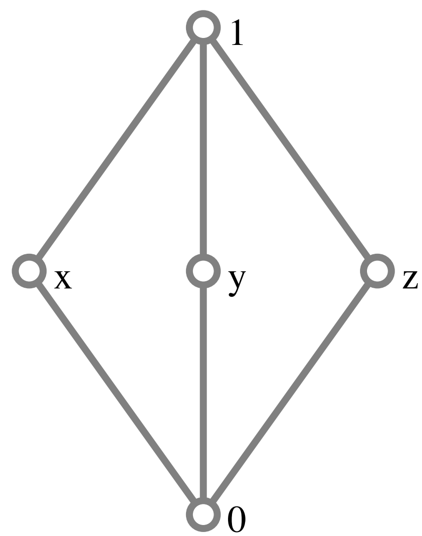
Distributive Lattices

These dual equational axioms are equivalent:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

Note, however, that not all lattices are distributive:



Characterizing Lattice Implication

- **Axiomatizing Heyting Algebras:**

$$x \leq y \to z \Leftrightarrow x \wedge y \leq z$$

- **Axiomatizing Boolean Algebras:**

$$x \leq (y \to z) \vee w \Leftrightarrow x \wedge y \leq z \vee w$$

Note: An alternative definition of a Boolean algebra can also be given using a negation operation:

$$x \leq \neg y \vee z \Leftrightarrow x \wedge y \leq z$$

Note: Implication operations are unique.

Heyting Algebras are Distributive!

$x \wedge y \leq (x \wedge y) \vee (x \wedge z)$	Obviously
$y \leq x \rightarrow ((x \wedge y) \vee (x \wedge z))$	Consequently
$z \leq x \rightarrow ((x \wedge y) \vee (x \wedge z))$	Similarly
$y \vee z \leq x \rightarrow ((x \wedge y) \vee (x \wedge z))$	Consequently
$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$	Consequently
$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$	Obviously
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	Finally

Implication Axiomatized Equationally

$$x \to x = 1$$

$$x \wedge (x \to y) = x \wedge y$$

$$y \wedge (x \to y) = y$$

$$x \to (y \wedge z) = (x \to y) \wedge (x \to z)$$

Note:

$$(x \vee y) \to z = (x \to z) \wedge (y \to z)$$

And a much further exposition on the axiomatics of Heyting algebras can be found at:

https://ncatlab.org/nlab/show/Heyting+algebra

nLab is a wiki for collaborative work on Mathematics, Physics, and Philosophy — especially (but far from exclusively) from the higher structures point of view: with a sympathy towards the tools and perspectives of homotopy theory/algebraic topology, homotopy type theory, higher category theory and higher categorical algebra.

What is a Complete Lattice?

$$\bigvee_{i\in I} x_i \leq y \Leftrightarrow (\forall i\in I)\ x_i \leq y$$

$$y \leq \bigwedge_{i\in I} x_i \Leftrightarrow (\forall i\in I)\ y \leq x_i$$

Key Observation: These lattice-theoretic laws are formally just the laws of the quantifiers ***some*** and ***all***. This means that ***abstract lattice structures*** can be used to give new ***interpretations*** *of logic.*

Note: $\bigwedge_{i\in I} x_i = \bigvee\{\, y \mid (\forall i\in I)\ y \leq x_i \,\}$

What is a Complete Heyting Algebra?

Theorem: Complete Heyting algebras are

($\wedge\bigvee$)-distributive:

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i)$$

and every such lattice **is** a Heyting algebra with:

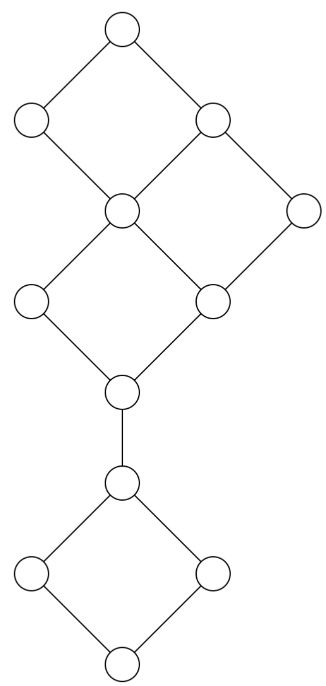
$$y \rightarrow z = \bigvee \{ x \mid x \wedge y \leq z \}$$

Note: The **dual distributive law** does **not** follow for complete Heyting algebras. **Example:** Take the lattice of **open sets** of the unit interval [0,1].

What About Finite Distributive Lattices?

Theorem: Every finite distributive lattice is a complete Heyting algebra, as is the dual lattice.

Note: A finite distributive lattice ***need not*** be isomorphic to its dual lattice.



Deciding Heyting Algebra Equations

Theorem: An equation holds in ***all*** Heyting algebras if, and only if, it holds in all ***finite*** Heyting algebras.

Corollary: The equational theory of Heyting algebras is thus primitive recursively ***decidable***.

Proof: Suppose the equation $\rho = \sigma$ ***fails*** in some Heyting algebra **H**. Think of the particular values of the ***variables*** needed. Since the expressions ρ and σ have only ***finitely*** many ***sub-expressions,*** under this substitution there are thus only finitely many ***values*** of these subexpressions. Let $\mathbf{H_0}$ be the finite ***sublattice*** of **H** generated by these values. It is a Heyting algebra! The subexpressions of ρ and σ that are ***implications*** will necessarily take on their ***correct*** **H** values. Hence, $\rho = \sigma$ ***fails*** in $\mathbf{H_0}$. **Q.E.D.**

Jaśkowski Lattices

Stanisław J. Surma, Andrzej Wroński, Stanisław Zachorowski.
On Jaśkowski-Type Semantics for the Intuitionistic Propositional Logic.
Studia Logica, vol. 34 (1975), pp. 145-148.

The remarkable result of Jaśkowski is the construction of a sequence of finite lattices adequate for intuitionistic propositional logic. This result was published in **1936**, but only with a very condensed sketch of proof. It was **17** years before a more detailed proof was published in by F. G. Rose, who worked out some modifications of the strategy suggested by Jaśkowski for eluding a lemma which he was unable to prove.

Definitions. For any lattice **L**, let $\mathbf{L}^n$ be the n-fold ***Cartesian power*** of **L**, and let $\mathbf{L}^\Delta$ be the lattice obtained from **L** by adding a ***new top element*** in the partial ordering. Let $\mathbf{J}_0$ be the one-element lattice, and, for n > 0, and let

$$\mathbf{J}_{n+1} = ((\mathbf{J}_n)^n)^\Delta.$$

Jaśkowski's Theorem. Each of the $\mathbf{J}_n$ are finite Heyting algebras. And any equation failing in some Heyting algebra will fail in some $\mathbf{J}_n$.

Gödel's Remark 1932. A Heyting algebra with m elements satisfies:

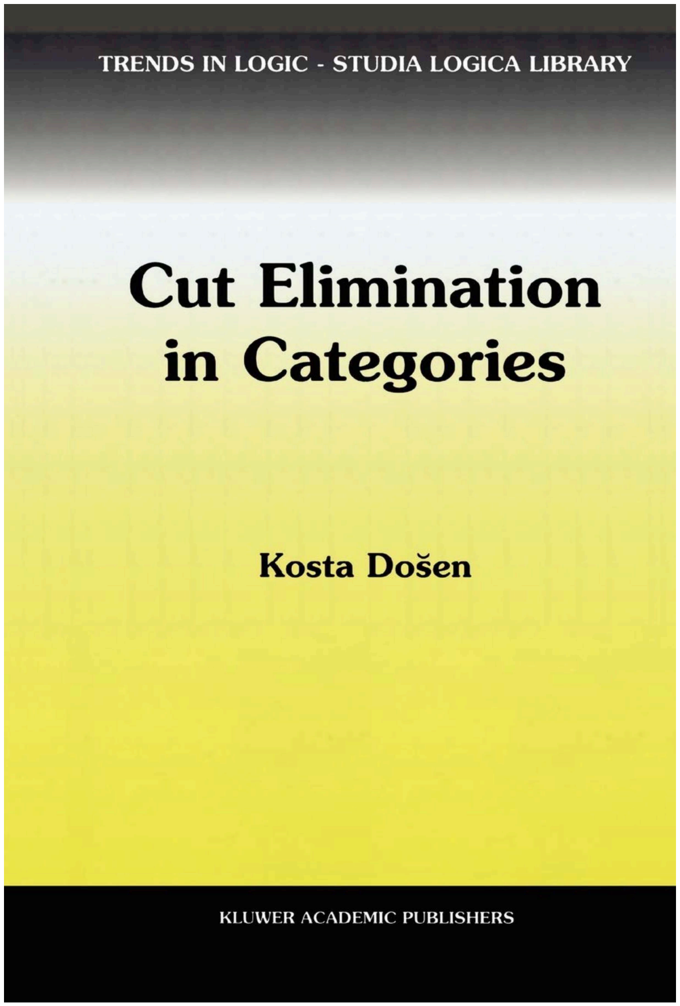
$$\bigvee_{0 \leqq i < j \leqq m} ((x_i \to x_j) \wedge (x_j \to x_i)) = 1$$

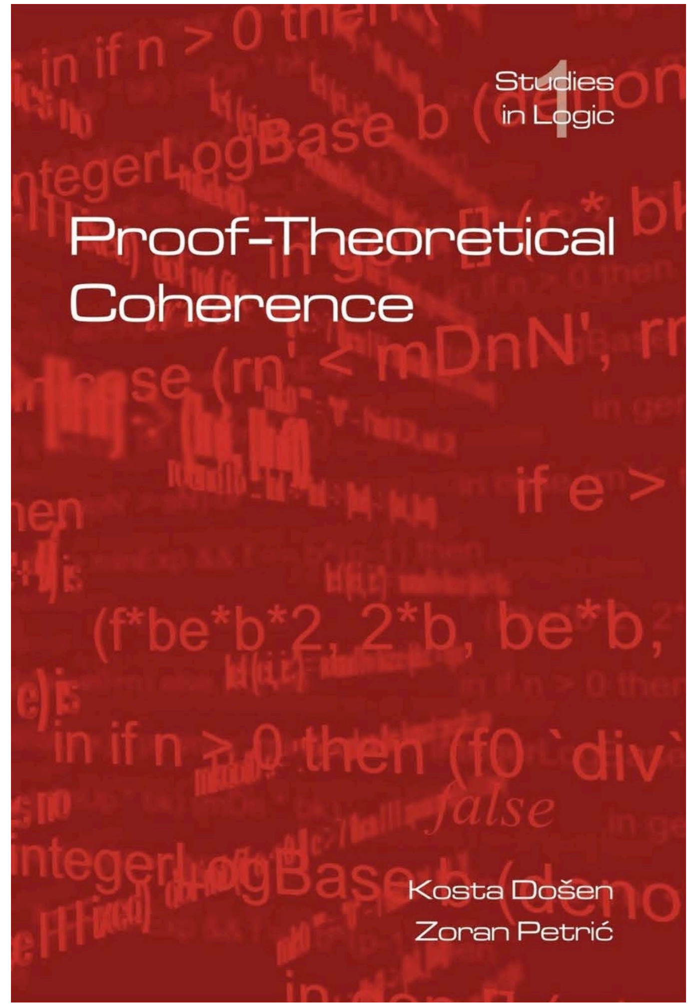
So what?

Recent Work of Kosta Dosen

1999

2004





Dosen References

Kosta Došen. **“Cut Elimination in Categories”**
Trends in Logic, vol. 6, 1999, xii + 241 pp.

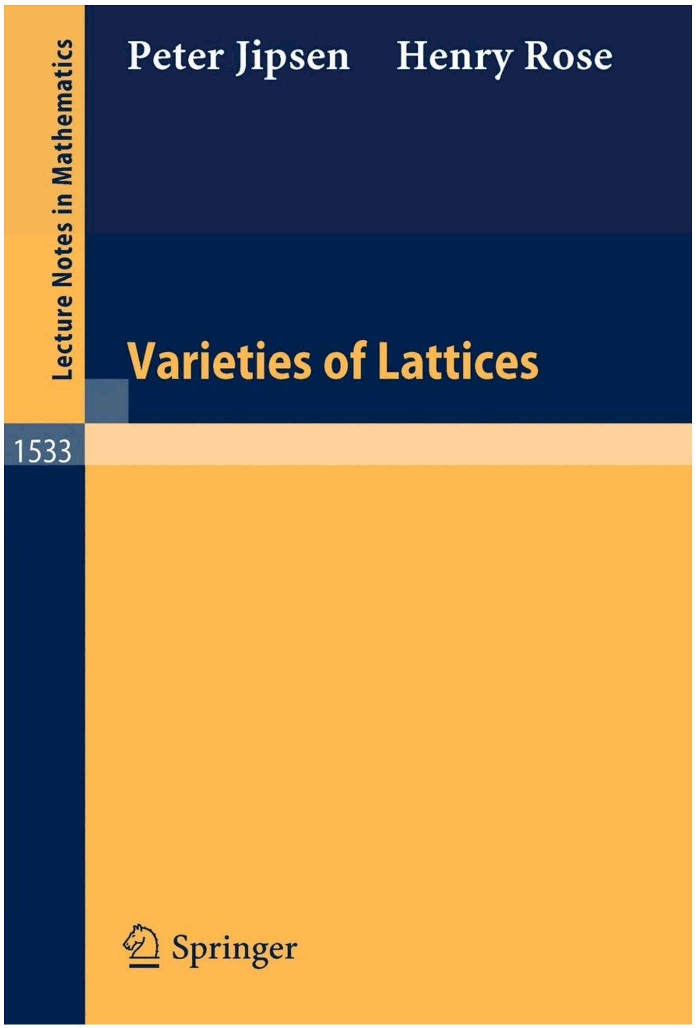
Proof theory and category theory were first drawn together by Lambek some 30 years ago but, until now, the most fundamental notions of category theory (as opposed to their embodiments in logic) have not been explained systematically in terms of proof theory. Here it is shown that these notions, in particular the notion of adjunction, can be formulated in such as way as to be characterized by composition elimination. Besides familiar topics, presented in a novel, simple way, the monograph also contains new results. It can be used as an introductory text in categorical proof theory.

Kosta Došen and Zoran Petric. **"Proof-Theoretical Coherence."**
Studies in Logic, College Publications, 2004, 392 pp.

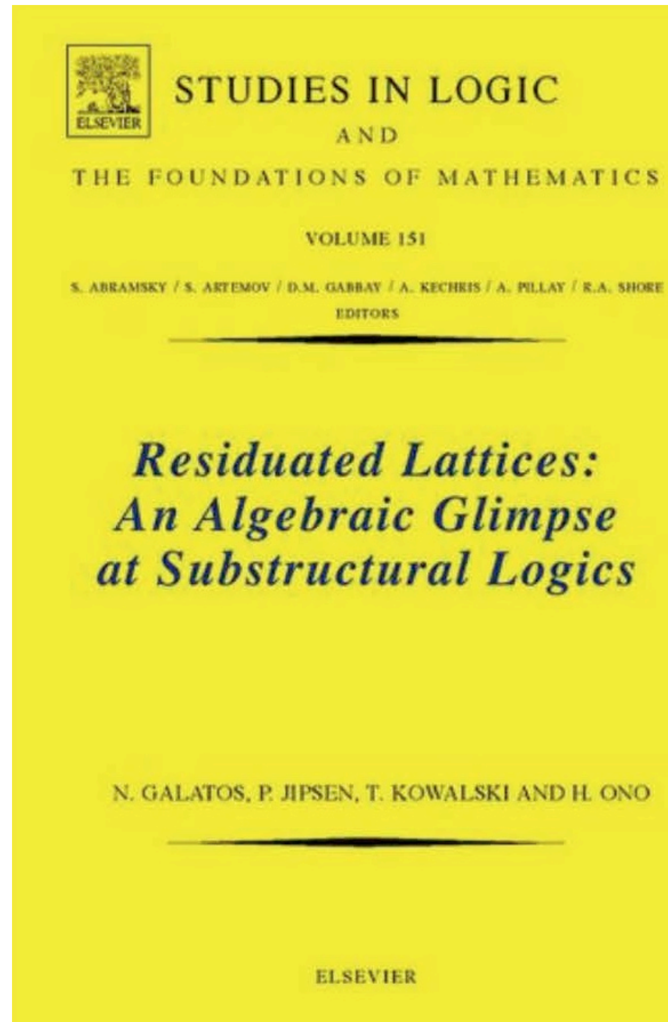
This book in categorial proof theory formulates in terms of category theory a generalization close to linear algebra of the notions of distributive lattice and Boolean algebra. These notions of distributive lattice category and Boolean category codify a plausible nontrivial notion of identity of proofs in classical propositional logic, which is in accordance with Gentzen's cut-elimination procedure for multiple-conclusion sequents modified by admitting new principles called union of proofs and zero proofs. These coherence results yield a simple decision procedure for equality of proofs. Coherence in the same sense is also proved for various more general notions of category that enter into the notions of distributive lattice category and Boolean category. Some of these coherence results, like those for monoidal and symmetric monoidal categories are well known, but are here presented in a new light.

Recent Work of Peter Jipsen

1992



2007



Jipsen References

Peter Jipsen and Henry Rose. **"Varieties of Lattices."**
Springer, Lecture Notes in Mathematics, vol. 1533, 1992, 176 pp.

The study of lattice varieties has experienced rapid growth in the last 30 years, but many of the interesting and deep results discovered in that period have so far only appeared in research papers. This monograph presents the main results about modular and nonmodular varieties, equational bases and the amalgamation property in a uniform way. The first chapter covers preliminaries in an accessible way to readers who has had an introductory course in universal algebra. Each subsequent chapter begins with an historical introduction which sites the original references before presenting the results with complete proofs. Numerous diagrams illustrate the beauty of lattice theory and aid in the visualization of many proofs.

Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono.
"Residuated Lattices: An Algebraic Glimpse at Substructural Logics."
Kindle Edition, 2007, 9793 KB.

The book has two purposes: The first and more obvious one is to present state of the art results in algebraic research into residuated structures related to substructural logics. The second, less obvious but equally important, is to provide a reasonably gentle introduction to algebraic logic. At the beginning, the second objective is predominant.

Within the more technical part of the book another transition process may be traced. Namely, the authors begin with logically inclined technicalities and end with algebraically inclined ones. Here, perhaps, algebraic rendering of Glivenko theorems marks the equilibrium point, at least in the sense that finiteness properties, decidability and Glivenko theorems are of clear interest to logicians, whereas semisimplicity and discriminator varieties are universal algebra par excellence.

Roy Dyckhoff's Investigations

Abstract. Gentzen solved the decision problem for intuitionistic propositional logic in his doctoral thesis and this paper reviews some of the subsequent progress. Solutions to the problem are of importance both for general philosophical reasons and because of their use in implementations of proof assistants based on intuitionistic logic. We tend (despite their importance) to avoid implementation issues in favour of relatively simple calculi where questions such as cut admissibility can be raised and, ideally by syntactic methods, answered. We also have our own implementations of several of the calculi mentioned here, using our own Prolog software, thus allowing sequent calculus rules to be coded clearly and proofs to be displayed using LaTeX, either as trees or linearly.

We are particularly interested in questions of:

1. termination (hence decidability)

2. bicompleteness (extractability of models from failed proof searches)

3. determinism (avoidance of backtracking)

4. simplicity (allows easier reasoning about systems).

Roy Dyckhoff. **"Intuitionistic decision procedures since Gentzen."**
In: Advances in Proof Theory, Springer, 2016, pp. 245-267.