Classical and Constructive Logic together: Ecumenical Systems

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Joint work with
Elaine Pimentel and Luiz Carlos Pereira

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Many Thanks!

Elaine Pimentel and Luiz Carlos Pereira

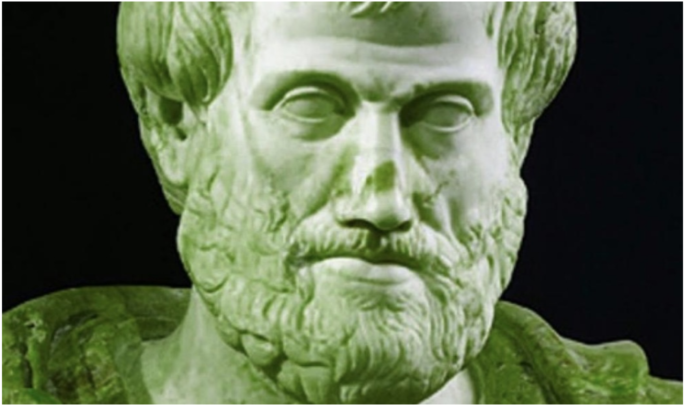
A collection of work together

- *Translations and Prawitz's Ecumenical System*. LC Pereira, E Pimentel, V de Paiva. Studia Logica, 1-16, 2024
- *An ecumenical notion of entailment*. E Pimentel, LC Pereira, V de Paiva. Synthese 198 (Suppl 22), 5391-5413, 2021.
- *A proof theoretical view of ecumenical systems*. E Pimentel, LC Pereira, V de Paiva. Logical and Semantic Frameworks with Applications (LSFA), 2019.

Logic

NOT

Logic



Immanuel Kant famously claimed that Aristotle had discovered all that there was to discover in logic.

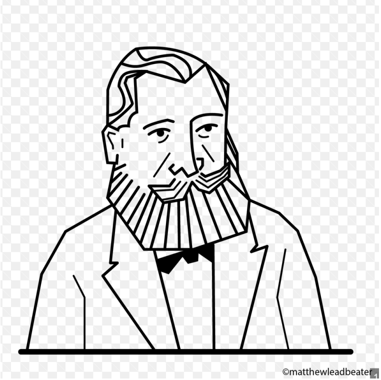
Logic

Around 2000 years of Aristotle's Laws of thought:

- (ID) every proposition implies itself (everything is identical to itself);
- (NC) no proposition is both true and false. (Nothing both holds and does not hold of any one thing at any one time);
- (EM) every proposition is either true or false. (Everything either holds or does not hold of any one thing at any one time)

From Wikipedia, Stephen Read has 5 big principles

Logic



The Fregean revolution at the end of the 19th century did not change the *status quo*. Only one logic: classical logic, now with quantifiers and symbols.

Change


The situation has changed drastically in the last hundred years: several logics presented themselves as extensions or rivals of classical logic.

Changes

- Adding **modal** operators to classical logic produces modal logics of various types (alethic, epistemic, deontic, temporal, program logics, etc). (Lewis 1912)
- These can be considered extensions of classical logic since the scope of logical-conceptual analysis is expanded. But modal operators can be regarded as restricted quantifiers too.
- By contrast questioning the validity of the fundamental principles produces new logics that are **alternatives**
- Intuitionistic logic emerged questioning the validity of the principle of the excluded-middle (Brouwer 1912).
- several paraconsistent logics questioned the unrestricted validity of the principle of non-contradiction.

Intuitionism

Here we interested in (*accidental*) intuitionists.
Intuitionists who want to use classical logic too!

More convincing to exhibit a term t such that
$\exists x.A(x)$ is true means $A(t)$ holds,
then to say that it is not the case that $(\forall x.\neg A(x))$

Moreover, (Krauss 1992) want a logical framework to help us identify where we do **have to** use classical reasoning

Intuitionism: Mathematical motivation

Simple Theorem There exist $x, y \notin \mathbb{Q}$ such that $x^y \in \mathbb{Q}$.

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Proof. Consider $a = \sqrt{2}^{\sqrt{2}}$.
If $a \in \mathbb{Q}$, then take $x = y = \sqrt{2}$.
If $a \notin \mathbb{Q}$, then take $x = a$ and $y = \sqrt{2}$. Then

$$x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$$

□

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Classical mathematician: cool!!!

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Classical mathematician: cool!!!

Intuitionistic mathematician: but $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ or $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$???

Accidental intuitionists

> *We should rather like to persuade classical mathematicians to carry out their proofs distinguishing between intuitionistic and classical logic operators depending on what they actually prove.[...] this way their reasoning stays constructively valid and therefore preserves the possibility of a computational interpretation.*

(Peter H. Krauss, 'A constructive interpretation of classical mathematics', 1992)

Accidental intuitionists

> *[...]use the idea of Hilbert and Poincaré that axioms and deduction rules define the meaning of the symbols of the language and it is then possible to explain that some judge the proposition* $(P \vee \neg P)$ *true and others do not because they do not assign the same meaning to the symbols* $\vee, \neg$*, etc. The need to distinguish several meanings of a common word is usual in mathematics. The proposition "there exists a number* x *such that* $2x = 1$*"is true or false depending on whether the word 'number' means 'natural number' or 'real number'.*

(Dowek, 'On the definition of the classical connectives and quantifiers', 2015)

Accidental intuitionists

Taking this idea seriously, we should not say that the proposition $(P \vee \neg P)$ has a classical proof but no constructive proof, but we should say that the proposition $(P \vee_c \neg_c P)$ has a proof and the proposition $(P \vee \neg P)$ does not, that is we should introduce two symbols for each connective and quantifier, for instance a symbol $\vee$ for the constructive disjunction and a symbol $\vee_c$ for the classical one, instead of introducing two judgments: "has a classical proof" and "has a constructive proof". (Dowek [2015])

Accidental intuitionists

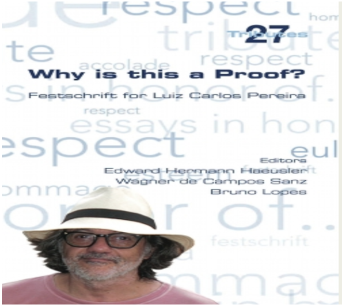
Proof checkers and proof assistants are used to formalize mathematical theorems and verify software, notably critical systems such as medical, industrial and transport systems. The development of their proofs both in the case of safety and security and in pure mathematics, has lead to the construction of large libraries. However, these libraries need to be made more usable and more sustainable. Today each library is specific to one proof system. The diversity of proof systems raises the question of their interoperability: how can proofs be rechecked and reused across systems?

(Adapted from Emilie Grienenberger's PhD thesis, 2025)

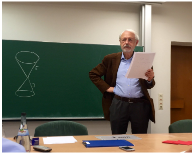
Intuitionists

Several possible ways of combining classical and constructive logics.

We concentrate on ‘Classical versus Intuitionistic Logic’, Dag Prawitz 2015



Prawitz



> *My interest in logic is first of all an interest in deductive reasoning. I see so-called classical first order predicate logic as an attempted* **codification** *of inferences occurring in actual deductive practice.[...]*
>
> *[Dummett] dismisses an* **ecletic attitute** *that finds an interest in both the classical and the intuitionistic codification.*

Prawitz, 'Classical versus Intuitionistic Logic' 2015

Prawitz on classical constants

> *Gentzen's introduction rules are of course accepted also in classical reasoning, but some of them can clearly not serve as explanations of meaning.*
>
> *[...]an existential sentence $\exists x A(x)$ may be rightly asserted classically without knowing how to find a proof of some instance $A(t)$. Hence, Gentzen's introduction rule for the existential quantifier, which allows one to infer $\exists x A(x)$ from $A(t)$, does not determine what is to count classically as a canonical proof of $\exists x A(x)$ and therefore does not either determine the classical meaning of the existential quantifier.*

Prawitz, 'Classical versus Intuitionistic Logic' 2015

Prawitz conclusion

Comparing the two codifications, it is clearly wrong to argue that classical logic is stronger than intuitionistic. What can be said is instead that the intuitionistic language is more expressive than the classical one, having access to stronger existence statements that cannot be expressed in the classical language. However, there is **no need** *to choose between the two codifications because we can have a more comprehensive one that codifies both classical and intuitionistic reasoning based on a uniform pattern of meaning explanations.*

Ecumenical system

Maybe a bad name...
Dictionary: **ecumenical (adjective)** promoting or relating to unity among the world's **Christian** Churches.

Here: a codification where two or more logics can coexist in peace

> *If they are sufficiently* **ecumenical** *and can use the other's vocabulary in their own speech, a classical logician and an intuitionist can both adopt the present mixed system, and the intuitionist must then agree that $A \vee_c \neg A$ is trivially provable for any sentence A, even when it contains intuitionistic constants, and the classical logician must admit that he has no ground for universally asserting $A \vee_i \neg A$, even when A contains only classical constants. (Prawitz 2015)*

Deviant logics

An easy argument (Quine, 1970):

1. If the deviant/revisionist logician does not accept the general validity of a classical principle of reasoning, then he gives new meanings to the concepts used in the formulation of the principle.
2. If the deviant logician gives new meanings to the concepts used in the formulation of the principle, then the deviant logician and the classical logician are not talking about the same thing (principle).
3. If they are talking about different things, they cannot disagree!!!
4. The deviant logician does not accept the general validity of the principle.

Ecumenical System

Prawitz seems to agree with Quine when he says:

> *When the classical and intuitionistic codifications attach different meanings to a constant, we need to use different symbols, and I shall use a subscript c for the classical meaning and i for the intuitionistic. The classical and intuitionistic constants can then have a peaceful coexistence in a language that contains both.*

Prawitz System

The ecumenical system defined by Prawitz has:

- two disjunctions ($\vee_c, \vee_i$),
- two implications ($\rightarrow_c, \rightarrow_i$),
- two existential quantifiers ($\exists_c, \exists_i$),

but only one conjunction, one negation, one constant for the absurd and one universal quantifier.

Why? Is this optimal? Which criteria can we use? How it compares to other ecumenical systems? Which proof-theoretic properties does it have?

Prawitz Ecumenical Natural Deduction

1. Gentzen's introduction and elimination rules for $\bot$, $\wedge$, $\neg$, and $\forall$: intro rule for $\bot$ is vacant, elim rule allows arbitrary sentence from $\bot$;
2. Gentzen's introduction and elimination rules for $\vee$, $\rightarrow$, and $\exists$, where now i is attached as a subscript to the logical constant;
3. New classical rules plus
4. Predicates

$$\frac{P_c(t) \qquad \neg P_i(t)}{\bot}$$

Ecumenical ND: classical implication

$$\dfrac{\begin{matrix}[A] & [\neg B] \\ & \Pi_1 \\ & \bot\end{matrix}}{A \to_c B} \to_c\text{-Int}$$

$$\dfrac{A \to_c B \qquad A \qquad \neg B}{\bot} \to_c\text{-Elim}$$

intuition: $(A \wedge \neg B) \to \bot \cong \neg(A \wedge \neg B) \cong \neg A \vee \neg\neg B \cong A \to_c B$

Ecumenical ND: classical disjunction

$$\dfrac{\begin{array}{cc}[\neg A] & [\neg B]\\ \multicolumn{2}{c}{\Pi_1}\\ \multicolumn{2}{c}{\bot}\end{array}}{A \vee_c B}\ \vee_c\text{-Int}$$

$$\dfrac{A \vee_c B \qquad \neg A \qquad \neg B}{\bot}\ \vee_c\text{-Elim}$$

intuition: want $A \vee B$ to be equiprovable with $\neg(\neg A \wedge \neg B)$ – De Morgan law intuitionitically not true – so define it such that $(\neg A \wedge \neg B) \to \bot$ is $A \vee_c B$

Ecumenical ND: existential

$$\dfrac{\begin{array}{c}[\forall x \neg A(x)] \\ \Pi \\ \bot\end{array}}{\exists_c x A(x)}$$

intuition: similar to disjunction

Ecumenical ND: full system

$$\dfrac{\begin{array}{c}[A, \neg B]\\ \Pi \\ \bot\end{array}}{A \to_c B}\ \to_c\text{-int}$$

$$\dfrac{\begin{array}{c}[\neg A, \neg B]\\ \Pi \\ \bot\end{array}}{A \vee_c B}\ \vee_c\text{-int}$$

$$\dfrac{\begin{array}{c}[\forall x.\neg A]\\ \Pi \\ \bot\end{array}}{\exists_c x.A}\ \exists_c\text{-int}$$

Classical

$$\dfrac{\begin{array}{c}[A]\\ \Pi \\ \bot\end{array}}{\neg A}\ \neg\text{-int}$$

$$\dfrac{A \quad B}{A \wedge B}\ \wedge\text{-int}$$

$$\dfrac{A(a/x)}{\forall x.A}\ \forall\text{-int}$$

Shared

$$\dfrac{\begin{array}{c}[A]\\ \Pi \\ B\end{array}}{A \to_i B}\ \to_i\text{-int}$$

$$\dfrac{A_j}{A_1 \vee_i A_2}\ \vee_i^j\text{-int}$$

$$\dfrac{A(a/x)}{\exists_i x.A}\ \exists_i\text{-int}$$

Intuitionistic

(Prawitz 2015)

28 / 45

Ecumenical system: simple theorems

The following are provable in the ecumenical system:

1. $\vdash (A \rightarrow_i B) \rightarrow_i (A \rightarrow_c B)$;
2. $\vdash (A \rightarrow_c \bot) \leftrightarrow_i (A \rightarrow_i \bot) \leftrightarrow_i (\neg A)$;
3. $\vdash (A \vee_c B) \leftrightarrow_i \neg(\neg A \wedge \neg B)$;
4. $\vdash (A \rightarrow_c B) \leftrightarrow_i \neg(A \wedge \neg B)$;
5. $\vdash (\exists_c x.A) \leftrightarrow_i \neg(\forall x.\neg A)$.

Ecumenical system: simple non-theorems

6. $\nvdash (A \rightarrow_c B) \rightarrow_i (A \rightarrow_i B)$ in general;
7. $\vdash A \vee_c \neg A$ but $\nvdash A \vee_i \neg A$ in general;
8. $\vdash (\neg\neg A) \rightarrow_c A$ but $\nvdash (\neg\neg A) \rightarrow_i A$ in general;
9. $\vdash (A \wedge (A \rightarrow_i B)) \rightarrow_i B$ but $\nvdash (A \wedge (A \rightarrow_c B)) \rightarrow_i B$ in general;
10. $\vdash \forall x.A \rightarrow \neg\exists_c x.\neg A$ but $\nvdash \neg\exists_c x.\neg A \rightarrow_i \forall x.A$ in general.

Entailment is intuitionistic, yay!
But note that lack of *Modus Ponens* for classical implication is BAD!

Which properties?

The Prawitz Ecumenical ND was codified by Pereira and Rodriguez (call it system E) who proved:

1. Normalization for the propositional fragment of the system
2. Soundness and completeness with respect to a Kripke-style semantics they defined

Pereira, Pimentel and de Paiva then

1. showed how to define a sequent-style calculus S corresponding to the natural deduction system E,
2. prove cut elimination for this calculus S.

Which properties...?

Which properties do we want of any *decent* logical system?

- I'd like Hilbert, sequent and ND presentations shown equivalent
- ND normalization (preferably strong)
- Cut elimination (preferably with subformula property)
- Curry-Howard terms (with subject reduction, confluence, etc)
- Related algebraic, relational and categorical semantics
- Interpolation?
- Good properties of the rules?

What about Curry-Howard?

- Grienenberger's thesis 'Combining Computational Theories'
- a different ecumenical system NE: duplicate all connectives and mark them with subscripts $(-)_i$ and $(-)_c$
- Also a third connective $\circ$ that embeds formulas into statements
- Like Prawitz a single negation, single true and single false $\bot$
- Ecumenical type theory developed
- to connect to type theories cube (for Dedukti and Rocq)
- how good is the proof of normalization?

While compatible with a double negation transformation, the intuitionistic and classical symbols in NE are primitive.

Which properties...?

Collaborators Pereira and Pimentel (with Marin and Sales) have more proof theoretical properties in mind. They want:

1. purity and separability of rules (using stoup or not)
2. harmony between rules
3. (Simpson-style) modal connectives as well
4. using relational semantics and polarities

A Problem

Given that

- we have two implications one classical and one intuitionistic,
- the negation of a proposition A can be understood as A implies $\bot$

why do we have only one negation? and why do we have a single constant for absurd?

Why don't we have

- a classical negation $\neg_c A$, understood as $(A \to_c \bot)$
- an intuitionistic negation $\neg_i A$, understood as $(A \to_i \bot)$?

Partial answer: Interderivability

$$\vdash (A \rightarrow_i \bot) \leftrightarrow_i (A \rightarrow_c \bot)$$

$$\dfrac{\dfrac{[A]^1 \qquad (A \rightarrow_i \bot)}{\bot}}{(A \rightarrow_c \bot)} \rightarrow_c\text{-Int, } 1$$

$$\dfrac{\dfrac{(A \rightarrow_c \bot) \qquad [A]^1 \qquad \widetilde{\neg\bot}}{\bot}}{(A \rightarrow_i \bot)} \rightarrow_i\text{-Int, } 1$$

tilde means $\neg\bot$ is a theorem, so it does not need to be discarded as premise.

A different answer

There is just one way to assert the negation of a proposition A, to wit, to produce a derivation of a contradiction from the assumption A.

Support from Proof Theory? Seldin's result:

Theorem [Seldin 1989]: Given a normal proof Π of $\neg A$ in classical first order or in intuitionistic first-order logic, then the last rule applied in Π is $\neg$-introduction, i.e. $\neg$-introduction is the only rule that allows us to prove the negation of a proposition

A different answer

It is true, we may have different ways to produce a contradiction from a given set of assumptions, a classical and an intuitionistic, but in both the same assertability condition holds: in order to assert $\neg A$, we should deduce a contradiction from A!

A different question

Question: Can we find a derivation of $\bot$ from A such that it is ‘essentially classic’, in the sense that it ‘essentially’ uses classical reasoning in the derivation of $\bot$ from A?

Glivenko Theorems

In the case of propositional logic, the answer is a strong no!
Given any classical derivation of $\bot$ from an assumption A, there is also an intuitionistic derivation of $\bot$ from the assumption A.
This is a consequence of Glivenko's theorems and the normalization theorem.

Glivenko's theorems

When we answer "No", we are thinking of Glivenko's second theorem: when we have a classical proof of $\neg A$, then, by the normalization theorem for classical propositional logic (Seldin's strategy), this proof can be transformed into a proof that assumes one of the following forms, where Π is an intuitionistic derivation:

Case 1

$$\dfrac{\begin{matrix}[\neg\neg A] \\ \Pi \\ \bot\end{matrix}}{\neg A}\ \bot_C$$

Case 2

$$\dfrac{\begin{matrix}[A] \\ \Pi \\ \bot\end{matrix}}{\neg A}\ \neg_{Int}$$

Two problems:

1. Is there a categorification of Seldin's normalization strategy?
2. What do we do in first-order?

Summing up

Some good reasons for just one negation:

- Interderivability
- Assertability conditions
- it works!

But plenty of interesting problems...

Also:
Many different ways of putting together classical and intuitionistic reasoning. Which one to choose? Why?

Related Work

- Krauss 1992
- Dowek 2015
- Liang and Miller 2014
- Girard LC and LU
- Caleiro&Ramos 2007
- Grienenberger 2025
- many others

Only Kripke semantics for most

I don't know of any system where the syntax and the (categorical) semantics work as well as for IPC. Is there one already in existence? Can we find one? What's the best we can do?

Thanks!

A collapse

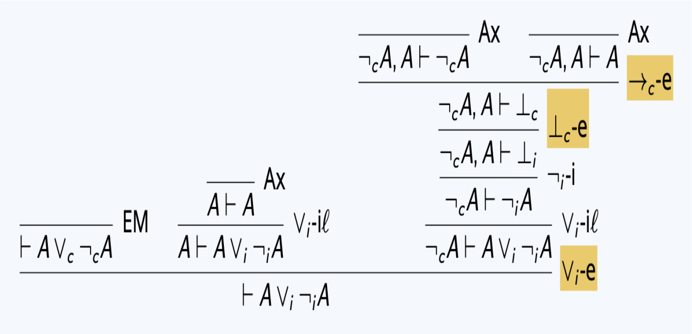


Figure 3.2: A problematic proof

From Grienenberger's thesis

Ecumenical connectives and rules: sequent version

Classical	Shared	Intuitionistic
$\dfrac{\Gamma, A, \neg B \vdash \bot}{\Gamma \vdash A \to_c B} \to_c R$	$\dfrac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A} \neg R$	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to R$
$\dfrac{\Gamma, \neg A, \neg B \vdash \bot}{\Gamma \vdash A \vee_c B} \vee_c R$	$\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge R$	$\dfrac{\Gamma \vdash A_j}{\Gamma \vdash A_1 \vee_i A_2} \vee_i R_j$
$\dfrac{\Gamma, \forall x.\neg A \vdash \bot}{\Gamma \vdash \exists_c x.A} \exists_c R$	$\dfrac{\Gamma \vdash A(y/x)}{\Gamma \vdash \forall x.A} \forall R$	$\dfrac{\Gamma \vdash A(a/x)}{\Gamma \vdash \exists_i x.A} \exists_i R$

(Pimentel, Pereira, de Paiva 2021)

Ecumenical connectives and rules: sequent version

$$\frac{\Gamma, A, \neg B \vdash \bot}{\Gamma \vdash A \to_c B} \to_c R \qquad \frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A} \neg R \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to R$$

$$\frac{\Gamma, \neg A, \neg B \vdash \bot}{\Gamma \vdash A \vee_c B} \vee_c R \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge R \qquad \frac{\Gamma \vdash A_j}{\Gamma \vdash A_1 \vee_i A_2} \vee_i R_j$$

$$\frac{\Gamma, \forall x. \neg A \vdash \bot}{\Gamma \vdash \exists_c x.A} \exists_c R \qquad \frac{\Gamma \vdash A(y/x)}{\Gamma \vdash \forall x.A} \forall R \qquad \frac{\Gamma \vdash A(a/x)}{\Gamma \vdash \exists_i x.A} \exists_i R$$

Classical **Shared** **Intuitionistic**

(Pimentel, Pereira, de Paiva 2021)

Ecumenical connectives and rules: sequent version

$$\frac{\Gamma, A, \neg B \vdash \bot}{\Gamma \vdash A \to_c B} \to_c R \qquad \frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A} \neg R \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to R$$

$$\frac{\Gamma, \neg A, \neg B \vdash \bot}{\Gamma \vdash A \vee_c B} \vee_c R \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge R \qquad \frac{\Gamma \vdash A_j}{\Gamma \vdash A_1 \vee_i A_2} \vee_i R_j$$

$$\frac{\Gamma, \forall x.\neg A \vdash \bot}{\Gamma \vdash \exists_c x.A} \exists_c R \qquad \frac{\Gamma \vdash A(y/x)}{\Gamma \vdash \forall x.A} \forall R \qquad \frac{\Gamma \vdash A(a/x)}{\Gamma \vdash \exists_i x.A} \exists_i R$$

Classical **Shared** **Intuitionistic**

(Pimentel, Pereira, de Paiva 2021)