

A loose Yoneda theory for equipments

Michael Lambert¹ **Nelson Niu**² Evan Patterson²

Topos Berkeley Seminar · October 13, 2026

¹Norwich University

²Topos Institute

1-Categorical Yoneda

[Yoneda '54] For a 1-category $\mathcal{C}$, object $a \in \mathcal{C}$, and functor $F: \mathcal{C} \rightarrow \mathbf{Set}$,

$$[\mathcal{C}, \mathbf{Set}](\mathcal{C}(a, -), F) \cong Fa$$

$$\theta \mapsto \theta_a(\mathrm{id}_a)$$

Applications:

- *forward*: tame large amount of data, *e.g. morphisms of polynomials*
- *backward*: express properties natively, *e.g. simplices of simplicial sets*

1-Categorical Yoneda, implicitly

[Yoneda '54] For a 1-category $\mathcal{C}$, object $a \in \mathcal{C}$, and functor $F: \mathcal{C} \rightarrow \mathbf{Set}$,

$$[\mathcal{C}, \mathbf{Set}](\mathcal{C}(a, -), F) \cong Fa$$

$$\theta \mapsto \theta_a(\mathrm{id}_a)$$

- $\mathcal{C}(a, -): \mathcal{C} \rightarrow \mathbf{Set}$ is a functor: a *representable functor*
- $a \mapsto \mathcal{C}(a, -)$ is a functor $\mathcal{C}^{\mathrm{op}} \rightarrow [\mathcal{C}, \mathbf{Set}]$: the *Yoneda embedding*
- naturality w.r.t. $f: a \rightarrow b$ in $\mathcal{C}$ and $\tau: F \Rightarrow G$ in $[\mathcal{C}, \mathbf{Set}]$:

$$\begin{array}{ccc} [\mathcal{C}, \mathbf{Set}](\mathcal{C}(a, -), F) & \xrightarrow{\sim} & Fa \\ \downarrow - \circ \mathcal{C}(f, -) & & \downarrow Ff \\ [\mathcal{C}, \mathbf{Set}](\mathcal{C}(b, -), F) & \xrightarrow{\sim} & Fb \end{array}$$

$$\begin{array}{ccc} [\mathcal{C}, \mathbf{Set}](\mathcal{C}(a, -), F) & \xrightarrow{\sim} & Fa \\ \downarrow \tau \circ - & & \downarrow \tau_a \\ [\mathcal{C}, \mathbf{Set}](\mathcal{C}(a, -), G) & \xrightarrow{\sim} & Ga \end{array}$$

1-Categorical Yoneda embedding

For a 1-category $\mathcal{C}$, object $a \in \mathcal{C}$, and object $b \in \mathcal{C}$,

$$[\mathcal{C}, \mathbf{Set}](\mathcal{C}(a, -), \mathcal{C}(b, -)) \cong \mathcal{C}(b, a)$$

$$\theta \mapsto \theta_a(\mathrm{id}_a)$$

$$\mathcal{C}(g, -) \hookleftarrow g$$

making the Yoneda embedding $\mathcal{C}^{\mathrm{op}} \rightarrow [\mathcal{C}, \mathbf{Set}]$ fully faithful.

Applications:

- identification/uniqueness via universal property, *e.g. initial object*
- colimit completion (once **density theorem** is established)

Bicategorical Yoneda

Idea: Replace hom-sets with hom-categories, but with unitality & associativity of composition up to isomorphism & functoriality up to isomorphism

[Street '80] For a bicategory $\underline{\mathbf{B}}$, object $a \in \underline{\mathbf{B}}$, and pseudofunctor $F: \underline{\mathbf{B}} \rightarrow \underline{\mathbf{Cat}}$,

$$\begin{aligned} [\underline{\mathbf{B}}, \underline{\mathbf{Cat}}](\underline{\mathbf{B}}(a, -), F) &\simeq Fa \\ \theta &\mapsto \theta_a(\text{id}_a) \end{aligned}$$

- $\underline{\mathbf{B}}(a, -): \underline{\mathbf{B}} \rightarrow \underline{\mathbf{Cat}}$ is a pseudofunctor: a *representable pseudofunctor*
- $a \mapsto \underline{\mathbf{B}}(a, -)$ is a pseudofunctor $\underline{\mathbf{B}}^{\text{op}} \rightarrow [\underline{\mathbf{B}}, \underline{\mathbf{Cat}}]$: the *Yoneda embedding*
- pseudonaturality in $a \in \underline{\mathbf{B}}$ & $F \in [\underline{\mathbf{B}}, \underline{\mathbf{Cat}}]$

Bicategorical Yoneda embedding

For a bicategory $\underline{\mathbf{B}}$, object $a \in \underline{\mathbf{B}}$, and object $b \in \underline{\mathbf{B}}$,

$$[\underline{\mathbf{B}}, \underline{\mathbf{Cat}}](\underline{\mathbf{B}}(a, -), \underline{\mathbf{B}}(b, -)) \simeq \underline{\mathbf{B}}(b, a)$$

$$\theta \mapsto \theta_a(\text{id}_a)$$

$$\underline{\mathbf{B}}(g, -) \hookleftarrow g$$

making the Yoneda embedding $\underline{\mathbf{B}}^{\text{op}} \rightarrow [\underline{\mathbf{B}}, \underline{\mathbf{Cat}}]$:

- essentially full on 1-cells
- fully faithful on 2-cells

Application: strictification of bicategories into 2-categories, because $\underline{\mathbf{Cat}}$ is strict

Double categories: tight & loose arrows

In a *double category*:

- **Tight** arrow composition satisfies unitality & associativity *strictly*
- **Loose** arrow composition satisfies unitality & associativity *up to isomorphism*

$$\begin{array}{ccc} a & \xrightarrow{m} & b \\ f \downarrow & \alpha & \downarrow g \\ c & \xrightarrow{n} & d \end{array}$$

Double categories: tight & loose arrows

In a *double category*:

- **Tight** arrow composition satisfies unitality & associativity *strictly*
- **Loose** arrow composition satisfies unitality & associativity *up to isomorphism*

Examples:

- $\text{Span}(\mathcal{C})$, when $\mathcal{C}$ has pullbacks (resp. pushouts): **morphisms & spans**
- $\text{Csp}(\mathcal{C})$, when $\mathcal{C}$ has pushouts: **morphisms & cospans**
- Rel : **functions & relations**
- Cat : **functors & profunctors**

Tight Yoneda theory for double categories

Let $\mathbb{D}$ be a double category & $a \in \mathbb{D}$ an object. We write $\text{Set} := \text{Span}(\text{Set})$.

[Paré '11] Yoneda theory for **tight** representable *lax* functors:

$$\mathbb{D}_0(a, -): \mathbb{D} \rightarrow \text{Set}$$

$$(x \in \mathbb{D}) \mapsto \{\text{tight arrows } a \rightarrow x \text{ in } \mathbb{D}\}$$

$$(x \xrightarrow{f} y) \mapsto (\mathbb{D}_0(a, x) \xrightarrow{f \circ -} \mathbb{D}_0(a, y))$$

$$(x \xrightarrow{m} y) \mapsto (\mathbb{D}_0(a, x) \xleftarrow{\text{dom}} \{\text{cells } \text{id}_a \Rightarrow m \text{ in } \mathbb{D}\} \xrightarrow{\text{cod}} \mathbb{D}_0(a, y))$$

Yoneda embedding full & faithful on tight arrows, *not* full on loose arrows

Loose Yoneda theory for double categories?

Let $\mathbb{D}$ be a double category & $a \in \mathbb{D}$ an object.

Loose representables?

- $(x \in \mathbb{D}) \mapsto \mathbb{D}(a, x)$, a *category* with:
 - *objects*: **loose** arrows $a \rightrightarrows x$ in $\mathbb{D}$
 - *morphisms*: **globular** cells bounded by tight identities $a = a$ & $x = x$
- $(x \xrightarrow{m} y) \mapsto (\mathbb{D}(a, x) \xrightarrow{-\odot m} \mathbb{D}(a, y))$, a *functor* (tight in $\mathbb{C}at$)
- $(x \xrightarrow{f} y) \mapsto (\mathbb{D}(a, f): \mathbb{D}(a, x) \rightrightarrows \mathbb{D}(a, y))$, a *profunctor* (loose in $\mathbb{C}at$)

$$(a \xrightarrow{p} x, a \xrightarrow{q} y) \mapsto \left\{ \text{cells } \begin{array}{ccc} a & \xrightarrow{p} & x \\ \parallel & & \downarrow f \\ a & \xrightarrow{q} & y \end{array} \text{ in } \mathbb{D} \right\}$$

Twisted functors

[LMP '26] For double categories $\mathbb{D}$ and $\mathbb{E}$, a *twisted functor* $F: \mathbb{D} \rightleftarrows \mathbb{E}$ sends:

- objects $x \in \mathbb{D}$ to objects $Fx \in \mathbb{E}$
- loose arrows $x \xrightarrow{m} y$ in $\mathbb{D}$ to tight arrows $Fx \xrightarrow{Fm} Fy$ in $\mathbb{E}$
- tight arrows $x \xrightarrow{f} y$ in $\mathbb{D}$ to loose arrows $Fx \xrightarrow{Ff} Fy$ in $\mathbb{E}$
- ...

Focus on case where:

- identities are preserved up to isocells
- loose-to-tight composition is preserved up to isocell: $F(m \odot m') \cong Fm' \circ Fm$
- F may be lax on tight-to-loose composition: **comparison cell**

$$F_{f,g}: Ff \odot Fg \Rightarrow F(g \circ f)$$

Twisted functors

[LMP '26] For double categories $\mathbb{D}$ and $\mathbb{E}$, a *twisted functor* $F: \mathbb{D} \looparrowright \mathbb{E}$ sends:

- objects $x \in \mathbb{D}$ to objects $Fx \in \mathbb{E}$
- loose arrows $x \xrightarrow{m} y$ in $\mathbb{D}$ to tight arrows $Fx \xrightarrow{Fm} Fy$ in $\mathbb{E}$
- tight arrows $x \xrightarrow{f} y$ in $\mathbb{D}$ to loose arrows $Fx \xrightarrow{Ff} Fy$ in $\mathbb{E}$
- ...

[LMP '26] There is a 2-category $\mathbf{Tw}(\mathbb{D}, \mathbb{E})$ with:

- twisted functors $\mathbb{D} \looparrowright \mathbb{E}$ as objects
- tight pseudonatural transformations as 1-cells
- modifications as 2-cells

Twisted functors

[LMP '26] For double categories $\mathbb{D}$ and $\mathbb{E}$, a *twisted functor* $F: \mathbb{D} \looparrowright \mathbb{E}$ sends:

- objects $x \in \mathbb{D}$ to objects $Fx \in \mathbb{E}$
- loose arrows $x \xrightarrow{m} y$ in $\mathbb{D}$ to tight arrows $Fx \xrightarrow{Fm} Fy$ in $\mathbb{E}$
- tight arrows $x \xrightarrow{f} y$ in $\mathbb{D}$ to loose arrows $Fx \xrightarrow{Ff} Fy$ in $\mathbb{E}$
- ...

[LNP] There is a double category $\mathsf{Tw}(\mathbb{D}, \mathbb{E})$ with:

- twisted functors $\mathbb{D} \looparrowright \mathbb{E}$ as objects
- tight (resp. loose) pseudonatural transformations as tight (resp. loose) arrows
- modifications as cells

Equipments: companions & conjoints

An **equipment** is a **double category** in which every *tight arrow* $f: x \rightarrow y$ has:

a universal **companion** loose arrow $f_! : x \rightrightarrows y$ with cell

$$\begin{array}{ccc} x & \xrightarrow{f_!} & y \\ f \downarrow & & \parallel \\ y & \rightrightarrows & y \end{array}$$

a universal **conjoint** loose arrow $f^* : y \rightrightarrows x$ with cell

$$\begin{array}{ccc} y & \xrightarrow{f^*} & x \\ \parallel & & \downarrow f \\ y & \rightrightarrows & y \end{array}$$

Equipments: examples

- In $\text{Span}(\mathcal{C})$, the arrow $f: x \rightarrow y$ has:
 - companion $x \xleftarrow{=} x \xrightarrow{f} y$ & conjoint $y \xleftarrow{f} x \xrightarrow{=} x$
- In $\text{Csp}(\mathcal{C})$, the arrow $f: x \rightarrow y$ has:
 - companion $x \xrightarrow{f} y \xleftarrow{=} y$ & conjoint $y \xrightarrow{=} y \xleftarrow{f} x$
- In $\mathbb{R}\text{el}$, the function $f: x \rightarrow y$ has:
 - companion & conjoint given by viewing f as a relation (i.e. its graph)
- In Cat , the functor $F: \mathcal{C} \rightarrow \mathcal{D}$ has:
 - companion $\mathcal{D}(F(-), =): \mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \mathbf{Set}$
 - conjoint $\mathcal{D}(-, F(=)): \mathcal{D}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$

Twisted functors between equipments

[LNP] For equipments $\mathbb{D}, \mathbb{E}$, there is an equivalence of 2-categories

$$\underline{\mathbf{Tw}}(\mathbb{D}, \mathbb{E}) \simeq [\underline{\mathbf{Loose}}(\mathbb{D}), \underline{\mathbf{Tight}}(\mathbb{E})]$$

$\underline{\mathbf{Tw}}(\mathbb{D}, \mathbb{E})$, 2-category with:

- twisted functors $\mathbb{D} \multimap \mathbb{E}$ as objects
- tight ps.nat. trans. as 1-cells
- modifications as 2-cells

$[\underline{\mathbf{Loose}}(\mathbb{D}), \underline{\mathbf{Tight}}(\mathbb{E})]$, 2-category with:

- pseudofunctors $\underline{\mathbf{Loose}}(\mathbb{D}) \rightarrow \underline{\mathbf{Tight}}(\mathbb{E})$ as objects
- ps.nat. trans. as 1-cells
- modifications as 2-cells

Twisted copresheaves (i.e. functors to $\mathbb{C}at$) on equipments

[LNP] For an equipment $\mathbb{D}$, there is an equivalence of 2-categories

$$\underline{\mathbf{Tw}}(\mathbb{D}, \mathbb{C}at) \simeq [\underline{\mathbf{Loose}}(\mathbb{D}), \underline{\mathbf{Cat}}]$$

$\underline{\mathbf{Tw}}(\mathbb{D}, \mathbb{C}at)$, 2-category with:

- twisted functors $\mathbb{D} \multimap \mathbb{C}at$ as objects
- tight ps.nat. trans. as 1-cells
- modifications as 2-cells

$[\underline{\mathbf{Loose}}(\mathbb{D}), \underline{\mathbf{Cat}}]$, 2-category with:

- pseudofunctors $\underline{\mathbf{Loose}}(\mathbb{D}) \rightarrow \underline{\mathbf{Cat}}$ as objects
- ps.nat. trans. as 1-cells
- modifications as 2-cells

Loose Yoneda for double categories from Yoneda for bicategories

[LNP] For an equipment $\mathbb{D}$, object $a \in \mathbb{D}$, and twisted functor $F: \mathbb{D} \rightarrow \mathbb{Cat}$,

$$\begin{aligned} \underline{\mathbf{Tw}}(\mathbb{D}, \mathbb{Cat})(\mathbb{D}(a, -), F) &\simeq Fa \\ \theta &\mapsto \theta_a(\mathrm{id}_a) \end{aligned}$$

- $\mathbb{D}(a, -): \mathbb{D} \rightarrow \mathbb{Cat}$ is a twisted functor: a ***twisted representable*** [LMP '26]
- $a \mapsto \mathbb{D}(a, -)$ is a twisted functor $\mathbb{D}^{\mathrm{co}} \rightarrow \mathbf{Tw}(\mathbb{D}, \mathbb{Cat})$: the ***twisted Yoneda embedding***, fully faithful in loose-to-tight direction
 - $\mathbb{D}^{\mathrm{co}}$ is $\mathbb{D}$ with all *loose* arrows flipped
- pseudonaturality in $a \in \mathbb{D}$ w.r.t. loose arrows & in $F \in \mathbf{Tw}(\mathbb{D}, \mathbb{Cat})$ w.r.t. tight natural transformations

Loose Yoneda for double categories

[LNP] For an equipment $\mathbb{D}$, object $a \in \mathbb{D}$, and twisted functor $F: \mathbb{D} \multimap \mathbb{C}at$,

$$\begin{aligned} \underline{\mathbf{Tw}}(\mathbb{D}, \mathbb{C}at)(\mathbb{D}(a, -), F) &\simeq Fa \\ \theta &\mapsto \theta_a(\mathrm{id}_a) \end{aligned}$$

- $\mathbb{D}(a, -): \mathbb{D} \multimap \mathbb{C}at$ is a twisted functor: a ***twisted representable*** [LMP '26]
- $a \mapsto \mathbb{D}(a, -)$ is a twisted functor $\mathbb{D}^{\mathrm{co}} \multimap \mathbf{Tw}(\mathbb{D}, \mathbb{C}at)$: the ***twisted Yoneda embedding***, fully faithful in loose-to-tight direction
 - $\mathbb{D}^{\mathrm{co}}$ is $\mathbb{D}$ with all *loose* arrows flipped
- pseudonaturality in $a \in \mathbb{D}$ w.r.t. loose & ***tight*** arrows & in $F \in \mathbf{Tw}(\mathbb{D}, \mathbb{C}at)$ w.r.t. tight & ***loose*** natural transformations

- Understanding universal properties w.r.t. loose arrows in equipments via loose Yoneda
 - Density theorem for colimits of twisted double diagrams
- Understanding twisted functors from equipments to $\mathbb{C}\text{at}$ as pseudofunctors to **Cat**, i.e. fibered/indexed categories
- Formal category theory: compact double categories, dualization