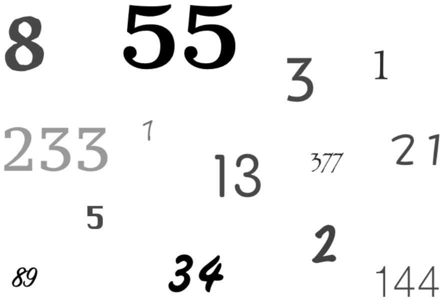
Where is the middle of a Fibonacci sequence?

Topos Institute • Berkeley, CA



Sam Vandervelde • Proof School • Aug 13, 2024

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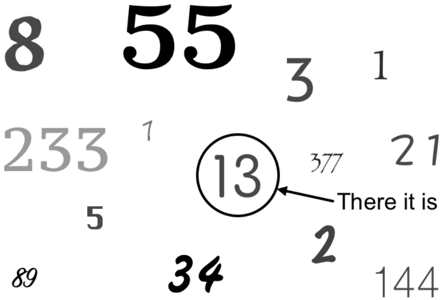

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Let's Review

How many values does it take to get a Fibonacci sequence going?

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How many values does it take to get a Fibonacci sequence going? **two**

2, 7,

Let's Review

How many values does it take to get a Fibonacci sequence going? **two**

2, 7, 9,

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How many values does it take to get a Fibonacci sequence going? **two**

2, 7, 9, 16,

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How many values does it take to get a Fibonacci sequence going? **two**

2, 7, 9, 16, 25,

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How many values does it take to get a Fibonacci sequence going? **two**

$$2, \ 7, \ 9, \ 16, \ 25, \ 41, \ \ldots$$

Let's Review

How many values does it take to get a Fibonacci sequence going? **two**

$$5,\ 2,\ 7,\ 9,\ 16,\ 25,\ 41,\ \ldots$$

Let's Review

How many values does it take to get a Fibonacci sequence going? **two**

-3, 5, 2, 7, 9, 16, 25, 41, . . .

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How many values does it take to get a Fibonacci sequence going? **two**

. . . , 8, -3, 5, 2, 7, 9, 16, 25, 41, . . .

Wanna guess which number is in the middle?

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. . . , 8, -3, 5, 2, 7, 9, 16, 25, 41, . . .

Wanna guess which number is in the middle?

Good guesses, but actually it's the 9.

You Gotta Be Kidding Me

Mad Libonacci

Time to practice. Is it a Fibonacci sequence?

(A) $0,\ -1,\ -1,\ -2,\ -3,\ -5,\ \ldots$

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(A) $0, \ -1, \ -1, \ -2, \ -3, \ -5, \ \ldots$ **YES**

(B) $\frac{1}{7}, \ \frac{1}{7}, \ \frac{2}{7}, \ \frac{3}{7}, \ \frac{5}{7}, \ \frac{8}{7}, \ldots$

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(C) $-17, 16, -1, 15, -14, 1, -13, \ldots$

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(C) $-17, 16, -1, 15, -14, 1, -13, \ldots$ **NO**

(D) $0, \ 0, \ 0, \ 0, \ 0, \ldots$

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(C) $-17, 16, -1, 15, -14, 1, -13, \ldots$ **NO**

(D) $0, \ 0, \ 0, \ 0, \ 0, \ldots$ **YES** (trivial)

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You're starting to get the hang of it. Now suppose that G_1, G_2, G_3, . . . is a Fibonacci sequence. Which of the following are also?

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(C) $G_1^2,\ \ G_2^2,\ \ G_3^2, \ldots$

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(C) $G_1^2,\ G_2^2,\ G_3^2, \ldots$ **NO**

(D) $G_2 - G_1,\ G_3 - G_2,\ G_4 - G_3, \ldots$

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(A) $G_1 + 1,\ G_2 + 1,\ G_3 + 1, \ldots$ **NO**

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(D) $G_2 - G_1,\ G_3 - G_2,\ G_4 - G_3, \ldots$ **YES**

Baby Steps

Now, back to our story. Here's a sample sequence to help illustrate the key idea we'll need for finding the middle of a Fibseq.

$$\ldots, 3.625, 3.75, 4, 4.5, 5.5, 7.5, 11.5, 19.5, \ldots$$

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What is the next number?

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What is the next number? **35.5**

Where is the middle number?

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$$\ldots, 3.625, 3.75, 4, 4.5, 5.5, 7.5, 11.5, 19.5, \ldots$$

What is the next number? **35.5**

Where is the middle number? **4.5 or 5.5**

Fibonacci Facts

What do we already know about Fibonaccis?

$$\ldots, -3,\ 2,\ -1,\ 1,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ \ldots$$

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The ratio of successive terms approaches φ, the golden ratio, which is given by

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I wonder why that is, anyway? **Discuss**

Fibonacci Facts

The Fibonacci numbers satisfy the recursion

$$F_{n+1} = F_n + F_{n-1}.$$

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If r represents the ratio, then

$$r = 1 + \frac{1}{r} \quad \Longrightarrow \quad r = \frac{1}{2}(1+\sqrt{5}).$$

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$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}.$$

But the ratio $\frac{F_{n+1}}{F_n}$ keeps changing, so it doesn't make sense to write

$$r = 1 + \frac{1}{r}.$$

Fibonacci Facts

It is a fact that for all n we have

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In other words, the golden ratio φ satisifies $\varphi^2 = \varphi + 1$. (Remember this for later!)

Fibonacci Facts

In the same way, for any nontrivial Fibonacci sequence G_1, G_2, G_3, ... it is the case that

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$$\tfrac{25}{16} \approx 1.56, \quad \tfrac{41}{25} \approx 1.64, \quad \tfrac{66}{41} \approx 1.61.$$

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But how does this help us to find the middle of a Fibonacci sequence?

What's the BIG Idea?

Given a nontrivial Fibonacci sequence, since

$$\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \varphi,$$

let's take a look at the value of

$$\delta_n = G_{n+1} - \varphi G_n.$$

What do you suppose will happen?

Crunching BIG Data

Here are some decimal approximations for δ_n.

G_n	δ_n
-11	25.798
8	-15.944
-3	9.854
5	-6.090
2	3.764

G_n	δ_n
7	-2.326
9	1.438
16	-0.889
25	0.549
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What do you notice? **Discuss**

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What do you notice? **Discuss** It appears that $\delta_{n+1} = -\frac{1}{\varphi} \cdot \delta_n$. I wonder why?

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To prove that $\delta_{n+1} = -\frac{1}{\varphi}\delta_n$, we simplify the quantity $\varphi\delta_{n+1} + \delta_n$. (What should happen?)

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= \; & \varphi(G_{n+1} + G_n - \varphi G_{n+1}) + (G_{n+1} - \varphi G_n) \\
= \; & \varphi(G_{n+1} - \varphi G_{n+1}) + G_{n+1} \\
= \; & G_{n+1}(\varphi - \varphi^2 + 1) \\
= \; & 0.
\end{aligned}$$

And You're Done!

A Mission Critical Definition

Let's revisit our approximations for δ_n.

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How could/should we use the values of δ_n to define the middle term? **Discuss**

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To be precise, look for $\_\_\_\_ < \delta_n < \_\_\_\_$.
(Recall that $\delta_{n+1} = -\frac{1}{\varphi} \cdot \delta_n$.)

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> Choose the value of δ_n that is closest to 1, in a multiplicative sense.

To be precise, look for $\underline{\frac{1}{\varphi}} < \delta_n < \underline{\varphi}$.
So we want $0.618 < \delta_n < 1.618$, roughly.

Short Ribsonacci

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- (So that $\frac{H}{G} \approx \varphi$ not too closely, and not too poorly, but just rightly.)
- Then G is the middle of the sequence!

I Call Dibsonacci

Let's play with our new tool. First, using only your intuition (no calculators allowed!), guess which term is the middle.

$$\ldots, \ -3, \ 4, \ 1, \ 5, \ 6, \ 11, \ 17, \ 28, \ 45, \ldots$$

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$$\ldots, \; -3, \; 4, \; 1, \; 5, \; 6, \; 11, \; 17, \; 28, \; 45, \ldots$$

$$\Uparrow$$

Duh, it's the one in the middle. (But let's perform the computations to see why.)

I Call Dibsonacci

Here are the δ-values. Sure enough!

-3	8.85
4	-5.47
1	3.38
5	-2.09
6	1.29
11	-0.80
17	0.49

Reality Check

Now see if you can do a few in your head. It will help to recall that $\varphi \approx 1.618$.

(A) $2, \; -1, \; 1, \; 0, \; 1, \; 1, \; 2, \; 3, \; 5, \; 8$

Reality Check

Now see if you can do a few in your head. It will help to recall that $\varphi \approx 1.618$.

(A) $2,\ -1,\ 1,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8$ **0**

(B) $4,\ -2,\ 2,\ 0,\ 2,\ 2,\ 4,\ 6,\ 10,\ 16$

Reality Check

Now see if you can do a few in your head. It will help to recall that $\varphi \approx 1.618$.

(A) $2, -1, 1, 0, 1, 1, 2, 3, 5, 8$ **0**

(B) $4, -2, 2, 0, 2, 2, 4, 6, 10, 16$ **2**

(C) $5, -2, 3, 1, 4, 1, 5, 9, 14, 23$

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(A) $2,\ -1,\ 1,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8$ **0**

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(C) $5,\ -2,\ 3,\ 1,\ 4,\ 1,\ 5,\ 9,\ 14,\ 23$ π

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(C) $5, \ -2, \ 3, \ 1, \ 4, \ 5, \ 9, \ 14, \ 23, \ 37$

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(D) $3, -1, 2, 1, 3, 4, 7, 11, 18, 29$

Reality Check

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(A) $2,\ -1,\ 1,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8$ **0**

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(C) $5,\ -2,\ 3,\ 1,\ 4,\ 5,\ 9,\ 14,\ 23,\ 37$ **5**

(D) $3,\ -1,\ 2,\ 1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29$ **1**

Nice work!

Mr. Pibbonacci

Make a guess: is there a Fibonacci sequence whose middle term is 7?

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How could we look for one? **Discuss**

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$$0.618 < H - 7\varphi < 1.618$$

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How could we look for one? **Discuss**

Let H be the next term after 7. We need

$$0.618 < H - 7\varphi < 1.618$$
$$\implies \quad 11.94 < H < 12.94,$$

so the only possibility is to take $H = 12$.

Mr. Pibbonacci

Here is that Fibonnaci sequence:

$$\ldots, -1,\ 3,\ 2,\ 5,\ \mathbf{7},\ 12,\ 19,\ 31,\ 50, \ldots$$

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Is there a Fibonacci sequence with middle term -2024? **Yes!**

How many such sequences are there?

Mr. Pibbonacci

Here is that Fibonnaci sequence:

$$\ldots, -1,\ 3,\ 2,\ 5,\ \mathbf{7},\ 12,\ 19,\ 31,\ 50, \ldots$$

Is there a Fibonacci sequence with middle term -2024? **Yes!**

How many such sequences are there? **one**

Because the next term H must satisfy the inequality $0.618 < H + 2024\varphi < 1.618$.

Math is Nice

Theorem

Given any integer k, there is precisely one Fibonacci sequence whose middle term is k.

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Given any integer k, there is precisely one Fibonacci sequence whose middle term is k.

This means that there is a natural way to number and order all nontrivial Fibonacci sequences! And “the” Fibonacci numbers, as sequence #0, comes right in the middle.

Don't Believe It

Glad we got that completely figured out!

Math is Not Nice After All?

Which sequence has a middle term of -1?

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What shall we do? **Discuss**

Math is Not Nice After All?

Which sequence has a middle term of -1?

$$0.618 < H - (-1)\varphi < 1.618,$$

$$\implies \quad -1 < H < 0. \qquad \textbf{Hmm.}$$

What shall we do? **Discuss**

Either way, sequence #(-1) is

$$\ldots, -2, 1, -1, 0, -1, -1, -2, -3, \ldots$$

Math is Not Nice After All?

Which sequence has a middle term of -1?

$$0.618 < H - (-1)\varphi < 1.618,$$
$$\Longrightarrow \quad -1 < H < 0. \quad \textbf{Hmm.}$$

What shall we do? **Discuss**

Either way, sequence #(-1) is

$$\ldots, -2, 1, -1, 0, -1, -1, -2, -3, \ldots$$

But which -1 is the correct middle term?

Math is Deep

−2	3	1	4	**5**	9	14	23	37	60	97	157
−4	4	0	4	**4**	8	12	20	32	52	84	136
−3	3	0	3	**3**	6	9	15	24	39	63	102
−2	2	0	2	**2**	4	6	10	16	26	42	68
−4	3	−1	2	**1**	3	4	7	11	18	29	47
−3	**2**	**−1**	**1**	**0**	**1**	**1**	**2**	**3**	**5**	**8**	**13**
−5	3	−2	1	**−1**	0	−1	−1	−2	−3	−5	−8
−4	2	−2	0	**−2**	−2	−4	−6	−10	−16	−26	−42
−3	1	−2	−1	**−3**	−4	−7	−11	−18	−29	−47	−76
−5	2	−3	−1	**−4**	−5	−9	−14	−23	−37	−60	−97
−4	1	−3	−2	**−5**	−7	−12	−19	−31	−50	−81	−131

How shall we place sequence #(-1)? OR $\rightarrow$

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−2	3	1	4	**5**	9	14	23	37	60	97	157
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It's surprisingly tricky to tell, but one of these placements makes more sense than the other.

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Let's add a vertical bar, then check out what happens to the right of the bar.

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Seriously. (You have been warned.)

Math is Deep

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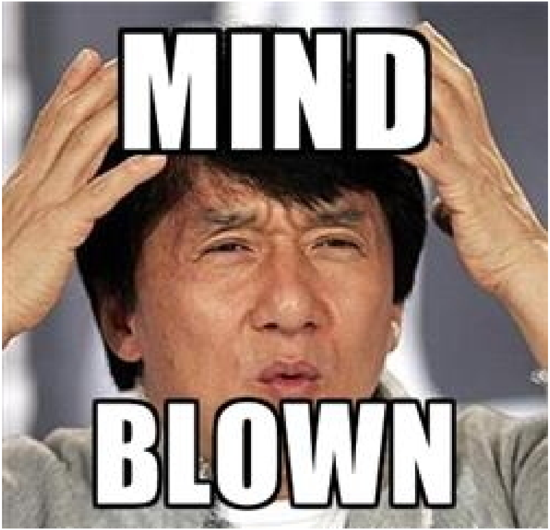
Here's the first array. What do you notice?

My Brain Hurts



An Exceptional Result

Theorem

Every integer appears exactly once to the right of the vertical bar, with the exception of 0 (not at all) and -1 (appears twice).

I wonder why that is?

An Exceptional Result

Theorem

Every integer appears exactly once to the right of the vertical bar, with the exception of 0 *(not at all) and* -1 *(appears twice).*

I wonder why that is? Perhaps I will show you some day. But for now. . .

What A Day

It's been great sharing cool math with you.

There Is Nothing To See

An Exceptional Proof

Suppose that G is an integer to the right of the vertical bar, and let H be the integer to its right. What can we say about

$$\boxed{H - \varphi G}$$

An Exceptional Proof

Suppose that G is an integer to the right of the vertical bar, and let H be the integer to its right. What can we say about

$$\boxed{H - \varphi G}$$

(Let's take a peek at that array again.)

Still Deep

−2	3	1	4	**5**	9	14	23	37	60	97	157
−4	4	0	4	**4**	8	12	20	32	52	84	136
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Identify the possible δ-values $H - \varphi G$ for a few representative integers G above.

An Exceptional Proof

To summarize, we know that if G is in the central column, then

$$\frac{1}{\varphi} < H - \varphi G \leq \varphi.$$

(By definition of the middle of a Fibonacci sequence.)

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To summarize, we know that if G is in the central column, then

$$\frac{1}{\varphi} < H - \varphi G \leq \varphi.$$

(By definition of the middle of a Fibonacci sequence.) For the next column, we have

$$-1 \leq H - \varphi G < -\frac{1}{\varphi^2}.$$

An Exceptional Proof

So we know that if G is in the column just to the left of the vertical bar, then

$$-1 \leq H - \varphi G < -\frac{1}{\varphi^2}.$$

(By the previous step.) For the next column, we have

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$$\frac{1}{\varphi^3} < H - \varphi G \leq \frac{1}{\varphi}.$$

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We know that if G is in the column just to the right of the vertical bar, then

$$\frac{1}{\varphi^3} < H - \varphi G \leq \frac{1}{\varphi}.$$

(By the previous step.) For the next column, we have

$$-\frac{1}{\varphi^2} \leq H - \varphi G < -\frac{1}{\varphi^4}.$$

An Exceptional Proof

We know that if G is situated two columns to the right of the vertical bar, then

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An Exceptional Proof

We know that if G is situated two columns to the right of the vertical bar, then

$$-\frac{1}{\varphi^2} \leq H - \varphi G < -\frac{1}{\varphi^4}.$$

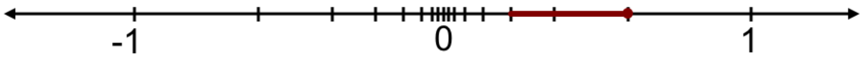
(By the previous step.) For the next column,

$$\frac{1}{\varphi^5} < H - \varphi G \leq \frac{1}{\varphi^3},$$

And so forth. . .

An Exceptional Proof

To track this argument visually, an integer G appears to the right of the vertical bar once for each integer H such that $H - \varphi G$ falls within the following ranges:



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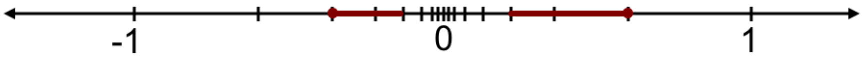

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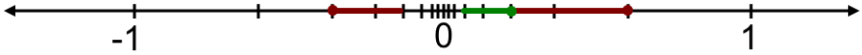

$$\frac{1}{\varphi^5} < H - \varphi G \leq \frac{1}{\varphi^3}.$$

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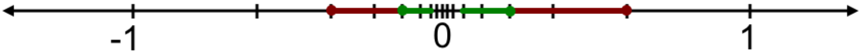

$$-\frac{1}{\varphi^4} \le H - \varphi G < -\frac{1}{\varphi^6}.$$

An Exceptional Proof

To track this argument visually, an integer G appears to the right of the vertical bar once for each integer H such that $H - \varphi G$ falls within the following ranges:

What is the set of all possible values?

An Exceptional Proof

To track this argument visually, an integer G appears to the right of the vertical bar once for each integer H such that $H - \varphi G$ falls within the following ranges:

We need $-\frac{1}{\varphi^2} \leq H - \varphi G \leq \frac{1}{\varphi}$, but not 0.

An Exceptional Proof

To track this argument visually, an integer G appears to the right of the vertical bar once for each integer H such that $H - \varphi G$ falls within the following ranges:

How long is this interval?

An Exceptional Proof

To track this argument visually, an integer G appears to the right of the vertical bar once for each integer H such that $H - \varphi G$ falls within the following ranges:

How long is this interval? **1** (Why?)

An Exceptional Proof

To track this argument visually, an integer G appears to the right of the vertical bar once for each integer H such that $H - \varphi G$ falls within the following ranges:

Because $\frac{1}{\varphi} - (-\frac{1}{\varphi^2}) = \frac{\varphi}{\varphi^2} + \frac{1}{\varphi^2}$

An Exceptional Proof

To track this argument visually, an integer G appears to the right of the vertical bar once for each integer H such that $H - \varphi G$ falls within the following ranges:

Because $\frac{1}{\varphi} - (-\frac{1}{\varphi^2}) = \frac{\varphi}{\varphi^2} + \frac{1}{\varphi^2} = \frac{\varphi^2}{\varphi^2} = 1.$

It All Makes Sense

So an integer G appears to the right of the vertbar once for each integer H satisfying

$$-\frac{1}{\varphi^2} \leq H - \varphi G \leq \frac{1}{\varphi}, \quad H - \varphi G \neq 0.$$

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How often does $G = 1776$ appear?

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How often does $G = 1776$ appear? **once**

Because $2873.25 \leq H \leq 2874.25$ has precisely **one** integer solution.

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How often does $G = -127$ appear? **once**
(Because $-205.87 \leq H \leq -204.87$.)

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$$-\frac{1}{\varphi^2} \leq H - \varphi G \leq \frac{1}{\varphi}, \quad H - \varphi G \neq 0.$$

How often does $G = -1$ appear?

It All Makes Sense

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$$-\frac{1}{\varphi^2} \leq H - \varphi G \leq \frac{1}{\varphi}, \quad H - \varphi G \neq 0.$$

How often does $G = -1$ appear? **Twice**

(Because $-2 \leq H \leq -1$.)

It All Makes Sense

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$$-\frac{1}{\varphi^2} \leq H - \varphi G \leq \frac{1}{\varphi}, \quad H - \varphi G \neq 0.$$

How many times does $G = 0$ appear?

It All Makes Sense

So an integer G appears to the right of the vertbar once for each integer H satisfying

$$-\frac{1}{\varphi^2} \leq H - \varphi G \leq \frac{1}{\varphi}, \quad H - \varphi G \neq 0.$$

How many times does $G = 0$ appear? **None**
(Because $-0.382 \leq H \leq 0.618$, $H \neq 0$.)

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And You're Done! ☐