Organising Compositionality in Classical Mechanics

Berkeley Seminar

Khyathi Komalan

TOPOS INSTITUTE

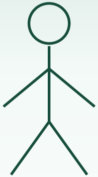
August 25, 2026

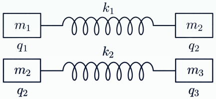
Can we compose classical mechanical systems?
Can we separate the parts of a mechanical system from the choice of how they are connected?

- Two springs, how may I connect them?
- Lagrangian and Hamiltonian descriptions of classical mechanics
- The Baez-Weisbart-Yassine approach and its limitations
- Our loose action construction
- Examples: One spring, two springs side by side, and two springs sharing a mass
- Future work: Hamiltonian evolution and representable maps

Suppose you give two people two springs

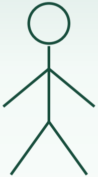
David Jaz

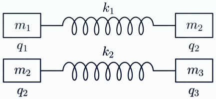




=

Khyathi

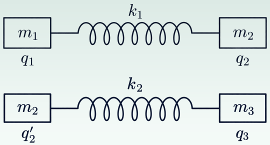




What will they make with these springs?

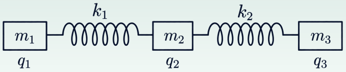
Same springs, different ways of putting them together

David Jaz: keeps them separate



The springs remain two separate systems.
Their middle positions q_2 and q_2' are independent, so altogether there are four position variables.

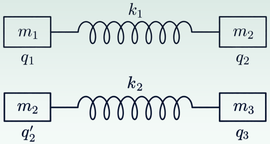
Khyathi: joins the middle masses



The right mass of the first spring and the left mass of the second become the same mass.
Thus $q_2 = q_2'$, leaving three position variables.

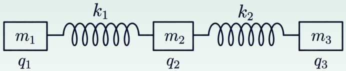
Same springs, different ways of putting them together

David Jaz: keeps them separate



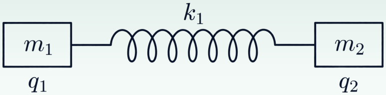
The springs remain two separate systems.
Their middle positions q_2 and q_2' are independent, so altogether there are four position variables.

Khyathi: joins the middle masses



The right mass of the first spring and the left mass of the second become the same mass.
Thus $q_2 = q_2'$, leaving three position variables.

II. Background: How do I represent a physical system?



Parameters

The masses m_1, m_2 and spring constant k_1 specify the physical ingredients.

Configuration

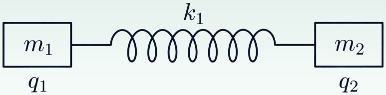
A point $q = (q_1, q_2) \in Q = \mathbb{R}^2$ encodes where the two masses are.

State

Velocity states live in TQ, momenta lives in T^*Q carrying the symplectic form $\omega_{\text{can}} = \sum_i dq_1 \wedge dp_i$

Dynamics

Finally, we need a function telling us how these states evolve.



Parameters

The masses m_1, m_2 and spring constant k_1 specify the physical ingredients.

Configuration

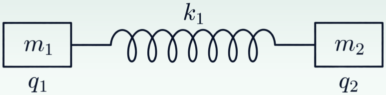
A point $q = (q_1, q_2) \in Q = \mathbb{R}^2$ encodes where the two masses are.

State

Velocity states live in TQ, momenta lives in T^*Q carrying the symplectic form $\omega_{\mathsf{can}} = \sum_i dq_1 \wedge dp_i$

Dynamics

Finally, we need a function telling us how these states evolve.

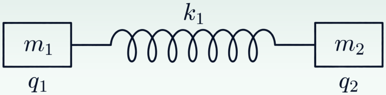


Parameters
The masses m_1, m_2 and spring constant k_1 specify the physical ingredients.

Configuration
A point $q = (q_1, q_2) \in Q = \mathbb{R}^2$ encodes where the two masses are.

State
Velocity states live in TQ, momenta lives in T^*Q carrying the symplectic form $\omega_{\text{can}} = \sum_i dq_1 \wedge dp_i$

Dynamics
Finally, we need a function telling us how these states evolve.



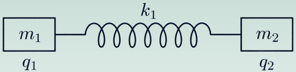
Parameters
The masses m_1, m_2 and spring constant k_1 specify the physical ingredients.

Configuration
A point $q = (q_1, q_2) \in Q = \mathbb{R}^2$ encodes where the two masses are.

State
Velocity states live in TQ, momenta lives in T^*Q carrying the symplectic form $\omega_{\text{can}} = \sum_i dq_1 \wedge dp_i$

Dynamics
Finally, we need a function telling us how these states evolve.

II. Background: How do I describe the dynamics?



Stretching the spring stores potential energy: $V = \frac{1}{2}k_1(q_1 - q_2)^2$.

Lagrangian mechanics

Describe the state using positions and velocities: $(q, v) \in TQ$.

$L = T - V$: motion contributes positively, while stretching contributes negatively.

$L = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{1}{2}k_1(q_1 - q_2)^2$.

The Euler-Lagrange equations use L to determine how the masses move.

Hamiltonian mechanics

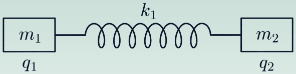
Describe the state using positions and momenta: $(q, p) \in T^*Q$.

$H = T + V$: it is the total kinetic and potential energy of the state.

$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2}k_1(q_1 - q_2)^2$.

Hamilton's equations use H to determine how position and momenta evolve with time.

II. Background: How do I describe the dynamics?



Stretching the spring stores potential energy: $V = \frac{1}{2}k_1(q_1 - q_2)^2$.

Lagrangian mechanics

Describe the state using positions and velocities: $(q, v) \in TQ$.

$L = T - V$: motion contributes positively, while stretching contributes negatively.

$L = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{1}{2}k_1(q_1 - q_2)^2$.

The Euler-Lagrange equations use L to determine how the masses move.

Hamiltonian mechanics

Describe the state using positions and momenta: $(q, p) \in T^*Q$.

$H = T + V$: it is the total kinetic and potential energy of the state.

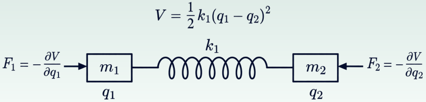
$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2}k_1(q_1 - q_2)^2$.

Hamilton's equations use H to determine how position and momenta evolve with time.

III. Background: What do we use here? Lagrangian submanifolds!

Instead of doing all that, we first encode the spring's position-force relation geometrically.

From potential to force



The spring has $V = \frac{1}{2}k_1(q_1 - q_2)^2$

Moving the masses changes this energy by dV_q, physically, the force covector is $-dV_q$

Thus the spring determines, at every position q, a particular covector dV_q.

Geometric structure

Collecting all position-covector pairs:

$$L_V = \{(q, dV_q) : q \in Q\} = \mathrm{graph}(dV) \subseteq T^*Q.$$

This is a submanifold of the spring's phase space T^*Q.

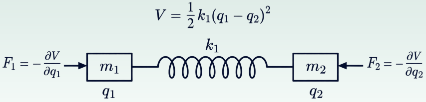
It is **Lagrangian** because $\omega_{\mathrm{can}}|_{L_V} = 0$ and $\dim L_V = \frac{1}{2}\dim T^*Q$.

Thus $\mathrm{graph}(dV)$ is the Lagrangian relation imposed by potential.

III. Background: What do we use here? Lagrangian submanifolds!

Instead of doing all that, we first encode the spring's position-force relation geometrically.

From potential to force



The spring has $V = \frac{1}{2}k_1(q_1 - q_2)^2$

Moving the masses changes this energy by dV_q, physically, the force covector is $-dV_q$

Thus the spring determines, at every position q, a particular covector dV_q.

Geometric structure

Collecting all position-covector pairs:

$$L_V = \{(q, dV_q) : q \in Q\} = \text{graph}(dV) \subseteq T^*Q.$$

This is a submanifold of the spring's phase space T^*Q.

It is **Lagrangian** because $\omega_{\text{can}}|_{L_V} = 0$ and $\dim L_V = \frac{1}{2}\dim T^*Q$.

Thus graph(dV) is the Lagrangian relation imposed by potential.

We now know the data behind one spring. What happens when we join springs?

The BWY approach

A system is an **augmented span** $(X, F_X) \leftarrow (S, F_S) \rightarrow (Y, F_Y)$.

The span describes how the system connects; the augmentation carries its potential V or Hamiltonian H.

Systems are composed by identifying shared parts using a pullback.

Why is this complicated?

Lagrangian and Hamiltonian mechanics preserve different geometric structure, so BWY need different categories for them.

Neither category has the ordinary pullbacks needed to compose spans.

They therefore introduce generalized F-pullbacks and span-tightness.

For this talk, you only need the idea that: **systems are spans, they compose by pullback**.

IV. Previous approach: Baez-Weisbart-Yassine

We now know the data behind one spring. What happens when we join springs?

The BWY approach

A system is an **augmented span** $(X, F_X) \leftarrow (S, F_S) \rightarrow (Y, F_Y)$.

The span describes how the system connects; the augmentation carries its potential V or Hamiltonian H.

Systems are composed by identifying shared parts using a pullback.

Why is this complicated?

Lagrangian and Hamiltonian mechanics preserve different geometric structure, so BWY need different categories for them.

Neither category has the ordinary pullbacks needed to compose spans.

They therefore introduce generalized F-pullbacks and span-tightness.

For this talk, you only need the idea that: **systems are spans, they compose by pullback**.

V. Our construction: separate systems from how they interact

Given fixed component systems, we want to vary their interaction without redefining the systems themselves.

Systems have interfaces

Think of a system S as something carrying an interface I through which it may interact.
Two systems can first be placed side by side:

$$S_1 \otimes S_2.$$

At this stage, we have specified the components, but not how they are connected.

Interactions act on systems

An interaction $\mu : I \rightsquigarrow J$ specifies how an interface is reorganised.
It acts on the system:

$$S \triangleleft \mu.$$

So for the two springs, different choices μ_A, μ_B give

$$(S_1 \otimes S_2) \triangleleft \mu_A, \quad (S_1 \otimes S_2) \triangleleft \mu_B.$$

Same components, but different modes of assembly.

V. Our construction: separate systems from how they interact

Given fixed component systems, we want to vary their interaction without redefining the systems themselves.

Systems have interfaces

Think of a system S as something carrying an interface I through which it may interact.
Two systems can first be placed side by side:

$$S_1 \otimes S_2.$$

At this stage, we have specified the components, but not how they are connected.

Interactions act on systems

An interaction $\mu : I \rightsquigarrow J$ specifies how an interface is reorganised.
It acts on the system:

$$S \triangleleft \mu.$$

So for the two springs, different choices μ_A, μ_B give

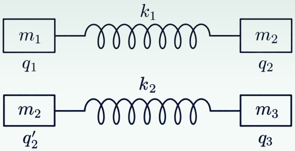
$$(S_1 \otimes S_2) \triangleleft \mu_A, \quad (S_1 \otimes S_2) \triangleleft \mu_B.$$

Same components, but different modes of assembly.

V. Our construction: what should a better construction do?

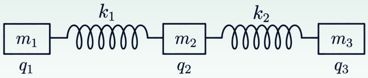
Keep the systems separate from the choice of how to connect them.

The systems



The two springs are unchanged.
Putting them side by side does not mean that $q_2 = q_2'$.

The interaction



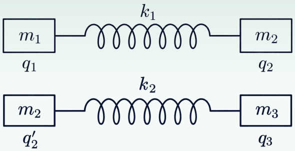
The choice that $q_2 = q_2'$ is separate data.
It acts on the two systems to produce the connected spring.

Mathematically, we want this action to be an ordinary **pullback**.

V. Our construction: what should a better construction do?

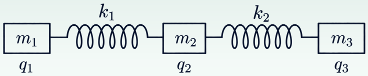
Keep the systems separate from the choice of how to connect them.

The systems



The two springs are unchanged. Putting them side by side does not mean that $q_2 = q'_2$.

The interaction



The choice that $q_2 = q'_2$ is separate data.
It acts on the two systems to produce the connected spring.

Mathematically, we want this action to be an ordinary **pullback**.

V. Our construction: first, we regain the power of pullbacks!

We keep the geometry of classical mechanics, but move to a setting where pullbacks always exist.

Presymplectic geometry

A **presymplectic space** is a smooth space X equipped with a closed 2-form ω_X.
A map $f : (X, \omega_X) \to (Y, \omega_Y)$ is **presymplectic** when $f^*\omega_Y = \omega_X$.
So the map preserves the 2-form geometry.

Why a smooth topos?

Pullbacks of manifolds need not be manifolds.
A smooth topos $\mathcal{H}$ enlarges the smooth world so that pullbacks always exist.
We work in $\mathcal{C} = \mathcal{H}/\Omega^2_{\mathrm{cl}}$.
Its objects are (X, ω_X), and its maps are the presymplectic maps defined earlier.

V. Our construction: first, we regain the power of pullbacks!

We keep the geometry of classical mechanics, but move to a setting where pullbacks always exist.

Presymplectic geometry

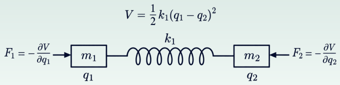
A **presymplectic space** is a smooth space X equipped with a closed 2-form ω_X.
A map $f : (X, \omega_X) \to (Y, \omega_Y)$ is **presymplectic** when $f^*\omega_Y = \omega_X$.
So the map preserves the 2-form geometry.

Why a smooth topos?

Pullbacks of manifolds need not be manifolds.
A smooth topos $\mathcal{H}$ enlarges the smooth world so that pullbacks always exist.
We work in $\mathcal{C} = \mathcal{H}/\Omega^2_{\mathrm{cl}}$.
Its objects are (X, ω_X), and its maps are the presymplectic maps defined earlier.

V. Our construction: why this slice captures the spring

The spring



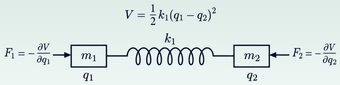
Configurations form $Q = \mathbb{R}^2$, with $V = \frac{1}{2}k_1(q_1 - q_2)^2$.
The differential $dV_q \in T^*_q Q$ measures the change of potential while $-dV_q$ is the force.
Thus the spring determines $dV : Q \to T^*Q,\ q \mapsto (q, dV_q)$.

Why this slice helps

In $\mathcal{C} = \mathcal{H}/\Omega^2_{\mathrm{cl}}$, we have $(Q, 0)$ and $(T^*Q, \omega_{\mathrm{can}})$.
Moreover, $(dV)^*\omega_{\mathrm{can}} = -d(dV) = 0$, so $dV : (Q, 0) \to (T^*Q, \omega_{\mathrm{can}})$ is a presymplectic map.

V. Our construction: why this slice captures the spring

The spring



Configurations form $Q = \mathbb{R}^2$, with $V = \frac{1}{2}k_1(q_1 - q_2)^2$.
The differential $dV_q \in T^*_q Q$ measures the change of potential while $-dV_q$ is the force.
Thus the spring determines $dV : Q \to T^*Q,\ q \mapsto (q, dV_q)$.

Why this slice helps

In $\mathcal{C} = \mathcal{H}/\Omega^2_{\mathrm{cl}}$, we have $(Q, 0)$ and $(T^*Q, \omega_{\mathrm{can}})$.
Moreover, $(dV)^*\omega_{\mathrm{can}} = -d(dV) = 0$, so $dV : (Q, 0) \to (T^*Q, \omega_{\mathrm{can}})$ is a presymplectic map.

V. Our construction: the categorical input

Spans and pullbacks

Let $\mathcal{C}$ be a category with pullbacks.
A span

$$X \leftarrow M \rightarrow Y$$

composes with

$$Y \leftarrow N \rightarrow Z$$

by pulling back over the shared interface Y:

$$X \leftarrow M \times_Y N \rightarrow Z.$$

Thus spans in $\mathcal{C}$ form the loose arrows of the double category $\mathrm{Span}(\mathcal{C})$.

Our presymplectic case

For

$$\mathcal{C} = \mathcal{H}/\Omega^2_{\mathrm{cl}},$$

all pullbacks exist since $\mathcal{C}$ is a topos.
Parallel composition is

$$(X, \omega_X) \otimes (Y, \omega_Y) = (X \times Y,\ \pi_X^* \omega_X + \pi_Y^* \omega_Y),$$

with unit $(1, 0)$.
This tensor preserves pullbacks in each variable.
Hence $\mathrm{Span}(\mathcal{C})$ is **symmetric monoidal**.

Spans and pullbacks

Let $\mathcal{C}$ be a category with pullbacks.
A span

$$X \leftarrow M \rightarrow Y$$

composes with

$$Y \leftarrow N \rightarrow Z$$

by pulling back over the shared interface Y:

$$X \leftarrow M \times_Y N \rightarrow Z.$$

Thus spans in $\mathcal{C}$ form the loose arrows of the double category $\mathrm{Span}(\mathcal{C})$.

Our presymplectic case

For

$$\mathcal{C} = \mathcal{H}/\Omega^2_{\mathrm{cl}},$$

all pullbacks exist since $\mathcal{C}$ is a topos.
Parallel composition is

$$(X, \omega_X) \otimes (Y, \omega_Y) = (X \times Y,\ \pi_X^* \omega_X + \pi_Y^* \omega_Y),$$

with unit $(1, 0)$.
This tensor preserves pullbacks in each variable.
Hence $\mathrm{Span}(\mathcal{C})$ is **symmetric monoidal**.

V. Our construction: spans give us systems and interactions

Libkind–Myers, Explication 6.4

Point $\mathrm{Span}(\mathcal{C})$ at $(1,0)$.
Then:
An **interface** is an object $I \in \mathcal{C}$,

A **system** on I is a span

$$(1,0) \leftarrow X \rightarrow I,$$

And an **interaction** $I \rightsquigarrow J$ is a span

$$I \leftarrow M \rightarrow J.$$

What does the interaction do?

Given

$$(1,0) \leftarrow X \rightarrow I \qquad \text{and} \qquad I \leftarrow M \rightarrow J,$$

the interaction acts by **pullback**:

$$(1,0) \leftarrow X \times_I M \rightarrow J.$$

Equivalently,

$$S \triangleleft \mu = \left[(1,0) \leftarrow X \times_I M \rightarrow J\right].$$

So the component system and the choice of how to couple it are separate pieces of data.

V. Our construction: spans give us systems and interactions

Libkind–Myers, Explication 6.4

Point $\mathrm{Span}(\mathcal{C})$ at $(1,0)$.
Then:
An **interface** is an object $I \in \mathcal{C}$,

A **system** on I is a span

$$(1,0) \leftarrow X \rightarrow I,$$

And an **interaction** $I \rightsquigarrow J$ is a span

$$I \leftarrow M \rightarrow J.$$

What does the interaction do?

Given

$$(1,0) \leftarrow X \rightarrow I \qquad \text{and} \qquad I \leftarrow M \rightarrow J,$$

the interaction acts by **pullback**:

$$(1,0) \leftarrow X \times_I M \rightarrow J.$$

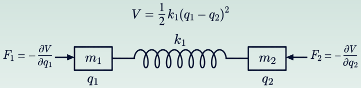
Equivalently,

$$S \triangleleft \mu = \big[(1,0) \leftarrow X \times_I M \rightarrow J\big].$$

So the component system and the choice of how to couple it are separate pieces of data.

VI. Examples: a single spring

A single spring



The configuration space is $Q = \mathbb{R}^2$ and

$$V_1(q_1, q_2) = \frac{1}{2}k_1(q_1 - q_2)^2.$$

Take the phase-space interface (T^*Q, ω_{12}), where $\omega_{12} = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$.
Define

$$x = dV_1 : Q \to T^*Q,$$

so $dV_1 = k_1(q_1 - q_2)(dq_1 - dq_2)$.
Thus x assigns to each configuration the covector describing the change of potential there, while its negative is the spring force.

The corresponding system

By Libkind-Myers Explication 6.4, we need a span in $\mathcal{C}$:

$$(1, 0) \xleftarrow{!} (Q, 0) \xrightarrow{x} (T^*Q, \omega_{12}).$$

The left leg is the unique map to $(1, 0)$, and $!^*0 = 0$.
For the right leg,

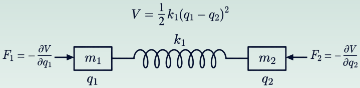
$$\begin{aligned} x^*\omega_{12} &= dq_1 \wedge k_1(dq_1 - dq_2) \\ &\quad + dq_2 \wedge [-k_1(dq_1 - dq_2)] \\ &= 0. \end{aligned}$$

Hence both legs are maps in $\mathcal{C}$, so the span is a system.

The left leg is trivial, but requiring it to be presymplectic forces the configuration-space form to be $0 :!^*0 = 0$.

VI. Examples: a single spring

A single spring



The configuration space is $Q = \mathbb{R}^2$ and

$$V_1(q_1, q_2) = \frac{1}{2}k_1(q_1 - q_2)^2.$$

Take the phase-space interface (T^*Q, ω_{12}), where $\omega_{12} = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$. Define

$$x = dV_1 : Q \to T^*Q,$$

so $dV_1 = k_1(q_1 - q_2)(dq_1 - dq_2)$.
Thus x assigns to each configuration the covector describing the change of potential there, while its negative is the spring force.

The corresponding system

By Libkind-Myers Explication 6.4, we need a span in $\mathcal{C}$:

$$(1, 0) \xleftarrow{!} (Q, 0) \xrightarrow{x} (T^*Q, \omega_{12}).$$

The left leg is the unique map to $(1, 0)$, and $!^*0 = 0$.
For the right leg,

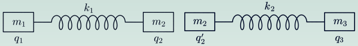
$$\begin{aligned} x^*\omega_{12} &= dq_1 \wedge k_1(dq_1 - dq_2) \\ &\quad + dq_2 \wedge [-k_1(dq_1 - dq_2)] \\ &= 0. \end{aligned}$$

Hence both legs are maps in $\mathcal{C}$, so the span is a system.

The left leg is trivial, but requiring it to be presymplectic forces the configuration-space form to be $0 :!^*0 = 0$.

VI. Examples: placing springs side by side

Two independent springs



Let

$$V_1 = \frac{1}{2}k_1(q_1 - q_2)^2, \quad V_2 = \frac{1}{2}k_2(q_2' - q_3)^2.$$

The corresponding systems are

$$s_{12} : (1,0) \leftarrow (\mathbb{R}^2, 0) \xrightarrow{dV_1} (T^*\mathbb{R}^2, \omega_{12}),$$

$$s_{23} : (1,0) \leftarrow (\mathbb{R}^2, 0) \xrightarrow{dV_2} (T^*\mathbb{R}^2, \omega_{23}).$$

Tensor product places them in parallel, without identifying any coordinates.

The corresponding system

Their configuration spaces combine to $\mathbb{R}^4$, with coordinates (q_1, q_2, q_2', q_3).
Set

$$W = V_1(q_1, q_2) + V_2(q_2', q_3).$$

Then

$$s_{12} \otimes s_{23} = \left[(1,0) \leftarrow (\mathbb{R}^4, 0) \xrightarrow{dW} (T^*\mathbb{R}^4, \omega_4)\right].$$

Under $T^*(\mathbb{R}^2 \times \mathbb{R}^2) \cong T^*\mathbb{R}^2 \times T^*\mathbb{R}^2$,

$$dW = (dV_1, dV_2).$$

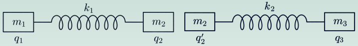
Since each dV_i preserves the chosen 2-forms,

$$(dW)^*\omega_4 = (dV_1)^*\omega_{12} + (dV_2)^*\omega_{23} = 0.$$

Hence this is again a system. Importantly, q_2 and q_2' remain independent.

VI. Examples: placing springs side by side

Two independent springs



Let

$$V_1 = \frac{1}{2}k_1(q_1 - q_2)^2, \quad V_2 = \frac{1}{2}k_2(q_2' - q_3)^2.$$

The corresponding systems are

$$s_{12} : (1,0) \leftarrow (\mathbb{R}^2, 0) \xrightarrow{dV_1} (T^*\mathbb{R}^2, \omega_{12}),$$

$$s_{23} : (1,0) \leftarrow (\mathbb{R}^2, 0) \xrightarrow{dV_2} (T^*\mathbb{R}^2, \omega_{23}).$$

Tensor product places them in parallel, without identifying any coordinates.

The corresponding system

Their configuration spaces combine to $\mathbb{R}^4$, with coordinates (q_1, q_2, q_2', q_3).
Set

$$W = V_1(q_1, q_2) + V_2(q_2', q_3).$$

Then

$$s_{12} \otimes s_{23} = \left[(1,0) \leftarrow (\mathbb{R}^4, 0) \xrightarrow{dW} (T^*\mathbb{R}^4, \omega_4)\right].$$

Under $T^*(\mathbb{R}^2 \times \mathbb{R}^2) \cong T^*\mathbb{R}^2 \times T^*\mathbb{R}^2$,

$$dW = (dV_1, dV_2).$$

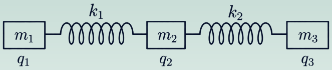
Since each dV_i preserves the chosen 2-forms,

$$(dW)^*\omega_4 = (dV_1)^*\omega_{12} + (dV_2)^*\omega_{23} = 0.$$

Hence this is again a system. Importantly, q_2 and q_2' remain independent.

VI. Examples: joining the middle masses

Joining the middle mass



Start with

$$(1,0) \leftarrow (\mathbb{R}^4, 0) \xrightarrow{dW} (T^*\mathbb{R}^4, \omega_4),$$

where q_2, q_2' are independent.
Define

$$\delta : \mathbb{R}^3 \to \mathbb{R}^4, \quad \delta(q_1, q_2, q_3) = (q_1, q_2, q_2, q_3).$$

This imposes $q_2 = q_2'$.
It gives the interaction

$$\mu_\delta : (T^*\mathbb{R}^4, \omega_4) \leftarrow (C_\delta, \omega_{C_\delta}) \to (T^*\mathbb{R}^3, \omega_3),$$

where $C_\delta = \delta^* T^*\mathbb{R}^4$.

The corresponding system

A point of C_δ is (q, α), with $\alpha \in T^*_{\delta(q)}\mathbb{R}^4$.
The left leg sends $(q, \alpha) \mapsto \alpha$; the right leg sends $(q, \alpha) \mapsto \delta_q^* \alpha$. So forces are pulled back along the constraint.
Acting by pullback gives

$$(s_{12} \otimes s_{23}) \triangleleft \mu_\delta \cong \left[(1,0) \leftarrow (\mathbb{R}^3, 0) \xrightarrow{d(W \circ \delta)} (T^*\mathbb{R}^3, \omega_3) \right].$$

Indeed, the apex is $\mathbb{R}^3$, and

$$\delta^*(dW) = d(W \circ \delta).$$

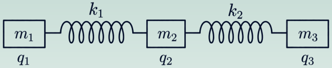
Finally,

$$W \circ \delta = \frac{1}{2} k_1 (q_1 - q_2)^2 + \frac{1}{2} k_2 (q_2 - q_3)^2.$$

Thus the interaction gives the force relation for two springs whose middle masses move together.

VI. Examples: joining the middle masses

Joining the middle mass



Start with

$$(1,0) \leftarrow (\mathbb{R}^4, 0) \xrightarrow{dW} (T^*\mathbb{R}^4, \omega_4),$$

where q_2, q_2' are independent.
Define

$$\delta : \mathbb{R}^3 \to \mathbb{R}^4, \quad \delta(q_1, q_2, q_3) = (q_1, q_2, q_2, q_3).$$

This imposes $q_2 = q_2'$.
It gives the interaction

$$\mu_\delta : (T^*\mathbb{R}^4, \omega_4) \leftarrow (C_\delta, \omega_{C_\delta}) \to (T^*\mathbb{R}^3, \omega_3),$$

where $C_\delta = \delta^* T^*\mathbb{R}^4$.

The corresponding system

A point of C_δ is (q, α), with $\alpha \in T^*_{\delta(q)}\mathbb{R}^4$.
The left leg sends $(q, \alpha) \mapsto \alpha$; the right leg sends $(q, \alpha) \mapsto \delta_q^* \alpha$. So forces are pulled back along the constraint.
Acting by pullback gives

$$(s_{12} \otimes s_{23}) \triangleleft \mu_\delta \cong \left[(1,0) \leftarrow (\mathbb{R}^3, 0) \xrightarrow{d(W \circ \delta)} (T^*\mathbb{R}^3, \omega_3) \right].$$

Indeed, the apex is $\mathbb{R}^3$, and

$$\delta^*(dW) = d(W \circ \delta).$$

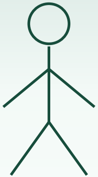
Finally,

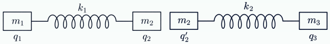
$$W \circ \delta = \frac{1}{2} k_1 (q_1 - q_2)^2 + \frac{1}{2} k_2 (q_2 - q_3)^2.$$

Thus the interaction gives the force relation for two springs whose middle masses move together.

A different assembly

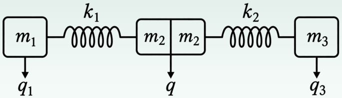
Kevin





glues the middle masses together

The resulting picture



Here the two middle particles are *distinct* point masses, each of mass m_2.
Gluing them means they move together, so

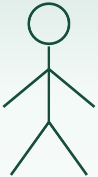
$$q_2 = q_2'.$$

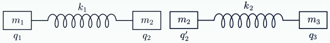
Since we are using point masses, they may occupy the same point.
So the middle body now has mass

$$m_2 + m_2 = 2m_2.$$

A different assembly

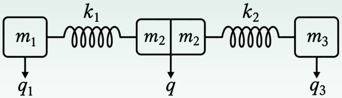
Kevin





glues the middle masses together

The resulting picture



Here the two middle particles are *distinct* point masses, each of mass m_2.
Gluing them means they move together, so

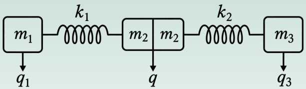
$$q_2 = q_2'.$$

Since we are using point masses, they may occupy the same point.
So the middle body now has mass

$$m_2 + m_2 = 2m_2.$$

VII. Future work: Kevin glues the middle masses

Kevin's choice



Start with two independent middle coordinates:

$$W = \frac{1}{2}k_1(q_1 - q_2)^2 + \frac{1}{2}k_2(q_2' - q_3)^2.$$

Now interpret q_2, q_2' as two distinct point masses, each of mass m_2.
Person C rigidly glues them. Since these are point masses, gluing means they occupy the same point:

$$q_2 = q_2'.$$

Physically, the middle body now has mass

$$m_2 + m_2 = 2m_2.$$

The corresponding system... or is it?

The same interaction imposes

$$\delta(q_1, q, q_3) = (q_1, q, q, q_3).$$

So acting by pullback again gives

$$(1, 0) \leftarrow (\mathbb{R}^3, 0) \xrightarrow{d(W\circ\delta)} (T^*\mathbb{R}^3, \omega_3).$$

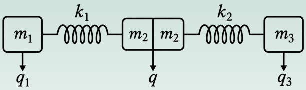
But

$$W \circ \delta = \frac{1}{2}k_1(q_1 - q)^2 + \frac{1}{2}k_2(q - q_3)^2.$$

This is the same span as the one-middle-mass example.
So the current construction sees the same position-force relation, but not the fact that the middle mass is now $2m_2$.

VII. Future work: Kevin glues the middle masses

Kevin's choice



Start with two independent middle coordinates:

$$W = \frac{1}{2}k_1(q_1 - q_2)^2 + \frac{1}{2}k_2(q_2' - q_3)^2.$$

Now interpret q_2, q_2' as two distinct point masses, each of mass m_2.
Person C rigidly glues them. Since these are point masses, gluing means they occupy the same point:

$$q_2 = q_2'.$$

Physically, the middle body now has mass

$$m_2 + m_2 = 2m_2.$$

The corresponding system... or is it?

The same interaction imposes

$$\delta(q_1, q, q_3) = (q_1, q, q, q_3).$$

So acting by pullback again gives

$$(1, 0) \leftarrow (\mathbb{R}^3, 0) \xrightarrow{d(W\circ\delta)} (T^*\mathbb{R}^3, \omega_3).$$

But

$$W \circ \delta = \frac{1}{2}k_1(q_1 - q)^2 + \frac{1}{2}k_2(q - q_3)^2.$$

This is the same span as the one-middle-mass example.
So the current construction sees the same position-force relation, but not the fact that the middle mass is now $2m_2$.

VII. Future work: why Hamiltonian data is needed

What dV shows

A span of the form

$$(1,0) \leftarrow (Q,0) \xrightarrow{dV} (T^* Q, \omega)$$

records the potential-gradient relation

$$q \mapsto dV(q).$$

For the spring, this is the force information:

$$-dV(q) = \text{force covector}.$$

But for the two middle-mass stories, the potential is the same:

$$V = \frac{1}{2}k_1(q_1 - q)^2 + \frac{1}{2}k_2(q - q_3)^2.$$

So dV cannot distinguish middle mass m_2 from middle mass $2m_2$.

What Hamiltonians show

Hamiltonians include kinetic energy:

$$H = T + V.$$

For the middle coordinate,

$$H_{\text{shared}} \supset \frac{P^2}{2m_2}, \quad H_{\text{glued}} \supset \frac{P^2}{2(2m_2)}.$$

Thus the two systems have the same dV, but different H.
Future direction: enrich objects to

$$(X, \omega_X, H_X).$$

Then Hamiltonian flow should give represented systems

$$\mathrm{Rep}_t(X_H) = \left[(1,0) \leftarrow (X,0) \xrightarrow{(\mathrm{id}, \phi_t^H)} X^\vee \otimes X\right].$$

Goal: show interactions compose these represented evolutions correctly.

VII. Future work: why Hamiltonian data is needed

What dV shows

A span of the form

$$(1,0) \leftarrow (Q,0) \xrightarrow{dV} (T^*Q, \omega)$$

records the potential-gradient relation

$$q \mapsto dV(q).$$

For the spring, this is the force information:

$$-dV(q) = \text{force covector}.$$

But for the two middle-mass stories, the potential is the same:

$$V = \frac{1}{2}k_1(q_1 - q)^2 + \frac{1}{2}k_2(q - q_3)^2.$$

So dV cannot distinguish middle mass m_2 from middle mass $2m_2$.

What Hamiltonians show

Hamiltonians include kinetic energy:

$$H = T + V.$$

For the middle coordinate,

$$H_{\text{shared}} \supset \frac{P^2}{2m_2}, \quad H_{\text{glued}} \supset \frac{P^2}{2(2m_2)}.$$

Thus the two systems have the same dV, but different H.
Future direction: enrich objects to

$$(X, \omega_X, H_X).$$

Then Hamiltonian flow should give represented systems

$$\mathrm{Rep}_t(X_H) = \left[(1,0) \leftarrow (X,0) \xrightarrow{(\mathrm{id}, \phi_t^H)} X^\vee \otimes X\right].$$

Goal: show interactions compose these represented evolutions correctly.

VIII. Conclusion

What we built

- A spring gives a system

$$(1,0) \leftarrow (Q,0) \xrightarrow{dV} (T^*Q,\omega).$$

- Tensor places systems side by side.
- Interactions are spans between interfaces.
- Acting by an interaction means pulling back.

So the components stay fixed while the assembly changes.

What comes next

- The present construction records geometry.
- It handles constraints such as

$$q_2 = q_2'.$$

- But it does not yet record inertia.
- Hamiltonian flavouring should add the missing mass-sensitive dynamics.

Thank You!