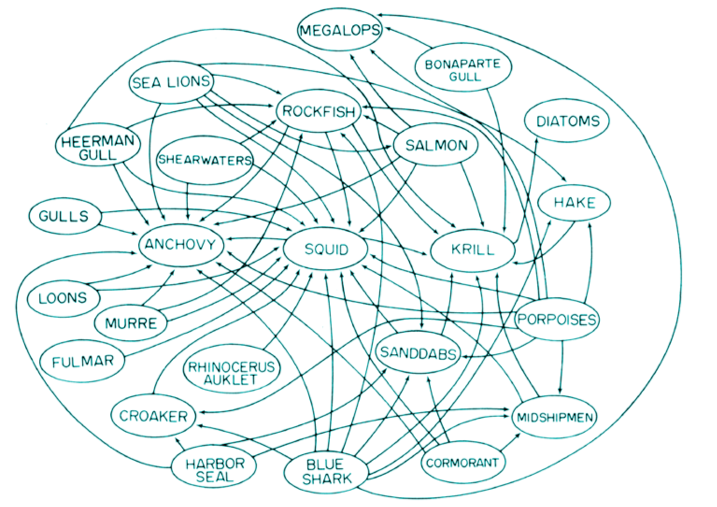
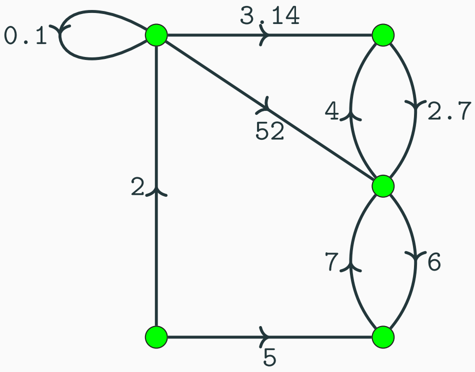
The Universal Property of the Algebraic Path Problem

EM-Cats - Jade Master - Aug 25 2021



Overview

- A distance graph may represent the data of an enriched category.
- This allows us to generalize "distance" to something else.
- Free enriched categories give a universal property for these generalizations.



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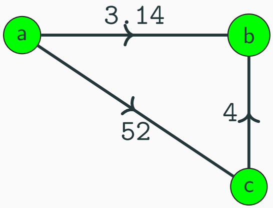
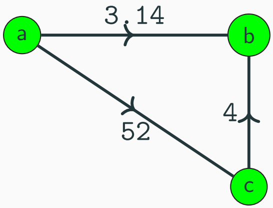
§1: Shortest Paths



The shortest path problem asks for the sequence of edges between a given pair of vertices with minimum total distance.

§1: Shortest Paths



The shortest path problem asks for the sequence of edges between a given pair of vertices with minimum total distance.

path	total length
(a,b)	3.14
(a,c) (c,b)	52+4=56

$$\text{shortest path} = \min_{\text{paths p}} \{\text{length}(p)\}$$

$$= \min\{3.14, 56\} = 3.14$$

What structure allows us to generalize this?

§1: Semirings

- A semiring $(R, +, \cdot)$ is like a ring except $+$ is only a monoid and need not have negatives.

Example

The natural numbers $\mathbb{N}$ form a semiring with the usual $+$ and $\cdot$.

Motivating Example

$[0, \infty]$ with min as the additive monoid and $+$ as the multiplicative monoid.

§1: Semirings

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The natural numbers $\mathbb{N}$ form a semiring with the usual $+$ and $\cdot$.

Motivating Example

$[0, \infty]$ with min as the additive monoid and $+$ as the multiplicative monoid.

Warning!

This example can be very confusing

addition	min
multiplication	$+$
additive identity	∞
multiplicative identity	0

§1: Weighted Graphs Are Matrices

M(i,j) = x	(i, j) has weight x
M(i,j) = add. identity	no edge drawn
$M(i, i) = x$	node i has weight x

Definition

Let R be a semiring. An R-matrix is a set of vertices X along with a weighting function $M : X \times X \to R$.

$$\begin{bmatrix} .1 & 3.14 & 52 & \infty \\ \infty & \infty & 2.7 & \infty \\ \infty & 4 & \infty & 6 \\ \infty & \infty & 9 & \infty \end{bmatrix}$$

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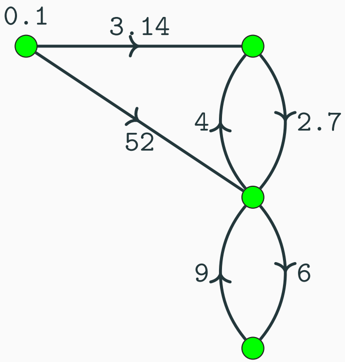

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Now we can use matrix operations to study weighted graphs.

§1: The Algebraic Path Problem

Let X be a set of vertices.

Let $M : X \times X \to R$ be an R-matrix.

Definition

An **edge** in M is a pair of vertices (a, b).

A **path** in M is a sequence of edges

$$p = \{(i, a_1), (a_1, a_2), \ldots, (a_n, j)\}.$$

The **weight** $w(p)$ of a path p is the product in R

$$M(i, a_1)M(a_1, a_2) \ldots M(a_n, j).$$

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For $i, j \in X$, Let

$$P_{ij} = \{\text{paths } p \text{ from } i \text{ to } j\}$$

The shortest path is

$$\min_{p \in P_{ij}} \{\text{length}(p)\}$$

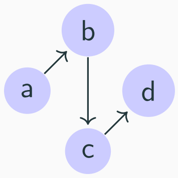
The **algebraic path problem** asks for

$$\sum_{p \in P_{ij}} w(p).$$

§1: Example: Connectivity

Let $\mathbb{B}$ be the semiring $(\{T, F\}, \vee, \wedge)$.

The APP for $\mathbb{B}$ detects connectivity.



M(i,j) = T	directed edge from i to j
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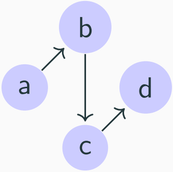

M(i,j) = T	directed edge from i to j
M(i,j) = F	no edge drawn

path p	weight w(p)
(a,b) (b,c) (c,d)	T $\wedge T \wedge T = T$
(a,c) (c,d)	F $\wedge T = F$
all other paths	F

$$\sum_{p \in P_{ad}} w(p) = \bigvee_{p \in P_{ad}} w(p)$$

$$= T \vee F \vee F \cdots = T$$

$$\sum_{p \in P_{da}} w(p) = \bigvee_{p \in P_{da}} w(p)$$

$$= F \vee F \vee F \cdots = F$$

There are lots of other examples!

semiring	**sum**	**product**	**solution of path problem**
$[0, \infty]$	inf	$+$	shortest paths in a weighted graph
$[0, \infty]$	sup	inf	maximum capacity in the tunnel problem
$[0, 1]$	sup	$\times$	most likely paths in a Markov process
$\{T, F\}$	or	and	transitive closure of a directed graph
$(\mathcal{P}(\Sigma^*), \subseteq)$	$\bigcup$	concatenation	decidable language of a NFA

§2: Why Matrices?

Idea

M^n has entries $M^n(i,j)$ given by the length of the shortest path from i to j with exactly n-steps.

When $R = [0, \infty]$,

$$
\begin{aligned}
M^2(i,j) &= \sum_{k \in X} M(i,k) M(k,j) \\
&= \min_{k \in X} \{ M(i,k) + M(k,j) \}
\end{aligned}
$$

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The pointwise minimum

$$F(M)(i,j) = \min_{n \geq 0}\{M^n(i,j)\}$$

gives solutions to the algebraic path problem.

For an arbitrary R

$$F(M)(i,j) = \sum_{n \geq 0} M^n(i,j)$$

Next will see how this is the formula for the "free R-enriched category on M".

§2: Free Categories

Idea

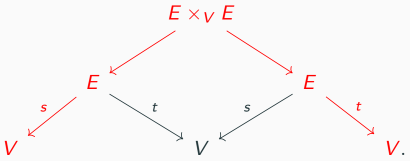
Paths of length n in G are given by iterated pullbacks of G with itself.

A graph is a diagram of sets and functions

$$E \overset{s}{\underset{t}{\rightrightarrows}} V$$

take the pullback with itself

$G^2 =$



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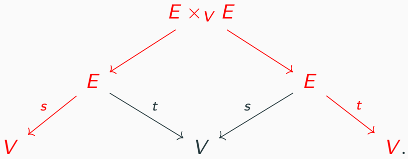

Edges of $G^2 = \{(e, e') \in E \times E \mid t(e) = s(e)\}$

taking the n-fold pulback gives G^n with

Edges of $G^n = \{\text{paths of length } n \text{ in } G\}$.

The coproduct

$$F(G) = \coprod_{n \geq 0} G^n$$

is the free category.

Proposition

There is an adjunction

$$\mathrm{Grph} \underset{U}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathrm{Cat}$$

$U(C) =$ the underlying graph of C

$F(G) = \coprod_{n \geq 0} G^n$.

Proposition

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$U(C) =$ the underlying graph of C

$F(G) = \coprod_{n \geq 0} G^n$.

Next we'll see . . .

- You can enrich in any monoidal category $(\mathcal{V}, \otimes)$
- This adjunction generalizes to

$$\mathcal{V}\mathrm{Grph} \underset{U}{\overset{F}{\underset{\bot}{\rightleftarrows}}} \mathcal{V}\mathrm{Cat}$$

  when $\mathcal{V}$ has countable coproducts preserved by $\otimes$
- Nice semirings R will be an example of such $\mathcal{V}$.

§3: Enriched Categories

Idea:

Instead of a set of morphisms, a $\mathcal{V}$-category has an object of $\mathcal{V}$.

A $\mathcal{V}$-category C is a set of objects X along with for all $x, y, z \in X$

- a $\mathcal{V}$-object $C(x, y)$,
- a $\mathcal{V}$-morphism

$$C(x, y) \otimes C(y, z) \to C(x, z)$$

- and a $\mathcal{V}$-morphism $I \to C(x, x)$.

Plus axioms.

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Plus axioms.

Examples:

- A category enriched in Set is an ordinary category.
- A category enriched in Cat is a 2-category.
- A category enriched in abelian groups is an pre-additive category.

A semiring $(R, +, \cdot)$ becomes a poset with
$a \leq b \iff \exists c$ s.t. $a + c = b$

V	R
objects	elements
morphisms	$\leq$
$\coprod$	$\sum$
$\otimes$	$\cdot$
distr. of $\coprod$ over $\otimes$	distr. of $+$ over $\cdot$

§3: Nice Posets

A semiring $(R, +, \cdot)$ becomes a poset with
$a \leq b \iff \exists c \text{ s.t. } a + c = b$

V	R
objects	elements
morphisms	$\leq$
$\coprod$	$\sum$
$\otimes$	$\cdot$
distr. of $\coprod$ over $\otimes$	distr. of $+$ over $\cdot$

Warning!

This gives $[0, \infty]$ the reverse of the usual ordering.

Quantales are posets nice enough for us.

Definition

A **quantale** is a monoidal closed poset with all coproducts.

Idea

Enriched category theory gets easier when enriching in a poset.

Definition

For a quantale R, an R-category is a matrix $C : X \times X \to R$ such that

- $1 \leq C$ (identities)
- $C^2 \leq C$ (composition)

For a monoidal closed category $\mathcal{V}$ with all coproducts

$$\mathcal{V}\mathrm{Grph} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{V}\mathrm{Cat} \quad (F \dashv U)$$

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Proposition

For a quantale R, there is an adjunction

$$\mathrm{RMat} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{RCat} \qquad (F \dashv U)$$

Realization!

The left adjoint F gives solutions to the algebraic path problem.

Big Idea

Let $M : X \times X \to R$ be an R-matrix and let $i, j \in X$. The entry of the free R-category on M

$$F(M)(i,j) = \sum_{n \geq 0} M^n(i,j)$$

is the solution to the algebraic path problem on M from i to j.

§3: Now We're in Business

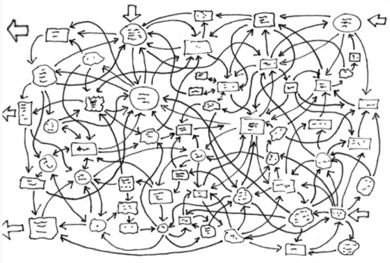
Big Idea

Let $M : X \times X \to R$ be an R-matrix and let $i, j \in X$. The entry of the free R-category on M

$$F(M)(i,j) = \sum_{n \geq 0} M^n(i,j)$$

is the solution to the algebraic path problem on M from i to j.

- R-matrices may be joined together using colimits
- Left adjoints preserve colimits
- Can this help us glue together solutions?

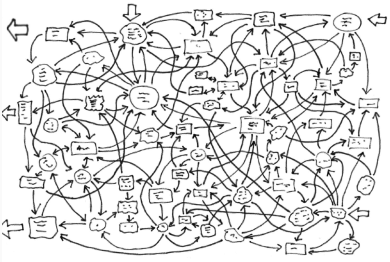


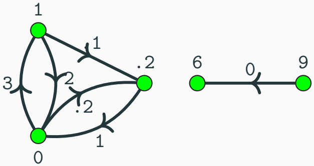
Algorithms for the algebraic path problem have $O(V^3)$ complexity.

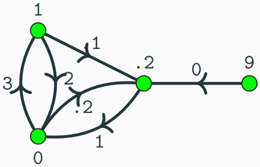
Question

Can solutions to the APP be built up from smaller components?

§4: Compositionality







Algorithms for the algebraic path problem have $O(V^3)$ complexity.

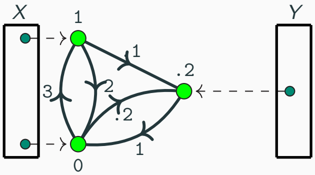
Question

Can solutions to the APP be built up from smaller components?

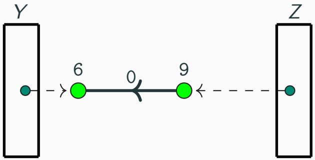
Idea

To glue graphs together first designate some of the vertices as inputs or outputs.

§4: Building Graphs with Composition



$G: X \to Y$



$H: Y \to Z$

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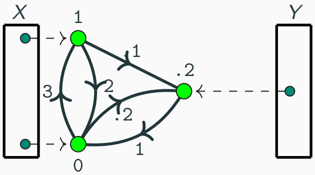
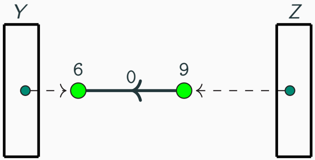
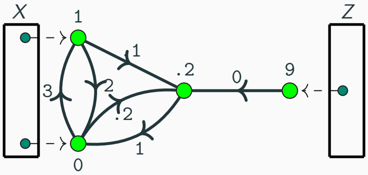


$G \colon X \to Y$



$H \colon Y \to Z$



$H \circ G \colon X \to Z$

- "Open R-matrices" are cospans in RMat
- They are glued together with pushouts
- For overlapping weights we use min

The Good News

Left adjoints preserve pushouts so

$$F(H \circ_{Mat} G) \cong F(H) \circ_{Cat} F(G)$$

where $\circ_{Mat}$ is the pushout of R-matrices and $\circ_{Cat}$ is the pushout of R-categories.

The Bad News

Pushouts of categories are hard.

The Good News Again

Under certain circumstances they get easier.

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The Good News Again

Under certain circumstances they get easier.

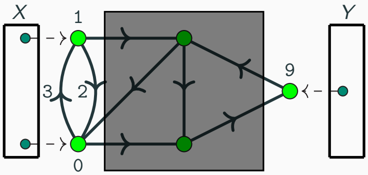
Theorem (JEM)

For "functional open R-matrices" $G\colon X \to Y$ and $H\colon Y \to Z$, there is an equality

$$\blacksquare(H \circ_{Mat} G) = \blacksquare(H)\blacksquare(G)$$

where the product on the right hand is matrix multiplication.

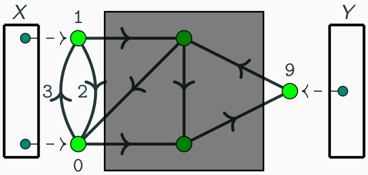
- What is a functional open R-matrix?
- What does $\blacksquare$ mean?



Idea

Focus on the inputs and outputs and forget about the rest.

§3: Black-boxing



Idea

Focus on the inputs and outputs and forget about the rest.

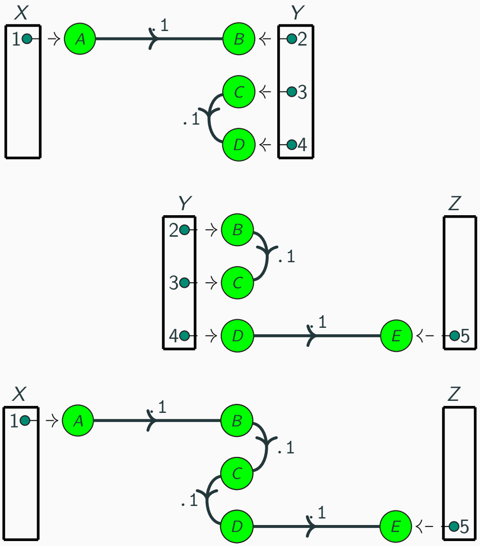
For an open R-matrix $G\colon X \to Y$ its **black-boxing** is the R-matrix

$$\blacksquare(G)\colon X \times Y \to R$$

with values

$$\blacksquare(G)(x, y) = \text{ solution of APP from } x \text{ to } y$$

§3: Matrix Multiplication does not Preserve Gluing



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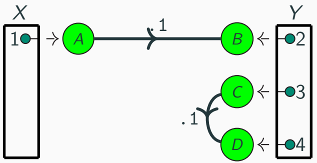

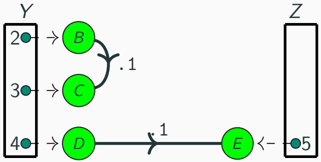

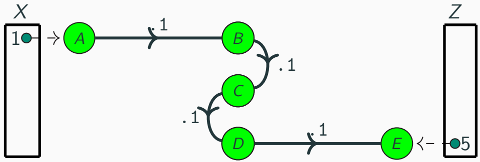

$$\blacksquare(G : X \to Y) = \begin{bmatrix} .1 & \infty & \infty \end{bmatrix}^T$$

$$\blacksquare(H : Y \to Z) = \begin{bmatrix} \infty & \infty & .1 \end{bmatrix}$$

$$\blacksquare(G)\blacksquare(H) = \begin{bmatrix} \infty \end{bmatrix}$$

On the other hand. . .

$$\blacksquare(H \circ_{Mat} G) = \begin{bmatrix} 0.4 \end{bmatrix}$$

Idea

A functional open R-matrix has no edges going into its inputs and no edges going out of its outputs.

Further Questions

Theorem

For functional open R-matrices $G\colon X \to Y$ and $H\colon Y \to Z$, there is an equality

$$\blacksquare(H \circ G) = \blacksquare(H)\blacksquare(G).$$

For more see..

- *The Open Algebraic Path Problem* arXiv:2005.06682
- *Open Petri Nets*, with John Baez
- *Composing Behaviors of Networks*, Ph.D. Thesis.

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- Can we refine functional matrices?
- Does Van-Kampen's theorem relate to the compositionality results?
- Are there non-posets $\mathcal{V}$ that have interesting free $\mathcal{V}$-categories?
- Can this be turned into useful code?