Skein Categories and Quantization

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I Quantization

II Skein Categories

III The A-polynomial

What's Quantization?

- The term quantization comes from Physics: Classical system $\rightsquigarrow$ Quantum system
- Observables (position, momentum, etc) go from being functions to operators — functions $\rightsquigarrow$ differential operators
- Mathematically, quantization is — commutative algebra $\rightsquigarrow$ non-commutative algebra

Example: Polynomials $\rightsquigarrow$ differential operators

$$\mathbb{C}[x,y] \qquad \mathbb{C}[\hbar^{\pm 1}]\langle x, \partial_x \mid \partial_x \cdot x - x\partial_x = 1 \rangle$$

Send $x \mapsto x$, $y \mapsto \hbar\partial_x$, but for general $f(x,y)$, the 'right' quantization might not be clear:

$$\begin{aligned} xy &\longrightarrow x\hbar\partial_x \\ \| \\ yx &\longrightarrow \hbar\partial_x x = x\hbar\partial_x + \hbar \end{aligned}$$

What and why we're quantizing today

- We're interested in quantizing a certain knot invariant $A(x,y)$ - the A-polynomial
- It's quantization is a q-difference operator $A_q(L,M)$, which acts on $f: \mathbb{Z} \to \mathbb{C}[q^{\pm 1}]$

shift: $Lf(n) = f(n+1)$, multiplication: $Mf(n) = q^n f(n)$

- Our motivation is the AJ conjecture, which predicts that for any knot

$$\star \quad A_q \cdot J(n,q) = 0$$

colored Jones polynomial

My Work:

Give a robust, computable construction of A_q such that

- $A_{q \to 1} = A$
- It's reasonable to expect $\star$ to hold

We do this using Skein Categories.

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Ribbon Categories

A **ribbon category** $\mathcal{A}$ has

- **monoidal structure**: $\otimes\ \mathcal{A}\times\mathcal{A}\to\mathcal{A}$, $\mathbb{1}$
- **braiding**: $c_{V,W}: V\otimes W \xrightarrow{\cong} W\otimes V$ such that

$$\begin{tikzcd}
 & W\otimes V\otimes U \arrow[r, "id\otimes c"] & W\otimes U\otimes V \arrow[r, "c\otimes id"] & U\otimes W\otimes V \\
V\otimes W\otimes U \arrow[ur, "c\otimes id"] \arrow[dr, "id\otimes c"'] & & & \\
 & V\otimes U\otimes W \arrow[r, "c\otimes id"'] & U\otimes V\otimes W \arrow[uur, "id\otimes c"'] &
\end{tikzcd}$$

- **duality**: coevaluation $b_V: \mathbb{1}\to V\otimes V^*$
  evaluation $d_V: V^*\otimes V\to\mathbb{1}$

$$\begin{tikzcd}
V\otimes V^*\otimes V \arrow[r, "id\otimes d"] & \mathbb{1}\otimes V \cong V \\
V\otimes\mathbb{1}\cong V \arrow[u, "b\otimes id"] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
V^*\otimes V\otimes V^* \arrow[r, "d\otimes id"] & \mathbb{1}\otimes V^* \cong V^* \\
V^*\otimes\mathbb{1}\cong V^* \arrow[u, "id\otimes b"] \arrow[ur] &
\end{tikzcd}$$

+ twist, compatibility conditions

Example: Rib

objects: $\mathbb{Z}_{\geq 0}^{\{+,-\}}$ **morphisms** ribbon graphs (framed tangles)

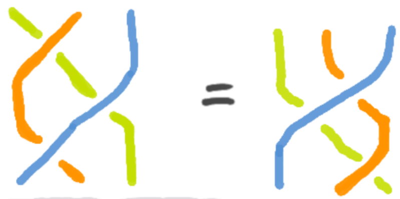
e.g. $\bullet^-$, $\bullet^+\bullet^-$, $\emptyset$

- **disjoint union**

- **crossing**: $c =$

$c^{-1} =$

$c\sqcup id$
$id\sqcup c$
$c\sqcup id$



$id\sqcup c$
$c\sqcup id$
$id\sqcup c$

- **cups and caps**

$b =$  $d =$ 

$id\sqcup d$
$b\sqcup id$

$d\sqcup id$
$id\sqcup b$

From Ribbons to Skeins

- **Colored Ribbons**: Rib_A ← Ribbon category

  objects: v w ← "colors"

  morphisms: Colored tangles:

  • can have coupons: $f: V \to W$

- There's a unique braided monoidal functor $F: Rib_A \longrightarrow \mathcal{A}$

$$F(\overset{+}{\underset{V_1}{\cdot}}\ \overset{+}{\underset{V_2}{\cdot}} \cdots \overset{+}{\underset{V_n}{\cdot}}) = V_1 \otimes V_2 \otimes \cdots \otimes V_n$$

- This induces skein relations on $Hom_{Rib_A}(X,Y) \ni T_1, T_2$

$$T_1 \sim T_2 \iff F(T_1) = F(T_2)$$

Def: Skein Category: $SkCat_A(\Sigma)$

First: For a disk $\mathbb{D}$

objects: Same as Rib_A

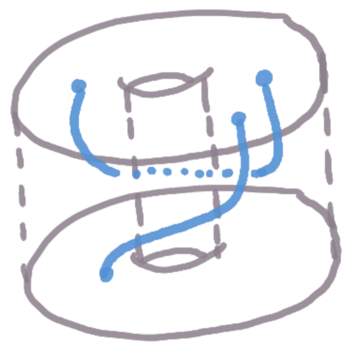
morphisms: Spanned by $f \in Rib_A(X,Y)$, subject to skein relations

For a general surface Σ:

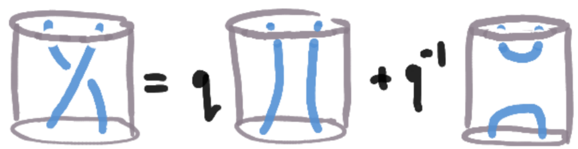
objects:

morphisms: spanned by ribbons in $\Sigma \times [0,1]$ with skein relations applied locally:

$\Sigma \times [0,1]$



$\mathbb{D} \times [0,1] \hookrightarrow \Sigma \times [0,1]$

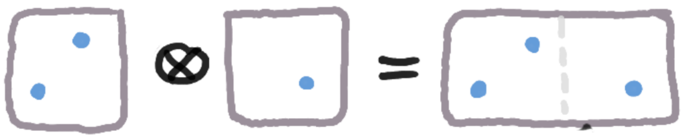


Properties of Skein Categories

- $\mathrm{SkCat}_{\mathcal{A}}$ is a functor $\underline{\mathrm{Surf}} \longrightarrow \underline{\mathrm{Cat}}$
  - $\underline{\mathrm{Surf}}$: · surfaces · embeddings
  - $\underline{\mathrm{Cat}}$: · Small $\mathbb{C}_q$-linear Categories · $\mathbb{C}_q$-linear Functors

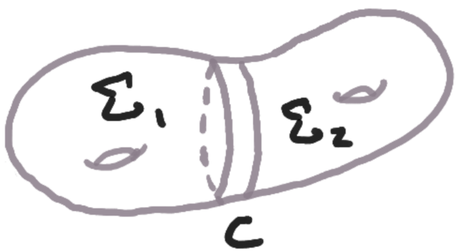
- By construction $F_{\mathcal{A}} : \mathrm{SkCat}_{\mathcal{A}}(\mathbb{D}) \xrightarrow{\simeq} \mathcal{A}$

  topologically, the monoidal structure is



  ($\mathrm{SkCat}_{\mathcal{A}}(\Sigma)$ isn't typically monoidal!)

- [Cooke 19] $\mathrm{SkCat}_{\mathcal{A}}$ satisfies excision: topological decomposition $\longrightarrow$ categorical decomposition



$$\longrightarrow \quad \mathrm{SkCat}_{\mathcal{A}}(\Sigma) \simeq \mathrm{SkCat}_{\mathcal{A}}(\Sigma_1) \underset{C}{\otimes} \mathrm{SkCat}_{\mathcal{A}}(\Sigma_1)$$

Skein Modules

- Let M be a 3-manifold, $\partial M = \Sigma$.
  X be an object in $\mathrm{SkCat}_A(\Sigma)$

Def The relative skein module $\mathrm{SkMod}(M,X)$ is the vector space spanned by tangles in M with boundary X

(modulo skein relations in $\mathbb{D}\times[0,1] \hookrightarrow M$)

$\mathrm{SkMod}(M,X)$

Def $\mathrm{SkMod}(M)$ is the functor

$$\mathrm{SkCat}_A(\partial M)^{op} \longrightarrow \mathrm{Vect}$$

On objects:

$$X \longmapsto \mathrm{SkMod}(M,X)$$

On morphisms:

- For quantization, we'll want to work with algebras and their modules

Some Category Theory

- Two functors $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "R"] & \mathcal{D} \arrow[l, shift left, "L"] \end{tikzcd}$ are **adjoint** if there are natural transformations

unit $\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "RL"] \arrow[r, bend right, "1_{\mathcal{D}}"'] \arrow[r, phantom, "\Uparrow\eta"] & \mathcal{D} \end{tikzcd}$ counit $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "1_{\mathcal{C}}"] \arrow[r, bend right, "LR"'] \arrow[r, phantom, "\Uparrow\varepsilon"] & \mathcal{C} \end{tikzcd}$ (Satisfying some conditions)

Equivalently, if $\mathrm{Hom}_{\mathcal{C}}(LX, Y) \cong \mathrm{Hom}_{\mathcal{D}}(X, RY)$
↑ natural isomorphism

- A functor $L: \mathcal{D} \to \mathcal{C}$ has a right adjoint if it's cocontinuous.
(+nice enough)

- If not: we can get cocontinuous extension

$$\hat{L}: \hat{\mathcal{D}} \to \hat{\mathcal{C}}$$

Def The **free cocompletion** of a category $\mathcal{C}$ is

$$\mathcal{C} \xrightarrow{\hat{}} \hat{\mathcal{C}} := \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Vect})$$ "presheaf category"

$$V \mapsto \hat{V} = \mathrm{Hom}_{\mathcal{C}}(-, V)$$

Fact: Free cocompletion $\hat{}: \mathrm{cat} \to \mathrm{Pr}^{loc}$ satisfies the universal property

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr, "\hat{}"'] & \mathcal{D} \\ & \hat{\mathcal{C}} \arrow[u, dashed, "\exists!\ \text{cocontinuous}"'] \end{tikzcd}$$

← cocomplete

· Locally Presentable Cats

· cocontinuous functors

- We can construct a monad from an adjoint pair.

Def A **monad** is $T: \mathcal{C} \to \mathcal{C}$, $\underset{\text{unit}}{\eta: Id_{\mathcal{C}} \to T}$, $\underset{\text{product}}{\mu: T^2 \to T}$ such that

associative $\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd}$ unital $\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[d, "T\eta"'] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$

- The **monad of an adjunction**

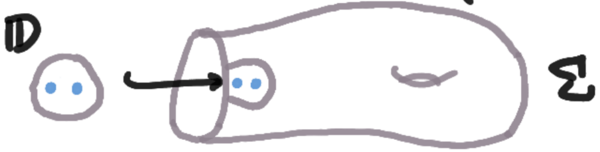
$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "R"] & \mathcal{D} \arrow[l, shift left, "L"] \end{tikzcd}$ unit $\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "RL"] \arrow[r, bend right, "1_{\mathcal{D}}"'] \arrow[r, phantom, "\Uparrow\eta"] & \mathcal{D} \end{tikzcd}$ counit $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "1_{\mathcal{C}}"] \arrow[r, bend right, "LR"'] \arrow[r, phantom, "\Uparrow\varepsilon"] & \mathcal{C} \end{tikzcd}$

is $T = RL: \mathcal{D} \to \mathcal{D}$, η, $\mu: (RL)(RL) \xrightarrow{\varepsilon} RL$

Internal Skein Algebras and Modules

- An embedding $\mathbb{D} \hookrightarrow \Sigma$ gives a functor

$$\mathcal{A} \simeq \mathrm{SkCat}_{\mathcal{A}}(\mathbb{D}) \xrightarrow{\text{act}} \mathrm{SkCat}_{\mathcal{A}}(\Sigma)$$



- $\widehat{\mathrm{act}} : \hat{\mathcal{A}} \to \widehat{\mathrm{SkCat}}_{\mathcal{A}}(\Sigma)$ has a right adjoint:

$$\widehat{\mathrm{act}}^R : \widehat{\mathrm{SkCat}}_{\mathcal{A}}(\Sigma) \to \hat{\mathcal{A}}$$

- Build a monad from $\widehat{\mathrm{act}}, \widehat{\mathrm{act}}^R$

Idea: $T = \widehat{\mathrm{act}}^R \widehat{\mathrm{act}} : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ respects the monoidal structure on $\hat{\mathcal{A}}$. (Lax Monoidal)

- It depends on Σ but lives over a disk

Def: The (internal) skein algebra:

$$\mathrm{SkAlg}^{\mathrm{int}}_{\mathcal{A}}(\Sigma) = \widehat{\mathrm{act}}^R \widehat{\mathrm{act}}(\bigcirc)$$

$\mathbb{D}$ with no points

The (internal) skein module $\mathrm{Sk}^{\mathrm{int}}_{\mathcal{A}}(M) \in \hat{\mathcal{A}}$ is the object in $\hat{\mathcal{A}}$ which represents the functor

$$\hat{\mathcal{A}} \xrightarrow{\widehat{\mathrm{act}}} \widehat{\mathrm{SkCat}}_{\mathcal{A}}(\Sigma) \xrightarrow{\widehat{\mathrm{SkMod}}} \mathrm{Vect}$$

i.e. for all $V \in \mathcal{A}$,

$$\mathrm{Hom}_{\hat{\mathcal{A}}}(\hat{V}, \mathrm{Sk}^{\mathrm{int}}_{\mathcal{A}}(M)) = \widehat{\mathrm{SkMod}}(M, \widehat{\mathrm{act}}(\hat{V}))$$

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The A-Polynomial is a Knot invariant

- knot invariants: Knot $K \cong S^1 \hookrightarrow S^3$ $\rightsquigarrow$ more rigid data, e.g. $\begin{cases} \text{number} \\ \text{polynomial} \\ \text{category} \end{cases}$

two depictions of the trefoil

$\rightsquigarrow$ $y + x^6$

the figure-8 knot

$\rightsquigarrow$ $y^2x^4 + y(-x^8 + x^6 + 2x^4 + x^2 - 1) + x^4$

- Ideally: they're easier to work with than the original knot, and/or independently interesting.
- The A-polynomial is defined using algebraic geometry. It was originally used to study incompressible surfaces in knot complements

Some Algebraic Geometry

- Historically: concerns polynomials
- Used to give geometric structure to families, e.g. representations, vector bundles.

Def: Affine set (zero set of polynomials)

$V \subseteq \mathbb{C}^n$ is called an affine set if there exists a set of polynomials $S \subseteq \mathbb{C}[x_1, \dots x_n]$ s.th.

$$V = \{\bar{x} \in \mathbb{C}^n \mid \text{for all } p \in S,\ p(\bar{x}) = 0\}$$

- The polynomials which vanish on V form an ideal:

$$I(V) = \{p \in \mathbb{C}[x_1, \dots x_n] \mid p|_V = 0\}$$

- V is called an affine variety if $I(V)$ is a prime ideal.

- Major Idea: Study spaces by looking at functions on them.

Def The coordinate ring $\mathcal{O}(V)$ is the ring of regular functions $V \to \mathbb{C}$

$$\mathcal{O}(\mathbb{C}^n) \simeq \mathbb{C}[x_1, \dots, x_n]$$

$$V \subset \mathbb{C}^n, \quad \mathcal{O}(V) \simeq \mathbb{C}[x_1, \dots, x_n] / I(V)$$

- A map between varieties $V_1 \longrightarrow V_2$ induces

$$\mathcal{O}(V_2) \longrightarrow \mathcal{O}(V_1)$$

so that

$$\underbrace{\mathcal{O}(V_2) \curvearrowright \mathcal{O}(V_1)}_{\text{this action is what we'll quantize.}}$$

Character Varieties

- Construct a variety from a general topological space.

- Let:

  X be a topological space, with

  $\pi_1(X) = \langle g_1, \ldots, g_n \rangle / \sim$ finitely gen.

  $SL_2\mathbb{C}$ denote the group of 2×2 matrices with determinant 1

  $B \subset SL_2\mathbb{C}$ be a fixed Borel Subgroup

  e.g. upper triangular matrices $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$

Def The (framed) character variety of X is the space of characters

$$Ch(X) := \{ \chi_\rho \mid \rho : \pi_1(X) \to B \}$$

where $\chi_\rho := tr \circ \rho : \pi_1(X) \to B \to \mathbb{C}$ is identified with the point $(\chi_\rho(g_1), \ldots, \chi_\rho(g_n)) \in \mathbb{C}^n$

for generators $g_1, \ldots, g_n$ of $\pi_1(X)$

Ex: $Ch(T^2) \cong \mathbb{C}^* \times \mathbb{C}^*$

$\pi_1(T^2) = \langle l, m \mid lm = ml \rangle \cong \mathbb{Z}^2$

The A-polynomial of a knot $K \subset S^3$

- The knot complement $S^3 \setminus K$ is a 3-manifold with torus boundary
- Inclusion $\partial(S^3 \setminus K) \cong T^2 \hookrightarrow S^3 \setminus K$ induces a map:

$$\mathrm{Ch}(S^3 \setminus K) \xrightarrow{\partial} \mathrm{Ch}(T^2) \subset \mathbb{C}^2 \qquad \text{"the boundary map"}$$

where $\partial(x)$ is the composition

$$\pi_1(T^2) \longrightarrow \pi_1(S^3 \setminus K) \xrightarrow{x} \mathbb{C}$$

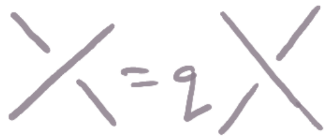
- This induces an action $\mathcal{O}(\mathrm{Ch}(T^2)) \cong \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \curvearrowright \mathcal{O}(\mathrm{Ch}(S^3 \setminus K))$.
- Fact: As an $\mathcal{O}(\mathrm{Ch}(T^2))$-module: $\mathcal{O}(\mathrm{Ch}(S^3 \setminus K)) \cong \mathbb{C}[x^{\pm 1}, y^{\pm 1}] / \langle A \rangle$ ★

Def: The A-polynomial

Up to normalization, the A-polynomial of a knot K is the generator A of the ideal ★

Skein Categories Quantize Character Varieties

- $\mathrm{SkAlg}^{int}_{\mathrm{Rep}_q T}(\Sigma)$ quantizes $\mathcal{O}(\mathrm{Ch}(\Sigma))$
- Use this to quantize the A-polynomial



q-difference operators

$$A \in \mathcal{O}(\mathrm{Ch}(T^2)) \simeq \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \rightsquigarrow \mathrm{SkAlg}^{int}_{A}(T^2) \simeq \mathbb{C}_q\langle L^{\pm 1}, M^{\pm 1} \mid LM = qML \rangle \ni A_q$$

- The knot complement is a 3-manifold and defines a skein module:

$$\mathcal{O}(\mathrm{Ch}(T^2)) \circlearrowright \mathcal{O}(\mathrm{Ch}(S^3 \setminus K)) \rightsquigarrow \mathrm{SkAlg}^{int}_{A}(T^2) \circlearrowright \mathrm{Sk}^{int}_{A}(S^3 \setminus K)$$

- As a $\mathrm{SkAlg}^{int}_{A}(T^2)$-module:

$$\mathrm{Sk}^{int}_{A}(S^3 \setminus K) \simeq \mathrm{SkAlg}^{int}_{A}(T^2)/I \simeq \mathbb{C}_q\langle L^{\pm 1}, M^{\pm 1}\rangle / \langle A_q \rangle$$

The quantum A-polynomial

Thank You!