A categorical study of quasi-uniform structures

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Overview

- Topology can be defined using open or closed sets. The approach using open sets is more familiar but the one using closed sets is equivalent.
- Classically, we think of closing subsets under limits. A closure operator is an abstraction formulation of this on the category **Top** of topological spaces. This allows an abstract formulation of continuous maps.
- We can then try to replace **Top** with a suitably nice category $\mathcal{C}$. This permits us to "do topology in category ". However this is hard.
- The topological structure we will look at is quasi-uniform structures. These allow to study things like uniform continuity, uniform convergence.

Structure of the talk

1. Quasi-uniform structure on a set
2. What is needed to do topology in categories?
3. Quasi-uniform structures on categories
4. Quasi-uniform structure determined by a subclass of $\mathcal{M}$
5. Other things the main theorem enables us to do

Quasi-uniform structure on a set

Idea of quasi-uniform structure

- Quasi-uniform structures are like uniform structures with one condition relaxed.

- A function $f : A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at a point if
  $\forall \varepsilon > 0,\ \exists \delta > 0 \mid \forall x \in D, 0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$

  This is about the neighbourhood of a point.

- $B_\delta = \{x \in \mathbb{R} \mid |x - a| < \delta\}$ is an open ball.

- $\{B_\delta \mid \delta > 0\}$ is a base of a topology on $\mathbb{R}$.

Quasi-uniform structure of the real line

- $f : A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is uniformly continuous if
  $\forall \varepsilon > 0,\ \exists \delta > 0 \mid \forall x, y \in D, 0 < |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$

- This is about x, y varying together.

  In fact $D_\delta = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid |x - y| < \varepsilon\}$ is a neighbourhood of the diagonal.

- $\mathcal{B}_\mathbb{R} = \{D_\delta \mid \varepsilon > 0\}$ is a base for a uniform structure $\mathcal{D}_\mathbb{R}$ on $\mathbb{R}$.

- We think of D_δ as a relation.

  The absolute value makes it symmetric.

  We will drop that symmetry condition to define a quasi-uniformity.

Definition of quasi-uniform structure on a set

- A quasi-uniform structure on a set X is non-empty collection $\mathcal{D}_X$ of subsets of $X \times X$ such that:

(U1) $\forall D \in \mathcal{D}_X,\ \triangle \subseteq D$ where $\triangle_X = \{(x,x) \mid x \in X\}$ (reflexivity).

(U2) $\forall D \in \mathcal{D}_X,\ \exists D' \in \mathcal{D}_X$ with $D' \circ D' \subseteq D$ (related to the triangle inequality)

(U3) $\forall D \in \mathcal{D}_X$ and $U \subseteq V,\ V \in \mathcal{U}_X$ (upwards closed).

(U4) $\forall D,\ D' \in \mathcal{D}_X,\ D \cap D' \in \mathcal{D}_X$ (finite intersections).

- A non-empty collection $\mathcal{B}_X$ of subsets of $X \times X$ is a base of a quasi-uniform structure on a set X if it satisfies axioms $(U1)$, $(U2)$ and $(U4)$ above.

From quasi-uniformity to topology

- If (X, d) is a metric space, $\mathcal{T}(d) = \{A \subseteq X \mid \forall x \in A, \ \exists\, \delta > 0 \text{ with } B(x, \delta) \subseteq A\}$.
- For any $D \in \mathcal{D}_X$, $D(x) = \{y \in X \mid (x, y) \in D\}$ is the uniform neigbourhood of x.
- Every quasi-uniform structure $\mathcal{D}_X$ on X induces a topology.
- $\mathcal{T}(\mathcal{D}) = \{A \subseteq X \mid \forall x \in A, \ \exists\ D \in \mathcal{D}_X \text{ with } D(x) \subseteq A\}$ where $D(x) = \{y \in X \mid (x, y) \in D\}$.

From topology to quasi-uniformity

- If $\mathcal{T}(\mathcal{D}) = \mathcal{T}_X$, then we say that the quasi-uniform structure $\mathcal{D}_X$ is compatible with a topology $\mathcal{T}_X$ or $\mathcal{T}_X$ admits $\mathcal{D}_X$.
- Example : $\mathcal{D}_{\mathbb{R}}$ is compatible with the topology of the real line.
- All topological spaces admit a quasi-uniform structure. This is better than uniform structures because only completely regular spaces admit a uniform structures.
- This means quasi-uniform structures are an alternative approach to the study of topological spaces

Quasi-uniform structure as a family of maps

Idea : We want to express quasi-uniform structures via maps so that we can look at them in other categories instead of **Top**.

A quasi-uniform structure on a set X is a family $\mathcal{U}_X = \{U : X \longrightarrow \mathcal{P}(X)\}$ of maps such that

$(U1)$ $\forall x \in X, \ \forall U \in \mathcal{U}_X, \ x \in U(x)$ (reflexivity).

$(U2)$ $\forall U \in \mathcal{U}_X, \ V \in \mathcal{U}_X \mid V \circ V \leq U$ i.e $y \in V(x) \Rightarrow V(y) \subseteq U(x)$ (related to the triangle inequality).

$(U3)$ $\forall U \in \mathcal{U}_X$ and $U \leq V$ $(i.e\ U(x) \subseteq V(x), \ \forall x \in X), \ V \in \mathcal{U}_X$ (upwards closed).

$(U4)$ $\forall U, \ V \in \mathcal{U}_X, \ U \cap V \in \mathcal{U}_X$ where $(U \cap V)(x) = U(x) \cap V(x)$ for each $x \in X$ (finite intersections).

- $U : X \longrightarrow \mathcal{P}(X) : x \longmapsto U(x)$= uniform neighbourhood of x

- It is easy to see that these maps can easily be extended to endomaps on $\mathcal{P}(X)$, $U : \mathcal{P}(X) \longrightarrow \mathcal{P}(X) : A \longmapsto \cup_{x \in A} U(x)$.

- Question: How can we deal with power sets in a general category $\mathcal{C}$?

  We also need images (for this definition) and pre-images for talking about continuity.

- Answer: We use factorization systems.

What is needed to do topology in categories?

Idea of factorization systems in Set

- Subsets are given by monics in *Set*
- Every $f : X \longrightarrow Y \in \textbf{Set}$ can be factored through its image
- $X \longrightarrow Y = X \longrightarrow Imf \longrightarrow Y$ giving epic/monic factorization.
- For any $X \in \textbf{Set}$, $\mathcal{P}(X)$ is a complete lattice with the union and intersection.
- For every $X \longrightarrow Y \in \textbf{Set}$, there is an image/pre-image adjunction in the power set lattice:

$$f(A) \subseteq B \Leftrightarrow A \subseteq f^{-1}(B)$$

for all $B \in \mathcal{P}(Y)$, $A \in \mathcal{P}(X)$

What is needed to do topology in categories?

- A category $\mathcal{C}$ endowed with an $(\mathcal{E},\mathcal{M})$-factorization system for morphisms.
- For any $X \in \mathcal{C}$, define sub$X = \{m \in \mathcal{M} \mid \text{cod}(m) = X\}$, collection of subobjects of X
- Sub$X$ ordered as follows: $n \leq m \Leftrightarrow \exists\, j \mid m \circ j = n$.
- Question: How can we make sure that sub$X$ is a complete lattice?
- Answer: We use $\mathcal{M}$-completeness.

$\mathcal{M}$-completeness

- $\mathcal{M}$-completeness is a technical condition involving wide pullbacks as a generalization of arbitrary intersections.

- If $\mathcal{C}$ is $\mathcal{M}$-complete then sub$X$ is a complete lattice with greatest element $1_X : X \longrightarrow X$.

- This also ensures that any $\mathcal{C}$-morphism, $f : X \longrightarrow Y$ induces an image/pre-image adjunction

$$f(m) \leq n \Leftrightarrow m \leq f^{-1}(n)$$

  for all $n \in \text{sub}Y$, $m \in \text{sub}X$ with $f(m)$ the $\mathcal{M}$-component of the $(\mathcal{E}, \mathcal{M})$-factorization of $f \circ m$.

- Comment: We now have everything we need to define quasi-uniform structures on a category $\mathcal{C}$.

Topology in **Top** vs Topology in a category $\mathcal{C}$

Topology in **Top**	Topology in a category $\mathcal{C}$
subsets	subobjects.
$\mathcal{P}(X)$	Sub$X$ lattice hence a category.
endomaps on $\mathcal{P}(X)$	$\mathcal{F}(\text{sub}X)$ = functor category on sub$X$

$U \leq V$ if $U(x) \subseteq V(x)$ for all element x	$U \leq V$ if $U(m) \leq V(m)$ for all subobject m.
Axioms $(U1)$-$(U4)$	Categorical version of $(U1)$-$(U4)$.
Continuous maps	Continuity with respect to the structure
Uniform continuity	$\mathcal{U}$-continuity

Quasi-uniform structures on categories

Definition (M. Iragi and D. Holgate, 2019)

- For all $U, V \in \mathcal{F}(\text{sub}X)$, we define $U \leq V$ if $U(m) \leq V(m)$ for all $m \in \text{sub}X$.
- A quasi-uniform structure on $\mathcal{C}$ with respect to $\mathcal{M}$ is a family $\mathcal{U} = \{\mathcal{U}_X \mid X \in \mathcal{C}\}$ with $\mathcal{U}_X$ a full subcategory of $\mathcal{F}(\text{sub}X)$ for each X such that:

($U1$) For any $U \in \mathcal{U}_X$, $1_X \leq U$ (reflexivity).

($U2$) For any $U \in \mathcal{U}_X$, there is $U' \in \mathcal{U}_X$ such that $U' \circ U' \leq U$ (related to triangle inequality).

($U3$) For any $U \in \mathcal{U}_X$ and $U \leq U'$, $U' \in \mathcal{U}_X$ (upwards closed).

($U4$) For any $U, U' \in \mathcal{U}_X, U \wedge U' \in \mathcal{U}_X$ (finite intersection).

($U5$) For any $\mathcal{C}$-morphism $f : X \longrightarrow Y$ and $U \in \mathcal{U}_Y$, there is $U' \in \mathcal{U}_X$ such that $f(U'(m)) \leq U(f(m))$ for all $m \in \text{sub}X$ (uniform continuity).

Totality of quasi-uniform structures as a lattice

- $QUNIF(\mathcal{C}, \mathcal{M})$ will denote the large collection of all quasi-uniform structures on $\mathcal{C}$ with respect to $\mathcal{C}$.
- $QUNIF(\mathcal{C}, \mathcal{M})$ is ordered as follows: $\mathcal{U} \leq \mathcal{V}$ if for all $X \in \mathcal{C}$ and $U \in \mathcal{U}_X$, there is $V \in \mathcal{V}_X$ such that $V \leq U$.
- A base for a quasi-uniformity $\mathcal{U}$ on $\mathcal{C}$ is a family $\mathcal{B} = \{\mathcal{B}_X \mid X \in \mathcal{C}\}$ with each $\mathcal{B}_X$ a full subcategory of $\mathcal{F}(\text{sub}X)$ for all $X \in \mathcal{C}$ satisfying all the axioms in the previous definition except $(U3)$.

Quasi-uniform structures and closure operators on **Top**.

- Every quasi-uniformity on X induces a closure operator on X.
- For a given topology on X, there are a number of quasi-uniformities on X which generate the topology.
- Quasi-uniformities and closure operators each make a complete lattice on **Top**.
- We have a monotone map from quasi-uniformities to closure operators and back on **Top**.
- Note that for **Top**, we can use interior operators instead of closure operators since sub$X$ is boolean.
- Our main result describes monotone maps between quasi-uniform structures and closure operators on $\mathcal{C}$.

Quasi-uniformities determined by a subclass of $\mathcal{M}$

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In **Top**	In $\mathcal{C}$
Embeddings	The class $\mathcal{M}$
Closed embeddings	subclass $\mathcal{N}$ of $\mathcal{M}$
Closed subsets of X	$\mathcal{N}_X = \{m \in \mathcal{N} \mid cod(m) = X\}$
Pre-images	Pullbacks
Pre-image of a closed subset is closed	$\mathcal{N}$ is closed under pullbacks

Quasi-uniformities determined by a subclass of $\mathcal{M}$

For all $X \in$ **Top**	For all $X \in \mathcal{C}$
Consider $\mathcal{N}_X$ all subsets of X	Consider $\mathcal{N}_X = \{m \in \mathcal{N} \mid cod(m) = X\}$
$\mathcal{N}_X$ is closed under arbitrary intersection	Require $\mathcal{N}_X$ to be closed under intersection.
Build a relation for each $L \subseteq \mathcal{N}_X$	Build a monotone map U^L for each $L \subseteq \mathcal{N}_X$
Make a base of a quasi-uniformity on **Top**	Make a base of a quasi-uniformity on $\mathcal{C}$

Main Theorem (M. Iragi, 2021)

- Let $\mathcal{N}$ be a fixed subclass of $\mathcal{M}$ closed under formation of pullbacks.
- For any $X \in \mathcal{C}$, let $\mathcal{N}_X = \{m \in \mathcal{N} \mid cod(m) = X\}$ be closed under arbitrary intersections and $\mathcal{A}_X$ the collection of all subclasses of $\mathcal{N}_X$ closed under arbitrary intersections.
- Define $U^L(m) = \bigwedge\{n \in L \mid m \leq n\}$ for any $m \in \mathrm{sub}X$
- Define $\mathcal{B}^\mathcal{A} = \{\mathcal{B}^\mathcal{A}_X \mid X \in \mathcal{C}\}$ where $\mathcal{B}^\mathcal{A}_X = \{U^L \mid L \in \mathcal{A}_X\}$

Then $\mathcal{B}^\mathcal{A}$ is a base for a quasi-uniformity $\mathcal{U}^\mathcal{A}$ on $\mathcal{C}$.

Closure operator on categories

- A closure operator c on $\mathcal{C}$ with respect to $\mathcal{M}$ is given by a family of maps

  $\{c_X\colon \text{sub}X \longrightarrow \text{sub}X \mid X \in \mathcal{C}\}$ such that:

  (C1) $m \leq c_X(m)$ for all $m \in \text{sub}X$;

  (C2) $m \leq n \Rightarrow c_X(m) \leq c_X(n)$ for all $m, n \in \text{sub}X$;

  (C3) every morphism $f : X \longrightarrow Y$ is c-continuous: $f(c_X(m)) \leq c_Y(f(m))$ for all $m \in \text{sub}X$.

- $CL(\mathcal{C}, \mathcal{M})$ = the collection of all closure operators on $\mathcal{C}$ with respect to $\mathcal{M}$.

- $c \in CL(\mathcal{C}, \mathcal{M})$ is said to be idempotent if $c_X(c_X(m)) = c_X(m)$ for all $m \in \text{sub}X$ and $X \in \mathcal{C}$.

From quasi-uniform structures to closure operators on $\mathcal{C}$.

- Let $\mathcal{U}$ be a quasi-uniform structure, then $c_X^{\mathcal{U}}(m) = \bigwedge\{U(m) : U \in \mathcal{U}_X\}$ is a closure operator on $\mathcal{C}$.

- $\mathcal{U} \in QUNIF(\mathcal{C}, \mathcal{M})$ is compatible with a closure operator c if $c_X(m) = c_X^{\mathcal{U}}(m)$

Quasi-uniformity generated by a closure operator

Let c be an idempotent closure operator on $\mathcal{C}$.

- $\mathcal{N} = \mathcal{M}^c$ = the class of c-closed elements of $\mathcal{M}$.
- For any $X \in \mathcal{C}$, consider $\mathcal{N}_X$ = c-closed subobjects of X.
- $\mathcal{N}_X$ is closed under arbitrary intersection.
- Build a monotone map c_X^L for each subclass L of $\mathcal{N}_X$ closed under arbitrary intersection.
- $\mathcal{B}_X^c = \{c_X^L \mid L \subseteq \mathcal{N}_X$ and L closed under arbitrary intersection$\}$ is a base of a quasi-uniformity on $\mathcal{C}$.
- $c_X(m) = c_X^{\mathcal{U}}(m)$.
- Note that we can also define interior operators but things do not work well.

Other things the main theorem enables us to do

Other things the main theorem enables us to do

- Galois connection between quasi-uniform structures and subclasses of $\mathcal{N}_X$ closed under pullbacks.
- Categorical description of all quasi-uniform structures compatible with a given topology.
- Every subcategory of $\mathcal{C}$ induces a quasi-uniform structure on $\mathcal{C}$
- Galois connection between quasi-uniform structures and subcategories of $\mathcal{C}$.
- Describe subcategories determined by a quasi-uniform on $\mathcal{C}$.
- We can apply this to many examples of $\mathcal{C}$ including groups, rings, topological groups.

Example: quasi-uniformity for groups

- $\mathcal{C} = \mathbf{Grp}$, the category of groups and group homomorphisms.
- $\mathcal{N}$ be the class of injective normal group homomorphisms.
- For any $G \in \mathbf{Grp}$, $\mathcal{N}_G =$ normal subgroups of G.
- Build a monotone map c_X^L for each subclass L of $\mathcal{N}_X$ closed under arbitrary intersection.
- $\mathcal{B}_X^c = \{c_X^L \mid L \subseteq \mathcal{N}_X$ and L closed under arbitrary intersection$\}$ is a family of closure operators on $\mathbf{Grp}$

Future work

Future work

- Completion of objects of a category using quasi-uniform structures.
- Categorical approach to quasi-uniform convergence topologies on function spaces.

For more on this topic, see e.g

- Quasi-uniform structures determined by closure operators with David Holgate, Topol. Appl. 295 (2021), 107669.
- Quasi-uniform and syntopogenous structures on categories with David Holgate, Topol. Appl. 263 (2019), 16-25.
- Topogenous and Nearness Structures on Categories with Holgate and Ando, Appl. Categor. Struct. 24(2016), 447-455.