Dialectica and Kolmogorov Problems

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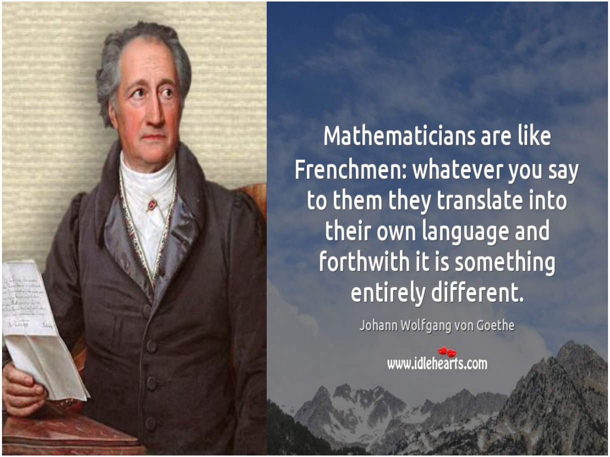
(joint work with Samuel G da Silva)

TOPOS INSTITUTE

Finding the Right Abstractions Workshop

Thanks for the invite!



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Asked 10 years, 5 months ago Active 10 years, 5 months ago Viewed 2k times

4

I've read about how a hard job in programming is writing code "at the right level of abstraction". Although abstraction is generally hiding details behind a layer, is writing something at the right level of abstraction basically getting the methodss to implement decision correct? For example, if I am going to write a class to represent a car, I will provide properties for the wheels etc, but if I am asked to write a class for a car with no wheels, I will not provide the wheels and thus no method to "drive" the car. Is this what is meant by getting the abstraction right?

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Thanks

oop

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- too lower-level!

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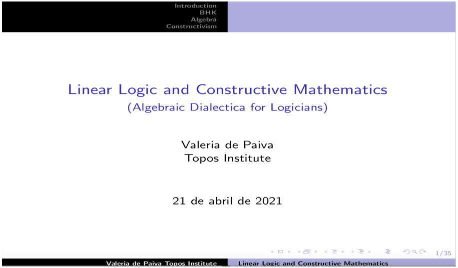

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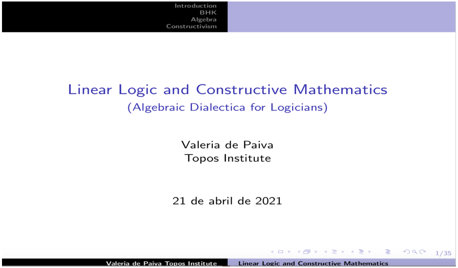
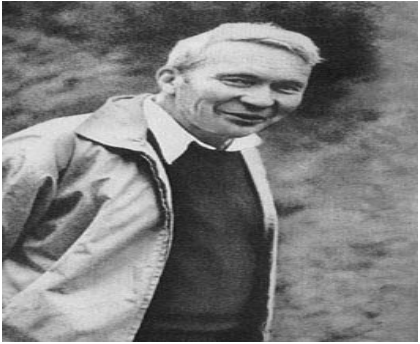


- A talk for philosophers of Mathematics!

Finding the right abstractions: today

- Finding right abstractions for **Reasoning**
- Category Theory and Curry-Howard for **Problems** (and solutions)
- 

  Andrei Kolmogorov. On the interpretation of intuitionistic logic (1932). Selected Works of A.N. Kolmogorov, Volume 1, Mathematics and Mechanics, ed. V. M. Tikhomirov, English translation Volosov, 1991.

Blass' Insight (1995)

> *A category used by de Paiva to model* linear logic *also occurs in Vojtas's analysis of* cardinal characteristics *of the continuum. Its morphisms have been used in describing reductions between* search problems *in complexity theory.*

(Questions and answers—a category arising in linear logic, complexity theory, and set theory. Advances in linear logic, 222:61–81, 1995, https://arxiv.org/abs/math/9309208)

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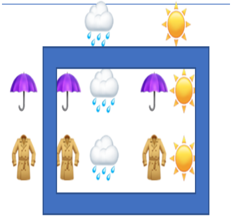
- You might prefer Scott's example:

$$C_1 = \begin{array}{c} \\ u \\ n \end{array} \begin{array}{c} \begin{array}{cc} r & s \end{array} \\ \begin{pmatrix} ur & us \\ nr & ns \end{pmatrix} \end{array}.$$



but it gets too hard real soon

Blass' Insight

- A morphism from $\mathbf{A}$ to $\mathbf{B} = (B^-, B^+, B)$ is a pair of functions $f^- : B^- \to A^-$ and $f^+ : A^+ \to B^+$ such that, for all $b \in B^-$ and all $a \in A^+$ $A(f^-(b), a) \Rightarrow B(b, f^+(a))$

$$
\begin{array}{ccc}
A^+ & (A) & A^- \\
{\scriptstyle (f^+)}\big\downarrow & \Downarrow & \big\uparrow{\scriptstyle (f^-)} \\
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- Think of an object **A** of $\mathcal{PV}$ as representing a **problem**. The elements of A^- are instances of the problem, specific questions of this type; the elements of A^+ are possible answers; and the relation A represents correctness: $A(x, y)$ means that y is a correct answer to the question x

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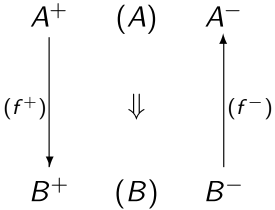

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- For Dialectica, we use $A = (U, X, \alpha)$ as we don't know whether something is a question u or an answer x

A few objects in Dialectica

1. The object $(\mathbb{N}, \mathbb{N}, =)$ where n is related to m iff $n = m$.
2. The object $(\mathbb{N}^{\mathbb{N}}, \mathbb{N}, \alpha)$ where f is α-related to n iff $f(n) = n$.
3. The object $(\mathbb{R}, \mathbb{R}, \leq)$ where r_1 and r_2 are related iff $r_1 \leq r_2$
4. The objects $(2, 2, =)$ and $(2, 2, \neq)$ with usual equality and inequality.

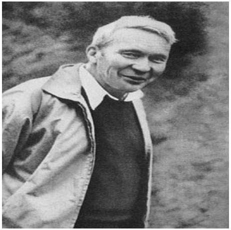
A healthy collection of results in Set Theory using Dialectica ideas. Samuel G. da Silva and Valeria de Paiva. Dialectica categories, cardinalities of the continuum and combinatorics of ideals. Logic Journal of the IGPL, 25(4):585–603, 06 2017.

A Theory of Problems

- What is the unknown?
- What are the data?
- What is the condition?

Veloso mentioned Kolmogorov as inspiration for his *theory of problems*.
Aspectos de uma teoria geral de problemas, by Paulo Veloso. Cadernos de Historia e Filosofia da Ciencia, 7:21–42, 1984.

A Theory of Problems



A **Kolgomorov problem** is a triple $P = (I, S, \sigma)$, where I and S are sets and $\sigma \subseteq I \times S$ is a set theoretical relation. Say that:

- I is the set of instances of the problem P;
- S is the set of possible solutions for the instances I; and
- σ is the problem condition, so $z \sigma s$ if "s σ-solves z".

Kolmogorov problems (1932) provide an alternative intuitive semantics for Propositional Intuitionistic Logic, part of the BHK interpretation.
(Kolmogorov: We never assume that a problem is solvable!)

Kolmogorov Problems

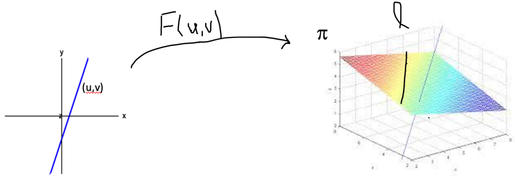
Kolmogorov problems are Dialectica sets, actually objects in $\mathrm{Dial}_2(\mathrm{Sets})^{\mathrm{op}}$. What about **morphisms**?
A morphism from an object $P' = (I', S', \sigma')$ to an object $P = (I, S, \sigma)$ is a pair of functions (f, F), where $f : I \to I'$ and $F : S' \to S$ are such that

$$(\forall z \in I)(\forall t \in S')[f(z)\sigma' t \Rightarrow z\sigma F(t)]$$

Looking at P and P' as Kolgomorov problems, existence of a Dialectica morphism from P' to P ensures a **reduction** of the problem P to the problem P'
This is our new paper: "Kolgomorov-Veloso Problems and Dialectica Categories"(just out Feb 2021!)

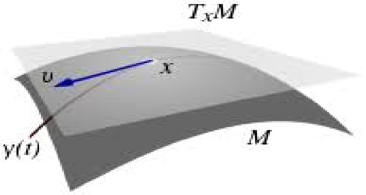
Kolmogorov Example 1: Analytical geometry

-
  - $F : \mathbb{R}^2 \to \pi$ is a coordinate system for a plane π. $P = F(u, v)$ with coordinates (u, v).
  - A line l of π is an equation of the form $ax + by = c$, (a, b real numbers, $a \neq 0$ or $b \neq 0$) and c in $\{0, 1\}$. E is the family of all equations of this form. $\mathcal{L}$ denotes the family of all lines in the plane π.
  - Call $eq(l)$ the canonical equation which represents the line l.
- Problem1: To decide "whether a given point (u, v) lies on a given line l" is the problem $(\mathcal{L}, \pi, \ni)$.
- Problem2: To decide "whether a given pair of real numbers (u, v) satisfies a given equation" is the problem $(E, \mathbb{R}^2, \chi)$, an equation $ax + by = c$ is χ-related to a pair (u, v) of reals if $au + bv = c$.



Kolmogorov Example 2: Tangent planes

- M is a two-dimensional differential manifold $M \subseteq \mathbb{R}^3$, for x a point in M there is a tangent plane $T_x(M)$ centred at x.
- a line l through x is the *normal line* of the plane π (through x) if l is perpendicular to all lines of π which go through x.
- $\mathcal{L}$ = family of all lines of $\mathbb{R}^3$, $\mathcal{P}$ = family of all planes of $\mathbb{R}^3$



- Problem1: find the normal lines through each point of the surface = $(M, \mathcal{L}, \sigma)$, where $x\,\sigma\,l$ means "l is the normal line of M through x" for every point $x \in M$ and every line $l \in \mathcal{L}$,
- Problem2: find orthogonal planes $(\mathcal{P}, \mathcal{P}, \xi)$, where $\pi\,\xi\,\rho$ means "π and ρ are orthogonal planes"

Kolmogorov Example 2: Tangent planes

Reducing problem 1 to problem 2:

- A map from $(M, \mathcal{L}, \sigma)$ to $(\mathcal{P}, \mathcal{P}, \xi)$ is given by the pair (f, F), where $f : M \to \mathcal{P}$ is given by $f(x) = T_x(M)$ for all $x \in M$.
- $F\colon M \times \mathcal{P} \to \mathcal{L}$ is $F(x, \rho) = \varphi(T_x(M), \rho, x)$, where $\varphi : \mathcal{P} \times \mathcal{P} \times \mathbb{R}^3 \to \mathcal{L}$ is given by $\varphi(\pi, \rho, x) =$ the line l contained in the plane $t(x, \rho)$ (where $t(x, \rho)$ is ρ itself if $x \in \rho$ or is the unique plane parallel to ρ passing through x) which is perpendicular to the intersecting line of π and $t(x, \rho)$ through x, if π and ρ are orthogonal planes;
  $F(x, \rho)$ is the normal line of $T_x(M)$ through x whenever ρ is a plane orthogonal to $f(x) = T_x(M)$.
- The map (f, F) needs to refer to x, so a reduction in this case is a morphism of the first kind of Dialectica category. This seems connected to automated differentiation and lenses.

Conclusions

- The categorical connection between Dialectica models, Kolmogorov's problems, Veloso's problems and Blass' problems shows that the use of categories really allows us to connect extremely different areas of mathematics, using simple methods.
- Main theorems of paper are about the Axiom of Choice (AC): how different versions of choice will lead to different classes of problems. While Kolmogorov and Blass do not require AC, Veloso's notion will enforce it.
- For me the surprise of starting with models of Linear Logic, end up with an abstract notion of problem that applies to Number Theory, Analysis, Geometry, Functional Programming, Multi-agents modelling in AI, and many others.

It often happens that there are similarities between the solutions to problems. Sometimes, these similarities point to more general phenomena that simultaneously explain several different pieces of mathematics. These more general phenomena can be very difficult to discover, but when they are discovered, they have a very important simplifying and organizing role, and can lead to the solutions of further problems, or raise new and fascinating questions.
T. Gowers, The Importance of Mathematics, 2000

THANKS!