Models of Enhanced 2-sketches & Algebras over Enhanced 2-monads

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Enhanced 2-sketches

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Example (Monoidal double categories)

$$C_1 \times_{C_0} C_1 \underset{\pi_2}{\overset{\pi_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xrightarrow{\;c\;}}}}} C_1 \underset{t}{\overset{s}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\;i\;}}}}} C_0$$

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c and i have to be *pseudo* monoidal, while others have to be *strict* monoidal

$\implies$ cannot be seen as a pseudocategory internal to monoidal categories

But, we could consider a 2-category of monoidal categories with *two* kinds of morphisms: pseudo $+$ strict monoidal functors

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$$\begin{array}{ccc} 1 & = & 1 \\ {\scriptstyle i_C}\downarrow & & \downarrow{\scriptstyle i_D} \\ C & \xrightarrow[F]{} & D \end{array} \qquad \begin{array}{ccc} C^2 & \xrightarrow{F \times F} & D^2 \\ {\scriptstyle m_C}\downarrow & & \downarrow{\scriptstyle m_D} \\ C & \xrightarrow[F]{} & D \end{array}$$

$$\begin{array}{ccc} C^2 & \xrightarrow{F \times F} & D^2 \\ {\scriptstyle (\pi_1)_C}\downarrow & & \downarrow{\scriptstyle (\pi_1)_D} \\ C & \xrightarrow[F]{} & D \end{array} \qquad \begin{array}{ccc} C^2 & \xrightarrow{F \times F} & D^2 \\ {\scriptstyle (\pi_2)_C}\downarrow & & \downarrow{\scriptstyle (\pi_2)_D} \\ C & \xrightarrow[F]{} & D \end{array}$$

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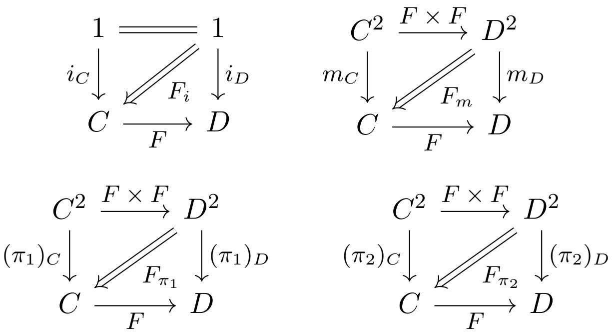

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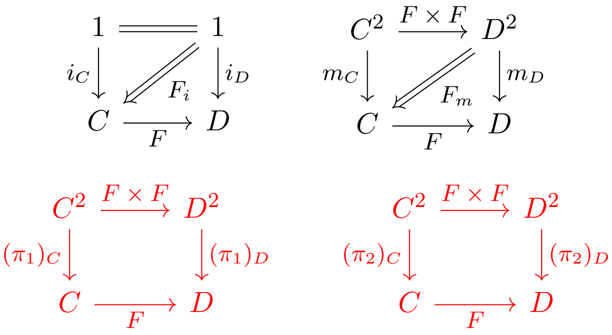

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$\rightsquigarrow$ enhanced limit 2-sketchs as *enriched* limit sketches by Kelly

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Definition (Lack, Shulman)

Let $\mathscr{F}$ be the full subcategory of the arrow category $\mathcal{C}at^{\mathbf{2}}$ of the category $\mathcal{C}at$, determined by the fully faithful and injective-on-objects functors, i.e., the *full embeddings*.

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Notation

$\mathcal{A}_\tau :=$ tight part;

$\mathcal{A}_\lambda :=$ loose part.

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$\rightsquigarrow$ A small enhanced 2-category $\mathbb{S}$, with a collection of $\mathscr{F}$-natural transformations

$$\{(W\colon \mathbb{J} \to \mathbb{F},\ D\colon \mathbb{J} \to \mathbb{S},\ s \in \mathbb{S},\ \gamma\colon W \to \mathbb{S}(s, D-))\}$$

which are called *weighted cones*.

Enhanced 2-sketches

Example (Pseudocategories)

The sketch $\mathbb{C}$ is generated by

- objects: C_0, C_1, C_2, C_3, C_4

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- objects: C_0, C_1, C_2, C_3, C_4
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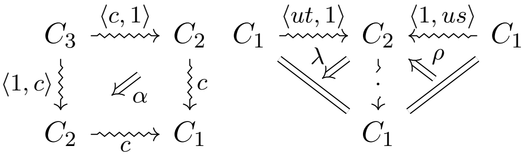

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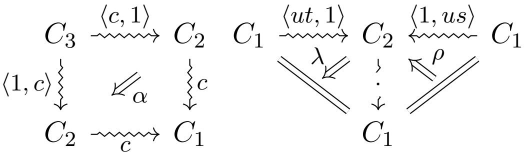

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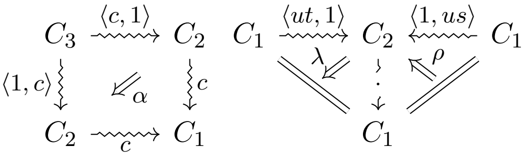

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Weighted cones: pullback cones C_i for $i = 2, 3, 4$

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The sketch $\mathbb{M}$ is generated by

- objects: $1, M, M^2, M^3, M^4$

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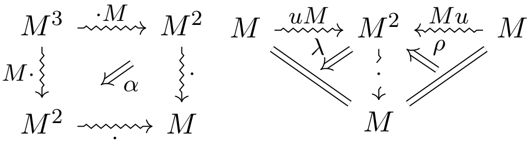

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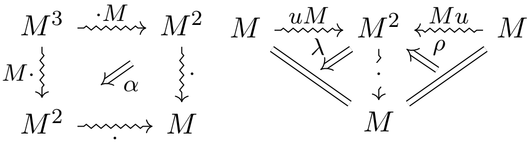

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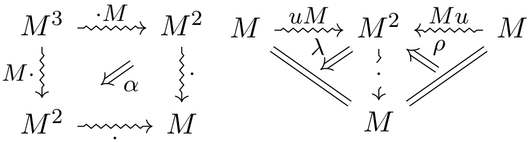

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Weighted cones: product cones M^i for $1 < i \leq 4$ and terminal object 1

Models of enhanced 2-sketches

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Definition

Let $\mathbb{S}$ and $\mathbb{T}$ be enhanced limit 2-sketches. A *model* of $\mathbb{S}$ in $\mathbb{T}$ is a morphism of sketches, i.e., an $\mathscr{F}$-functor $\mathbb{S} \to \mathbb{T}$ that sends weighted cones to weighted cones.

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$[\mathbb{S}, \mathbb{T}]_{w',w} :=$ $\mathscr{F}$-functors, tight & loose (w', w)-natural transformations, modifications.

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$[\mathbb{S}, \mathbb{T}]_{w',w} :=$ $\mathscr{F}$-functors, tight & loose (w', w)-natural transformations, modifications.

$\mathbb{M}\mathrm{od}_{w',w}(\mathbb{S}, \mathbb{T}) :=$ full sub-$\mathscr{F}$-category of $[\mathbb{S}, \mathbb{T}]_{w',w}$ determined by the models.

Example (Lax double functors)

Let A and B be (pseudo) double categories, described as pseudocategories (in the previous slide) in $\mathcal{C}at$, i.e., $A, B \colon \mathbb{C} \rightrightarrows \mathcal{C}at$.

Models of enhanced 2-sketches

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Let A and B be (pseudo) double categories, described as pseudocategories (in the previous slide) in $\mathcal{C}at$, i.e., $A, B \colon \mathbb{C} \rightrightarrows \mathcal{C}at$. A lax double functor is a loose (s, l)-natural transformation $A \to B$.

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$\mathbb{M}\mathrm{od}_{s,l}(\mathbb{C}, \mathcal{C}at)$ is the enhanced 2-category of (pseudo) double categories, strict double functors, lax double functors, and double transformations.

Models of enhanced 2-sketches

Example (Lax monoidal functors)

Let A and B be monoidal categories, described as pseudomonoids in $\mathcal{C}at$, i.e., $A, B \colon \mathbb{M} \rightrightarrows \mathcal{C}at$.

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These all *models of enhanced 2-sketches*, i.e., $\mathbb{M}\mathrm{od}_{s,w}(\mathbb{T}, \mathbb{K})$.
$\rightsquigarrow$ Understand features of models of enhanced 2-sketches, e.g. limits in $\mathbb{M}\mathrm{od}_{s,w}(\mathbb{T}, \mathbb{K})$

Monadicity of models

Idea

Monadicity of models

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Construct the left adjoint for

$$\mathbb{M}\mathrm{od}(\mathbb{T}, \mathbb{K}) \xrightarrow{i^*} \mathbb{M}\mathrm{od}(\mathcal{T}_\tau, \mathbb{K})$$

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(1) construct the reflection of $\mathbb{Mod}(\mathbb{T}, \mathbb{K}) \hookrightarrow [\mathbb{T}, \mathbb{K}]$;

(2) composing with the left Kan extension $\mathbb{Mod}(\mathcal{T}_\tau, \mathbb{K}) \xrightarrow{\mathrm{Lan}_i} [\mathbb{T}, \mathbb{K}]$.

Monadicity of models: Constructing the reflection

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Remark

Lack & Rosický: $\mathcal{R}$-injectivity

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When $\mathcal{V} = \mathbf{Set}$ and $\mathcal{R} = \{\text{bijections}\}$, we recover the ordinary notion of orthogonal objects.

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Example

When $\mathcal{V} = \mathbf{Set}_{\Delta}$ and $\mathcal{R} = \{\text{weak equivalences}\}$, the objects which are orthogonal to the horn inclusions with respect to $\mathcal{R}$ are precisely the Kan complices.

Monadicity of models: Constructing the reflection

Definition (K.)

Let $\mathcal{C}, \mathcal{D}$ be $\mathcal{V}$-categories, and $U\colon \mathcal{D} \to \mathcal{C}$ be a $\mathcal{V}$-functor.

Monadicity of models: Constructing the reflection

Definition (K.)

Let $\mathcal{C}, \mathcal{D}$ be $\mathcal{V}$-categories, and $U\colon \mathcal{D} \to \mathcal{C}$ be a $\mathcal{V}$-functor. A functor $F\colon \mathcal{C}_0 \to \mathcal{D}_0$ is an *unenriched left adjoint of U with respect to $\mathcal{R}$* if and only if there is a collection

$$\{c \xrightarrow{\eta_c} UFc\}_{c \in \operatorname{ob} \mathcal{C}}$$

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in $\mathcal{V}_0$ satisfying unenriched naturality, such that for any $c \in \operatorname{ob} \mathcal{C}$ and any $d \in \operatorname{ob} \mathcal{D}$,

$$\mathcal{D}(Fc, d) \xrightarrow{U_{Fc,d}} \mathcal{C}(UFc, Ud) \xrightarrow{\mathcal{C}(\eta_c, Ud)} \mathcal{C}(c, Ud)$$

belongs to $\mathcal{R}$.

Monadicity of models: Constructing the reflection

Example (Projective covers)

Let $\mathcal{V} = \mathbf{Set}$, $\mathcal{R} = \{\text{surjections}\}$, C be a category with enough projectives, and $D = C^{\twoheadrightarrow}$.

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be a functor that sends each $c \in \operatorname{ob} C$ to a projective cover of c. Then, F is a left adjoint of U with respect to $\mathcal{R}$.

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Proposition (General Enriched Orthogonal Sub-category Theorem)

The inclusion

$$\mathcal{I}^{\perp^{\mathcal{R}}} \overset{I}{\hookrightarrow} \mathcal{K}$$

admits an unenriched left adjoint with respect to $\mathcal{R}$.

Monadicity of models: Constructing the reflection

Proof.

Apply the enriched small object argument by Jurka:

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in $\mathcal{K}$, such that for $K \in \mathrm{ob}\,\mathcal{K}$ of and $X \in \mathcal{I}^{\perp^{\mathcal{R}}}$,

$$\mathcal{I}^{\perp^{\mathcal{R}}}(LK, X) \xrightarrow{I_{LK,X}} \mathcal{K}(ILK, IX) \xrightarrow{\mathcal{K}(\eta_K, IX)} \mathcal{K}(K, IX)$$

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$$\begin{array}{ccc} K & \xrightarrow{\;\;!_K\;\;} & 1 \\ & \searrow^{\eta_K} \quad \nearrow_{!_{\mathrm{colim}_{\alpha<\lambda} D^K(\alpha)}} & \\ & \mathrm{colim}_{\alpha<\lambda} D^K(\alpha) & \end{array}$$

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Define

$$\begin{aligned} L\colon \mathcal{K}_0 &\to \mathcal{I}_0^{\perp^{\mathcal{R}}} \\ K_1 &\mapsto \operatorname*{colim}_{\alpha<\lambda} D^{K_1}(\alpha) \\ k \downarrow \ &\mapsto \qquad\quad \downarrow Lk \\ K_2 &\mapsto \operatorname*{colim}_{\alpha<\lambda} D^{K_2}(\alpha), \end{aligned}$$

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As I is fully faithful, we verify that the pre-composition

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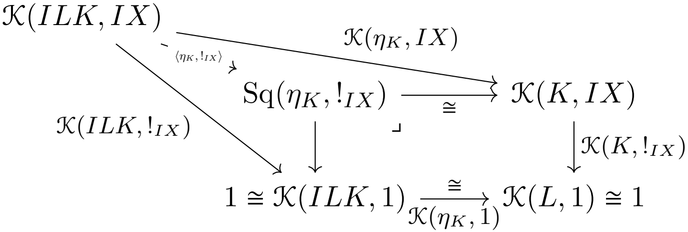

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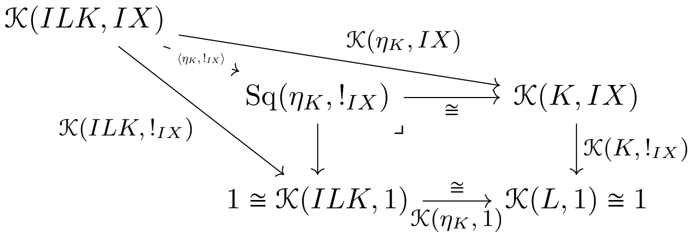

η_K's lifting property $\implies \langle \eta_K, !_{IX} \rangle \in \mathcal{R} \implies \mathcal{K}(\eta_K, IX) \in \mathcal{R}$. □

Monadicity of models: Constructing the reflection

Corollary (Special Enriched Orthogonal Sub-category Theorem)

If $\mathcal{R}$ contains precisely the isomorphisms in $\mathcal{V}$, and $\mathcal{K}$ has powers by $\mathcal{V}$, then $\mathcal{I}^{\perp}$ is a reflective sub-$\mathcal{V}$-category of $\mathcal{K}$.

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Lemma (K.)

Let $\mathcal{S}$ be a $\mathcal{V}$-sketch with weighted cones $\{(W_i, D_i, s_i, \gamma_i)\}_{i \in I}$, and $\mathcal{K}$ be a complete $\mathcal{V}$-category.

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$\rightsquigarrow$ models are orthogonal objects

Monadicity of models: Constructing the reflection

Theorem (K.)

Let $\mathcal{S}$ be a limit $\mathcal{V}$-sketch, and $\mathcal{K}$ be a locally presentable $\mathcal{V}$-category. The (fully faithful) inclusion

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of $\mathcal{V}$-models of $\mathcal{S}$ in $\mathcal{K}$ into $\mathcal{V}$-functors is reflective.

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Furthermore, if i is essentially surjective, then this adjunction is monadic.

Monadicity of models

Proof.

Let $F \in \mathcal{M}\mathrm{od}(\mathcal{T}, \mathcal{K})$ and $G \in \mathcal{M}\mathrm{od}(\mathcal{S}, \mathcal{K})$.

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$$\mathcal{M}\mathsf{od}(\mathcal{T}, \mathcal{K})(L \cdot \mathrm{Lan}_i G, F) \cong [\mathcal{T}, \mathcal{K}](\mathrm{Lan}_i G, F)$$

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Monadicity: If i is essentially surjective, we can apply the Enriched Monadicity Theorem by Dubuc. □

Equivalence to algebras over an enhanced 2-monad

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Definition (Lack, Shulman)

An *enhanced 2-monad* is an $\mathscr{F}$-monad $T\colon \mathbb{A} \to \mathbb{A}$ on an enhanced 2-category $\mathbb{A}$,

Equivalence to algebras over an enhanced 2-monad

Definition (Lack, Shulman)

An *enhanced 2-monad* is an $\mathscr{F}$-monad $T\colon \mathbb{A} \to \mathbb{A}$ on an enhanced 2-category $\mathbb{A}$, i.e., a monad in the 2-category $\mathscr{F}\text{-}\mathcal{C}at$ of enhanced 2-categories.

Equivalence to algebras over an enhanced 2-monad

Definition (Lack, Shulman)

An *enhanced 2-monad* is an $\mathscr{F}$-monad $T\colon \mathbb{A} \to \mathbb{A}$ on an enhanced 2-category $\mathbb{A}$, i.e., a monad in the 2-category $\mathscr{F}\text{-}\mathcal{C}at$ of enhanced 2-categories.

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This means that T consists of an $\mathscr{F}$-functor $T\colon \mathbb{A} \to \mathbb{A}$, together with a multiplication μ and a unit η, which are both $\mathscr{F}$-natural transformations, satisfying associativity and unity:

$$
\begin{array}{ccc}
TTT & \xrightarrow{T\mu} & TT \\
{\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\
TT & \xrightarrow[\mu]{} & T
\end{array}
\qquad
\begin{array}{ccccc}
T & \xrightarrow{\eta T} & TT & \xleftarrow{T\eta} & T \\
 & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu} & \diagup\!\!\diagup & \\
 & & T & &
\end{array}
$$

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Let (T, μ, η) be an enhanced 2-monad.

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Let (T, μ, η) be an enhanced 2-monad.A *strict T-algebra* consists of a pair $(A, TA \xrightarrow{a} A)$, where $A \in \mathrm{ob}\,\mathbb{A}$, satisfying the multiplication and unity conditions:

$$
\begin{array}{ccc}
TTA & \xrightarrow{Ta} & TA \\
{\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle a} \\
TA & \xrightarrow[a]{} & A
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & TA \\
& \searrow^{=} & \downarrow{\scriptstyle a} \\
& & A
\end{array}
$$

Equivalence to algebras over an enhanced 2-monad

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Definition (Lack, Shulman)

A *w-T-morphism* f from $(A, TA \xrightarrow{a} A)$ to $(B, TB \xrightarrow{b} B)$ is a w-morphism

$$\begin{array}{ccc} TA & \overset{Tf}{\rightsquigarrow} & TB \\ {\scriptstyle a}\downarrow & w & \downarrow{\scriptstyle b} \\ A & \underset{f}{\rightsquigarrow} & B \end{array}$$

Equivalence to algebras over an enhanced 2-monad

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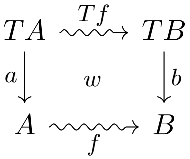

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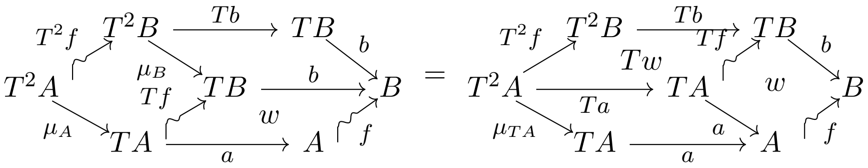

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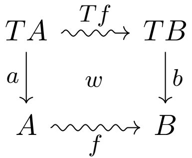

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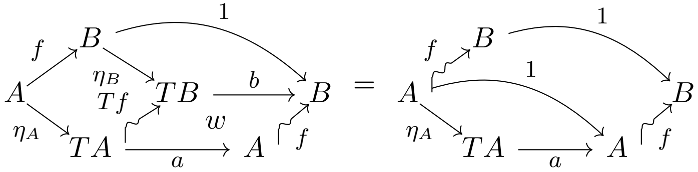

Equivalence to algebras over an enhanced 2-monad

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Definition

A *strict T-morphism* f from $(A, TA \xrightarrow{a} A)$ to $(B, TB \xrightarrow{b} B)$ is a s-T-morphism where f is tight:

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

Equivalence to algebras over an enhanced 2-monad

We need tools from 2-enhanced 2-categories:

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An object of $\mathscr{F}_2$ is a pair of composable full embeddings

$$A_\tau \xhookrightarrow{j_{A_\tau}} A_\theta \xhookrightarrow{j_{A_\theta}} A_\lambda.$$

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$$A_\tau \xhookrightarrow{j_{A_\tau}} A_\theta \xhookrightarrow{j_{A_\theta}} A_\lambda.$$

A morphism f from j_A to j_B is given by $f_\tau \colon A_\tau \to B_\tau$, $f_\theta \colon A_\theta \to B_\theta$, and $f_\lambda \colon A_\lambda \to B_\lambda$ such that

$$\begin{array}{ccccc} A_\tau & \xhookrightarrow{j_{A_\tau}} & A_\theta & \xhookrightarrow{j_{A_\theta}} & A_\lambda \\ {\scriptstyle f_\tau}\downarrow & & \downarrow{\scriptstyle f_\theta} & & \downarrow{\scriptstyle f_\lambda} \\ B_\tau & \xhookrightarrow[j_{B_\tau}]{} & B_\theta & \xhookrightarrow[j_{B_\theta}]{} & B_\lambda \end{array}.$$

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Definition

A *2-enhanced 2-category* $\mathbb{A}$ is a category enriched in $\mathscr{F}_2$.

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- objects $x, y, \cdots$
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$$\mathbb{A}(x,y)_\tau \hookrightarrow \mathbb{A}(x,y)_\theta \hookrightarrow \mathbb{A}(x,y)_\lambda$$

  of the tight part into the fit part into the loose part

Equivalence to algebras over an enhanced 2-monad

We will be using a theorem by Bourke:

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Suppose G has a left $\mathscr{F}$-adjoint. If both $\mathbb{A}_{\tau,\theta}$ and $\mathbb{A}_{\tau,\lambda}$ admit $\overline{w}$-limits of loose morphisms,

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$$\begin{array}{ccc} \mathbb{A}_{\tau,\lambda} & \overset{E_w}{\dashrightarrow} & \mathrm{T}\text{-}\mathbb{Alg}_{s,w} \\ & {}_{H}\searrow \quad \swarrow & \\ & \mathbb{B} & \end{array}$$

If G is $\mathscr{F}$-monadic, $\mathbb{B}$ admits $\overline{w}$-limits of loose morphisms, and that H is w-doctrinal,

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Suppose G has a left $\mathscr{F}$-adjoint. If both $\mathbb{A}_{\tau,\theta}$ and $\mathbb{A}_{\tau,\lambda}$ admit $\overline{w}$-limits of loose morphisms, then

$$\begin{array}{ccc} \mathbb{A}_{\tau,\lambda} & \xrightarrow{E_w} & \mathrm{T}\text{-}\mathbb{A}\mathrm{lg}_{s,w} \\ & H \searrow \quad \mathbb{B} \quad \swarrow & \end{array}$$

If G is $\mathscr{F}$-monadic, $\mathbb{B}$ admits $\overline{w}$-limits of loose morphisms, and that H is w-doctrinal, then E_w is an equivalence.

Equivalence to algebras over an enhanced 2-monad

We are able to establish our main result:

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Theorem (K.)

Let $\mathbb{K}$ be a locally presentable enhanced 2-category.

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Equivalence to algebras over an enhanced 2-monad

We are able to establish our main result:

Theorem (K.)

Let $\mathbb{K}$ be a locally presentable enhanced 2-category. Let $\mathbb{T}$ be an enhanced limit 2-sketch with tight weighted cones. There is an enhanced 2-monad T on the enhanced 2-category $\mathbb{M}\mathrm{od}(\mathcal{T}_\tau, \mathbb{K})$ of models restricted to the tights such that

$$\mathbb{M}\mathrm{od}_{s,w}(\mathbb{T}, \mathbb{K}) \simeq \mathrm{T}\text{-}\mathbb{A}\mathrm{lg}_{s,w}$$

is an equivalence of enhanced 2-categories.

Equivalence to algebras over an enhanced 2-monad

Proof.

$\mathbb{M}\mathrm{od}(\mathbb{T}, \mathbb{K})$ is an $\mathscr{F}_2$-category:

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$\mathbb{M}\mathrm{od}(\mathbb{T}, \mathbb{K})$ is an $\mathscr{F}_2$-category:

- tight: $\mathscr{F}$-natural transformations

Equivalence to algebras over an enhanced 2-monad

Proof.

$\mathbb{M}\mathrm{od}(\mathbb{T}, \mathbb{K})$ is an $\mathscr{F}_2$-category:

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- fit: 2-natural transformations (between the loose parts)

Equivalence to algebras over an enhanced 2-monad

Proof.

$\mathbb{Mod}(\mathbb{T}, \mathbb{K})$ is an $\mathscr{F}_2$-category:

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- fit: 2-natural transformations (between the loose parts)
- loose: loose w-natural transformations

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Note that

$$\begin{array}{ccc} \mathbb{Mod}(\mathbb{T}, \mathbb{K}) & \hookrightarrow & \mathbb{Mod}_{s,w}(\mathbb{T}, \mathbb{K}) \\ & {\scriptstyle i^*} \searrow \quad \swarrow {\scriptstyle i^*_{s,w}} & \\ & \mathbb{Mod}(\mathcal{T}_\tau, \mathbb{K}) & \end{array} .$$

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To apply Bourke's theorem, we need to verify

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- i^* is monadic:

Equivalence to algebras over an enhanced 2-monad

Proof.

$$\mathbb{Mod}(\mathcal{T}_\tau, \mathbb{K}) \underset{i^*}{\overset{L \cdot \mathrm{Lan}_i}{\underset{\perp}{\rightleftarrows}}} \mathbb{Mod}(\mathbb{T}, \mathbb{K}) \ .$$

Equivalence to algebras over an enhanced 2-monad

Proof.

$$\mathbb{Mod}(\mathcal{T}_\tau, \mathbb{K}) \underset{i^*}{\overset{L \cdot \mathrm{Lan}_i}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbb{Mod}(\mathbb{T}, \mathbb{K}) \ .$$

- $\mathbb{Mod}(\mathbb{T}, \mathbb{K})$, $\mathbb{Mod}_{s,w}(\mathbb{T}, \mathbb{K})$, $\mathbb{Mod}(\mathcal{T}_\tau, \mathbb{K})$ all admit $\overline{w}$-limits of loose morphisms:

Equivalence to algebras over an enhanced 2-monad

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  reflectivity & direct verification

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- $i^*_{s,w}$ is w-doctrinal $\iff$ locally faithful + reflects identity 2-morphisms + satisfies weak w-doctrinal adjunction:

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  For the first two, note that i is identity-on-objects.

Equivalence to algebras over an enhanced 2-monad

Proof.

Let $F, G \in \mathbb{M}\mathrm{od}(\mathbb{T}, \mathbb{K})$,

Equivalence to algebras over an enhanced 2-monad

Proof.

Let $F, G \in \mathbb{Mod}(\mathbb{T}, \mathbb{K})$,and $\alpha \colon F \to G$ be an $\mathscr{F}$-natural transformation, and $\beta \colon G \cdot i \to F \cdot i$ be a 2-natural transformation (between the loose parts).

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Suppose that $(\varepsilon, \alpha \cdot i \dashv \beta, \eta)$ is an adjunction in $\mathbb{Mod}(\mathcal{T}_\tau, \mathbb{K})$.

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We construct a loose lax natural transformation $\overline{\beta}\colon G \to F$:

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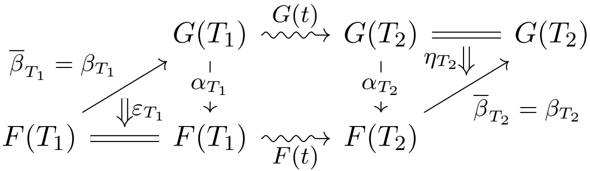
We construct a loose lax natural transformation $\overline{\beta}\colon G \to F$:

- 1-component $\overline{\beta}_T := \beta_T$
- 2-component $\overline{\beta}_t$ at $t\colon T_1 \rightsquigarrow T_2 :=$



Equivalence to algebras over an enhanced 2-monad

Proof.

η is a modification, we have, for a tight morphism $t\colon T_1 \to T_2$,

$$
\begin{array}{ccccc}
G(T_1) & \xrightarrow{G(t)} & G(T_2) & & \\
\| & & \| & \searrow^{\alpha_{T_2}} & \\
& & \overset{\Rightarrow}{\eta_{T_2}} & & F(T_2) \\
\| & & \| & \swarrow_{\beta_{T_2}} & \\
G(T_1) & \xrightarrow[G(t)]{} & G(T_2) & &
\end{array}
\quad = \quad
\begin{array}{ccccccc}
G(T_1) & & \xrightarrow{G(t)} & & G(T_2) & & \\
\| & \searrow^{\alpha_{T_1}} & & \alpha_t & & \searrow^{\alpha_{T_2}} & \\
\| & \overset{\Rightarrow}{\eta_{T_1}} & F(T_1) & \xrightarrow[F(t)]{} & & & F(T_2) \\
\| & \swarrow_{\beta_{T_1}} & & \beta_t & & \swarrow_{\beta_{T_2}} & \\
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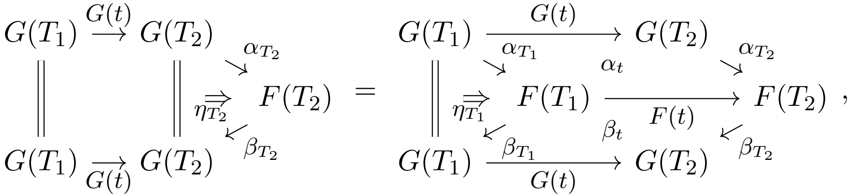
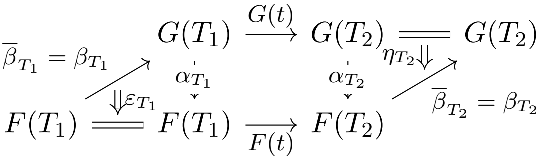


$\Longrightarrow$



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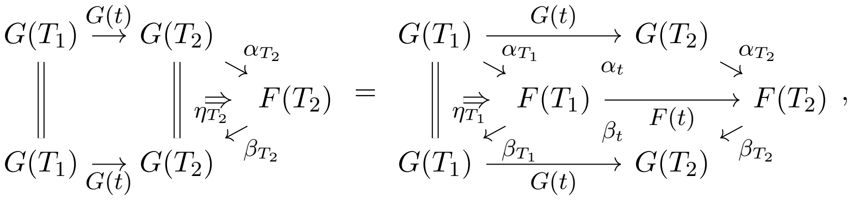

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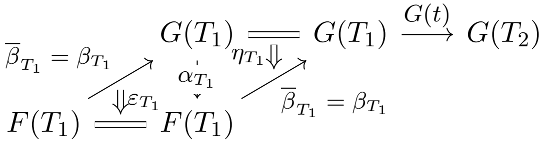

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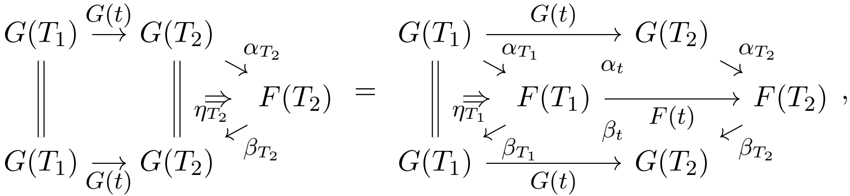
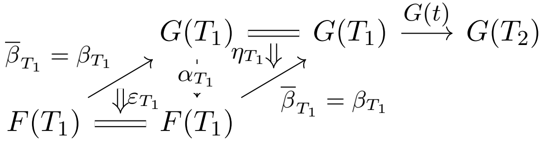


$\Longrightarrow$



which is the identity.

Equivalence to algebras over an enhanced 2-monad

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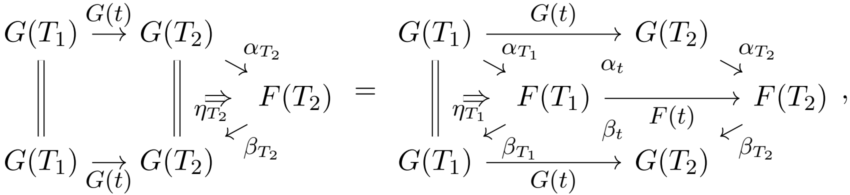

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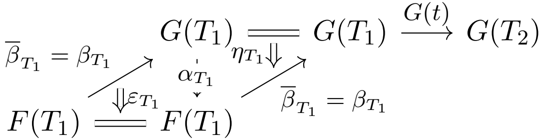

which is the identity. So $\overline{\beta}$ is a loose lax natural transformation.

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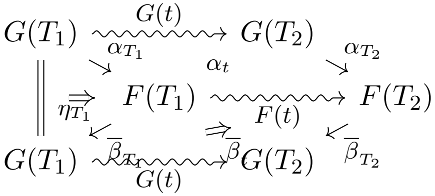

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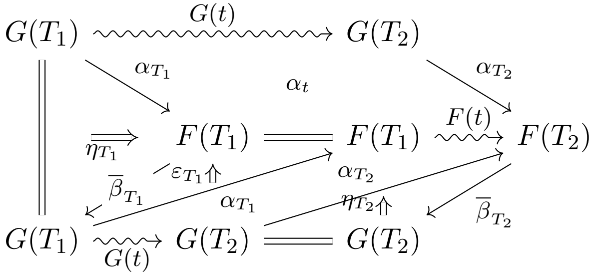

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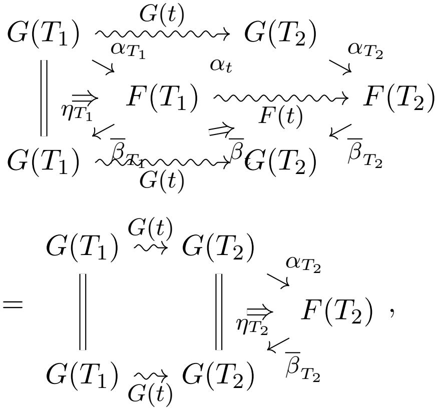

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By Bourke's theorem,

$$\begin{array}{ccc} \mathbb{M}\mathrm{od}_{s,w}(\mathbb{T}, \mathbb{K}) & \xrightarrow{\simeq} & \mathrm{T}\text{-}\mathbb{A}\mathrm{lg}_{s,w} \\ & {\scriptstyle i^*_{s,w}} \searrow \quad \swarrow {\scriptstyle U_{s,w}} & \\ & \mathbb{M}\mathrm{od}(\mathcal{T}_\tau, \mathbb{K}) & \end{array}.$$

□

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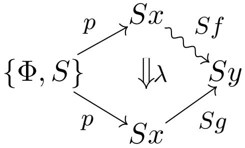

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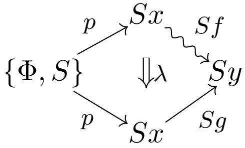

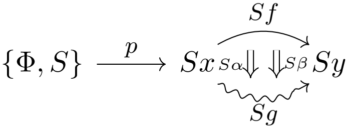

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