Coend calculus in a compact closed virtual equipment

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Contents

1. Background on virtual double categories

Virtual double categories (VDCs) are a generalization of weak double categories.

Definition.

A VDC $\mathbb{X}$ consists of

(1) an ordinary category $\mathcal{X}$ of objects $A, B, C, \ldots$ and tight arrows $f : A \to B$,

(2) loose arrows $p : A \nrightarrow B$

(3) 2-cells

$$\begin{tikzcd}
A_0 \arrow[r, "p_1", "\shortmid"{marking}] \arrow[d, "f"'] & A_1 \arrow[r, phantom, "\cdots"] & A_{n-1} \arrow[r, "p_n", "\shortmid"{marking}] & A_n \arrow[d, "g"] \\
B_0 \arrow[rrr, "q"', "\shortmid"{marking}] \arrow[rrru, phantom, "\alpha"] & & & B_1
\end{tikzcd}$$

(4) a composition law for 2-cells

$$\begin{tikzcd}[column sep=small]
\bullet \arrow[r, "\shortmid"{marking}] \arrow[d] & \bullet \arrow[r, phantom, "\cdots"] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] \arrow[d] & \bullet \arrow[r, phantom, "\cdots"] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[d] \arrow[r, phantom, "\cdots"] & \bullet \arrow[r, "\shortmid"{marking}] \arrow[d] & \bullet \arrow[r, phantom, "\cdots"] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[d] \\
\bullet \arrow[rrr, "\shortmid"{marking}] \arrow[d] \arrow[rrru, phantom, "\alpha_1"] & & & \bullet \arrow[rrr, "\shortmid"{marking}] \arrow[rrru, phantom, "\alpha_2"] & & & \bullet \arrow[r, phantom, "\cdots"] & \bullet \arrow[rrr, "\shortmid"{marking}] \arrow[rrru, phantom, "\alpha_n"] & & & \bullet \arrow[d] \\
\bullet \arrow[rrrrrrrrrr, "\shortmid"{marking}] \arrow[rrrrrrrrrru, phantom, "\beta"] & & & & & & & & & & \bullet
\end{tikzcd}$$

(5) identity 2-cells

$$\begin{tikzcd}
A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[d, equal] \\
A \arrow[r, "p"', "\shortmid"{marking}] \arrow[ru, phantom, "="] & B
\end{tikzcd}$$

for each loose arrow p

s.t. the composition of 2-cells is associative and unital wrt the identity 2-cells.

Example. weak double categories

VDCs provide a convenient framework to do formal category theory.

But need some additional structure $\rightsquigarrow$ virtual equipment.

Definition.

A virtual equipment is a VDC $\mathbb{X}$ together with the data of :

- for each object A, a loose unit
$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, equal] & \\ A \arrow[rr, "e_A"', "\shortmid"{marking}] & & A \end{tikzcd}$$

  which is opcartesian,

  i.e., every 2-cell α of the outer shape factors uniquely through a 2-cell α' as below:

$$\alpha = \begin{tikzcd}
\bullet \arrow[r, "\shortmid"{marking}] \arrow[d, equal] & \bullet & \cdots \arrow[d, phantom, "="] & \bullet \arrow[rr, "\shortmid"{marking}] & & A \arrow[dl, equal] \arrow[dr, equal] \arrow[rr, "\shortmid"{marking}] & & \bullet & \cdots \arrow[d, phantom, "="] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "\shortmid"{marking}] \arrow[d] & \bullet & \cdots & \bullet \arrow[r, "\shortmid"{marking}] & A \arrow[rr, "e_A"', "\shortmid"{marking}] & \arrow[d, phantom, "\alpha'"] & A \arrow[r, "\shortmid"{marking}] & \bullet & \cdots & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[d] \\
\bullet \arrow[rrrrrrrrrr, "\shortmid"{marking}] & & & & & & & & & & \bullet
\end{tikzcd}$$

- for every niche diagram
$$\begin{tikzcd} A \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "q"', "|" marking] & D \end{tikzcd},$$
a loose restriction
$$\begin{tikzcd} A \arrow[r, "q(g,f)", "|" marking] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "|" marking] & D \end{tikzcd}$$
which is cartesian,

i.e., every 2-cell α of the outer shape factors uniquely through a 2-cell α' as below:

$$\alpha = \begin{tikzcd} \bullet \arrow[r, "|" marking] \arrow[d] & \bullet \cdots \bullet \arrow[r, "|" marking] \arrow[d, phantom, "\alpha'"] & \bullet \arrow[d] \\ A \arrow[rr, "q(g,f)"', "|" marking] \arrow[d, "f"'] & & B \arrow[d, "g"] \\ C \arrow[rr, "q"', "|" marking] & & D \end{tikzcd}$$

For convenience, we will later suppose that our virtual equipment is strict,

i.e., it is compatible with compositions and identities.

$$
\begin{tikzcd}
A \arrow[r, "q(k,h)(g,f)", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[d, "g"] \\
C \arrow[r, "q(k,h)"', "\shortmid"{marking}] \arrow[d, "h"'] & D \arrow[d, "k"] \\
E \arrow[r, "q"', "\shortmid"{marking}] & F
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "q(kg,hf)", "\shortmid"{marking}] \arrow[d, "gf"'] & B \arrow[d, "kh"] \\
E \arrow[r, "q"', "\shortmid"{marking}] & F
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "p(1_B,1_A)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[d, equal] \\
A \arrow[r, "p"', "\shortmid"{marking}] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
A \arrow[r, "p"', "\shortmid"{marking}] & B
\end{tikzcd}
$$

2. Compact closure

Motivation.

In ordinary category theory, compact closed categories are usually defined as monoidal categories in which every object A is equipped with a dual object A^* together with an evaluation morphism $I \to A \otimes A^*$ and a coevaluation morphism $A^* \otimes A \to I$ satisfying a certain adjunction property.

Similarly, in two-dimensional category theory, compact closed bicategories are usually defined as monoidal bicategories with dualizable objects, now subject to a pseudo adjunction property.

Example: The monoidal bicategory $\mathcal{P}rof$ is compact closed: As dual of a small category $\mathcal{C}$, one may take its opposite category $\mathcal{C}^{op}$.

Idea: Generalize this for VDCs.

Problem: This universal property does not involve the tight arrows (the functors), so $\mathcal{C}^{op}$ is determined only up to equivalence of profunctors (Morita equivalence).

For VDCs, we want some double categorical universal property of $\mathcal{C}^{op}$.
However, it is not clear whether there exists a natural such property.

$\rightarrow$ Need another approach to define the notion of a compact closed VDC.

Idea: Use the description of compact closed categories as $*$-autonomous categories
whose involution preserves the monoidal structure.

Benefit: This is naturally expressed as structure rather than as a property.

Definition.

A monoidal structure on a VDC $\mathbb{X}$ consists of

(1) a functor of VDCs $\otimes : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ called the monoidal product,

(2) a monoidal unit

$$\begin{tikzcd} & I \arrow[dr, equal] & \\ I \arrow[ur, equal] \arrow[rr, "e_I"', "\shortmid" marking] & & I, \end{tikzcd}$$

(3) invertible transformations

$$a : (- \otimes -) \otimes - \overset{\sim}{\Longrightarrow} - \otimes (- \otimes -) \qquad \text{associator}$$

$$\ell : I \otimes - \overset{\sim}{\Longrightarrow} \mathrm{id}_{\mathbb{X}} \qquad \text{left unitor}$$

$$r : - \otimes I \overset{\sim}{\Longrightarrow} \mathrm{id}_{\mathbb{X}} \qquad \text{right unitor}$$

satisfying the usual pentagon and triangle axioms.

Note: A monoidal VDC is a pseudomonoid in the strict 2-category of VDCs.

Definition.

A compact right closed VDC consists of

(1) a VDC $\mathbb{X}$,

(2) a monoidal structure on $\mathbb{X}$,

loose opposite VDC

(3) an involution of VDCs $(-)^\circ : \mathbb{X} \xrightarrow{\cong} \mathbb{X}^{\mathrm{op}\ell}$ preserving the monoidal product $\otimes$ and the monoidal unit, so in particular, we have

$$(A \otimes B)^\circ = A^\circ \otimes B^\circ \quad \text{and} \quad I^\circ = I,$$

(4) bijective "transposition" operations on loose arrows

$$\{\text{loose arrows } A \otimes B \to C \text{ in } \mathbb{X}\} \overset{1:1}{\longleftrightarrow} \{\text{loose arrows } A \to B^\circ \otimes C \text{ in } \mathbb{X}\}$$

$$p \longmapsto p^{>}$$

and on 2-cells

$$\left\{ \text{2-cells in } \mathbb{X} \text{ with frame } \begin{tikzcd} A_0 \otimes B \arrow[r, "r_1 \otimes B"] \arrow[d, "f \otimes g"'] & \cdots \arrow[r, "r_n \otimes B"] & A_n \otimes B \arrow[r, "p", "\shortmid"{marking}] & C_0 \arrow[r, "s_1", "\shortmid"{marking}] & \cdots \arrow[r, "s_m", "\shortmid"{marking}] & C_m \arrow[d, "h"] \\ A' \otimes B' \arrow[rrrrr, "q"', "\shortmid"{marking}] & & & & & C' \end{tikzcd} \right\}$$

$$\overset{1:1}{\longleftrightarrow} \left\{ \text{2-cells in } \mathbb{X} \text{ with frame } \begin{tikzcd} A_0 \arrow[r, "r_1", "\shortmid"{marking}] \arrow[d, "f"'] & \cdots \arrow[r, "r_n", "\shortmid"{marking}] & A_n \arrow[r, "p^{>}", "\shortmid"{marking}] & B^{\circ} \otimes C_0 \arrow[r, "B^{\circ} \otimes s_1", "\shortmid"{marking}] & \cdots \arrow[r, "B^{\circ} \otimes s_m", "\shortmid"{marking}] & B^{\circ} \otimes C_m \arrow[d, "g^{\circ} \otimes h"] \\ A' \arrow[rrrrr, "q^{>}"', "\shortmid"{marking}] & & & & & (B')^{\circ} \otimes C' \end{tikzcd} \right\}$$

$$\alpha \mapsto \alpha^{>}$$

subject to a natural property wrt the composition of 2-cells.

A compact left closed VDC is defined analogously.

Thus:

$$A \otimes B \nrightarrow C \quad \overset{1:1}{\longleftrightarrow} \quad B \nrightarrow C \otimes A^{\circ}.$$

An involutive monoidal VDC is compact closed iff it is equipped with both a compact right closure and a compact left closure which are compatible in an appropriate sense.

<u>Example</u>.

Let $\mathcal{E}$ be a category with pullbacks.

The VDC $\mathbb{S}\mathrm{pan}(\mathcal{E})$ of spans in $\mathcal{E}$ is defined by:

(1) underlying category of objects and tight arrows is $\mathcal{E}$,

(2) loose arrows $A \nrightarrow B$ are spans $A \xleftarrow{u} X \xrightarrow{v} B$ in $\mathcal{E}$,

(3) 2-cells

$$\begin{tikzcd} A_0 \arrow[r, "\shortmid" marking] \arrow[d] & A_1 \cdots A_{n-1} \arrow[r, "\shortmid" marking] \arrow[d, phantom, "\alpha"] & A_n \arrow[d] \\ B_0 \arrow[rr, "\shortmid" marking] & & B_1 \end{tikzcd}$$

are commutative diagrams in $\mathcal{E}$

$$\begin{tikzcd}
A_0 \arrow[d] & X_1 \arrow[l] \arrow[dr, phantom, "\circlearrowright"] & X_1 \times_{A_1} X_2 \times_{A_2} \cdots \times_{A_{n-1}} X_n \arrow[l, "\pi_1"'] \arrow[r, "\pi_n"] \arrow[d, "\dot{\alpha}"] & X_n \arrow[r] \arrow[dl, phantom, "\circlearrowright"] & A_n \arrow[d] \\
B_0 & & Y \arrow[ll] \arrow[rr] & & B_1
\end{tikzcd}$$

for some morphism $\dot{\alpha}$ on the n-ary pullback of

$$X_1 \xrightarrow{v_1} A_1 \xleftarrow{u_2} X_2 \xrightarrow{v_2} A_2 \xleftarrow{u_3} X_3 \quad \cdots \quad X_{n-1} \xrightarrow{v_{n-1}} A_{n-1} \xleftarrow{u_n} X_n,$$

(4) composition of 2-cells via universal property of pullbacks,

(5) identity 2-cell of a span $A \xleftarrow{u} X \xrightarrow{v} B$ is

$$\begin{tikzcd}
A \arrow[d, "id_A"'] & X \arrow[l, "u"'] \arrow[r, "v"] \arrow[d, "id_X"] & B \arrow[d, "id_B"] \\
A & X \arrow[l, "u"] \arrow[r, "v"'] & B.
\end{tikzcd}$$

The products in $\mathcal{E}$ induce a cartesian monoidal structure on $\mathcal{S}pan(\mathcal{E})$.
For example, $(A \xleftarrow{u} X \xrightarrow{v} B) \times (A' \xleftarrow{u'} X' \xrightarrow{v'} B') = (A \times A' \xleftarrow{u \times u'} X \times X' \xrightarrow{v \times v'} B \times B')$.

We define an involution

$$(-)^\circ : \mathbb{S}\mathrm{pan}(\mathcal{E}) \longrightarrow \mathbb{S}\mathrm{pan}(\mathcal{E})^{\mathrm{op}\ell}$$

by reversing the spans: $(A \xleftarrow{u} X \xrightarrow{v} B)^\circ := (B \xleftarrow{v} X \xrightarrow{u} A)$.

This makes $\mathbb{S}\mathrm{pan}(\mathcal{E})$ into an involutive monoidal VDC.

Compact right closure:

We observe:

$\{$ loose arrows $A \otimes B \to\!\!\!\!\!+ C$ in $\mathbb{S}\mathrm{pan}(\mathcal{E})\}$ $=$ $\{$ spans $A \times B \longleftarrow X \longrightarrow C$ in $\mathcal{E}$ for some $X \in \mathrm{Ob}(\mathcal{E})\}$

$\cong$ $\{$ triples of morphisms $X \to A, X \to B, X \to C$ in $\mathcal{E}$ for some $X \in \mathrm{Ob}(\mathcal{E})\}$

universal property of the product

$\cong$ $\{$ spans $A \longleftarrow X \longrightarrow B \times C$ in $\mathcal{E}$ for some $X \in \mathrm{Ob}(\mathcal{E})\}$

$=$ $\{$ loose arrows $A \to\!\!\!\!\!+ B \otimes C = B^\circ \otimes C$ in $\mathbb{S}\mathrm{pan}(\mathcal{E})\}$.

This bijection can be extended to 2-cells.

Compact left closure: analogous.

This makes $\mathbb{S}\mathrm{pan}(\mathcal{E})$ into a compact closed VDC.

3. Coend calculus

Aim: Define a notion of a coend for tight arrows in a compact closed strict virtual equipment $\mathbb{X}$.

$\rightsquigarrow$ Motivated by the construction of ends in categories enriched over the bicategory of spans due to Betti and Walters.

Need special loose arrows of the form $h_A : I \nrightarrow A^\circ \otimes A$.

Construction: Start with the loose unit arrow $e_A : A \nrightarrow A$.

$$\begin{tikzcd} I \otimes A \arrow[r, "A(1, \ell_A)", "|"'] \arrow[d, "\ell_A"'] \arrow[dr, phantom, "\mathrm{cart}"] & A \arrow[d, equal] \\ A \arrow[r, "e_A"', "|"] & A \end{tikzcd}$$

It corresponds to a loose arrow $I \otimes A \nrightarrow A$.

Using the compact right closed structure on $\mathbb{X}$, we obtain a loose arrow

$$h_A : I \nrightarrow A^\circ \otimes A.$$

Example:

In $\mathbf{Span}(\mathcal{E})$, we can think of h_A as the span $* \longleftarrow A \xrightarrow{\langle id_A, id_A \rangle} A \times A.$

Definition.

Let $f : A^\circ \otimes A \to C$ be a tight arrow in $\mathbb{X}$.

A coend of f is an h_A-weighted colimit of f. More explicitly, this consists of a tight arrow

$$\textstyle\int^A f : I \to C$$

and a 2-cell

$$\begin{tikzcd} I \arrow[r, "h_A", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\xi"] & A^\circ \otimes A \arrow[d, equal] \\ I \arrow[r, "C(f, \int^A f)"', "|" marking] & A^\circ \otimes A \end{tikzcd}$$

satisfying the following universal property. Here, the loose target arrow $C(f, \int^A f) : I \nrightarrow A^\circ \otimes A$ is obtained by restriction :

$$\begin{tikzcd} I \arrow[r, "C(f, \int^A f)", "|" marking] \arrow[d, "\int^A f"'] \arrow[dr, phantom, "\mathrm{cart}"] & A^\circ \otimes A \arrow[d, "f"] \\ C \arrow[r, "e_C"', "|" marking] & C \end{tikzcd}$$

Universal property for $(\int^A f, \xi)$:

Every 2-cell

$$
\begin{tikzcd}
C = X_0 \arrow[r, "\shortmid" marking] & X_1 \cdots X_{k-1} \arrow[r, "\shortmid" marking] & X_k = I \arrow[r, "h_A", "\shortmid" marking] & A^\circ \otimes A \\
C \arrow[u, equal] \arrow[rrr, "C(f,1)"', "\shortmid" marking] & \arrow[u, phantom, "\alpha"] & & A^\circ \otimes A \arrow[u, equal]
\end{tikzcd}
$$

$A^\circ \otimes A$,

factors uniquely as follows:

$$
\begin{tikzcd}
C = X_0 \arrow[r, "\shortmid" marking] & X_1 \cdots X_{k-1} \arrow[r, "\shortmid" marking] & X_k = I \arrow[r, "h_A", "\shortmid" marking] & A^\circ \otimes A \\
C \arrow[u, equal] \arrow[rr, "C(\int^A f, 1)"', "\shortmid" marking] & \arrow[u, phantom, "\alpha'"] & I \arrow[u, equal] \arrow[r, "C(f, \int^A f)"', "\shortmid" marking] \arrow[ur, phantom, "\xi"] & A^\circ \otimes A \arrow[u, equal] \\
C \arrow[u, equal] \arrow[rrr, "C(f,1)"', "\shortmid" marking] & & & A^\circ \otimes A \arrow[u, equal]
\end{tikzcd}
$$

obtained by a kind of composition exploiting the universal property of restrictions

In other words, the loose arrow $C(\int^A f, 1) : C \nrightarrow I$ (the conjoint of $\int^A f$) is a right lift of $C(f,1) : C \nrightarrow A^\circ \otimes A$ (the conjoint of f) through $h_A : I \nrightarrow A^\circ \otimes A$.

$$\begin{tikzcd} C \arrow[r, "C(\int^A f, 1)", "|"] \arrow[d, equal] & I \arrow[r, "h_A", "|"] & A^\circ \otimes A \arrow[d, equal] \\ C \arrow[rr, "C(f,1)"', "|"] & & A^\circ \otimes A \end{tikzcd}$$

More general, one can define a coend for a tight arrow of the form $f : X \otimes A^\circ \otimes A \otimes Y \to C$.
This is a colimit of f weighted by the loose arrow

$$X \otimes Y \cong X \otimes I \otimes Y \xrightarrow{X \otimes h_A \otimes Y} X \otimes A^\circ \otimes A \otimes Y.$$

Theorem. (Fubini rule).

Let $f : A^\circ \otimes A \otimes B^\circ \otimes B \to C$ be a tight arrow in $\mathbb{X}$.

We suppose that f admits a coend along A $\quad \int^A f : B^\circ \otimes B \to C$

and a coend along B $\quad \int^B f : A^\circ \otimes A \to C$.

If $\int^A f$ or $\int^B f$ admits a coend, then so does the other and both coends are isomorphic:

$$\int^B \int^A f \cong \int^A \int^B f.$$

Moreover, if the involutive monoidal structure on $\mathbb{X}$ is symmetric, then we may also take the coend of $\quad A^\circ \otimes B^\circ \otimes A \otimes B \cong A^\circ \otimes A \otimes B^\circ \otimes B \xrightarrow{f} C \quad$ along $A \otimes B$, and we have

$$\int^{A \otimes B} f \cong \int^B \int^A f \cong \int^A \int^B f.$$

4. Future work

Presheaf objects:

Given a pseudomonoid M in a compact closed strict virtual equipment $\mathbb{X}$,
we would like to construct an induced pseudomonoid structure on its presheaf object $\mathcal{P}M$.

Strategy: Get inspirations from the usual coend calculus of the Day convolution monoidal structure on a presheaf category $\mathrm{Cat}(\mathcal{C}^{op}, \mathrm{Set})$.

This would show that compact closed strict virtual equipments form an appropriate framework for the more complex coend calculus from enriched category theory.

Furthermore, we might use this to investigate convolution monoidal structures.
In the literature, there exist constructions of convolution monoidal structures in several different contexts. The coend calculus in the virtual double categorical setting might give rise to interesting generalizations.

Thank you very much for your attention!

I am pleased about all comments and questions.