Homotopy-coherent Bousfield–Kan

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Topos Oxford Seminar

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Plan

pictures + history 1. Who is this so-called "wiggly Delta" ?

these slides

story + maths 2. What did Bousfield and Kan do?

maths 3. Some new results

[4. Homotopy homotopy limits]

better slides

1. Who is  ?

Secret character in two stories

homotopy theory

Bousfield–Kan

If $X^\bullet_\bullet$ is a Reedy fibrant cosimplicial simplicial set then

$\exists \ \mathrm{holim}\, X^\bullet_\bullet \xrightarrow{\sim} \mathrm{Tot}\, X^\bullet_\bullet$

non-algebraic complex geometry

Green–Toledo–Tong

If $\mathcal{F}^\bullet \in \mathrm{Ccoh}(X)$ for holomorphic X then

$\exists \ \mathcal{R}^\bullet_\bullet \xrightarrow{\sim} i^*\mathcal{F}^\bullet_\bullet$

Motto: holomorphic stuff is so homotopical

"Green"-Toledo-Tong

We now present Green's construction. He starts with a twisting cochain for $\mathcal{F}$ as in [13] and successively modifies it to obtain a complex of simplicial vector bundles. We recall that a twisting cochain for $\mathcal{F}$ means

1) over each $U_\alpha \in \mathfrak{U}$ a free resolution

$$0 \to E_\alpha^{-n} \xrightarrow{a_\alpha^{0,1}} \cdots \xrightarrow{a_\alpha^{0,1}} E^0 \to \mathcal{F}|U_\alpha \to 0,$$

2) over each U_σ, $\sigma = \langle\alpha_0 \ldots \alpha_k\rangle$, a bundle map $a^{k,1-k}_{\alpha_0\ldots\alpha_k} : E^\bullet_{\alpha_k}|U_\sigma \to E^\bullet_{\alpha_0}|U_\sigma$ of degree $1-k$,

3) the cochain $a = a^{0,1} + a^{1,0} + a^{2,-1} + \ldots$ satisfies the identities $\delta a + a \cdot a = 0$, $a^{1,0}_{\alpha\alpha} = 1$.

We refer to [7], §1 for details of the meaning of $\delta a + a \cdot a = 0$, and for the simple theorem that a coherent sheaf can be resolved by a twisting cochain.

To be able to go from a twisting cochain to a complex of simplicial vector bundles we need a concept that interpolates between them, namely a *simplicial twisting cochain* on $N\mathfrak{U}$. This means:

(STC 1) For each vertex α and each $\sigma \geqslant \langle\alpha\rangle$ a free resolution: $E^\bullet_{\sigma,\alpha}$ of $\mathcal{F}|U_\sigma$: $0 \to E^{-n}_{\sigma,\alpha} \xrightarrow{{}^\sigma a^{0,1}_\alpha} \cdots \xrightarrow{{}^\sigma a^{0,1}_\alpha} E^0_{\sigma,\alpha} \to \mathcal{F}|U_\sigma \to 0$

(STC 2) For each $\sigma = \langle\alpha_0 \ldots \alpha_p\rangle$ a twisting cochain ${}^\sigma a = {}^\sigma a^{0,1} + {}^\sigma a^{1,0} + a^{2,-1} + \ldots$ acting on the resolutions $E^\bullet_{\sigma,\alpha_i}$.

(STC 3) Whenever $\sigma < \tau$ and α a vertex of σ, a short exact sequence of free modules $0 \to E^\bullet_{\sigma,\alpha}|U_\tau \to E^\bullet_{\tau,\alpha} \to E^\bullet_{\sigma,\tau,\alpha} \to 0$ as in (SB 4), §1.

(STC 4) Whenever $\sigma < \tau$ and $\alpha_0, \ldots, \alpha_k$ any vertices of σ (repetitions allowed !), the matrix ${}^\tau a^{k,1-k}_{\alpha_0\ldots\alpha_k}$ is of the form:

i) for $k = 0$, ${}^\tau a^{0,1}_\alpha = {}^\sigma a^{0,1}_\alpha \oplus$ elementary sequences,

ii) for $k = 1$ ${}^\tau a^{1,0}_{\alpha\beta} = \begin{bmatrix} {}^\sigma a^{1,0}_{\alpha\beta} & * \\ 0 & 1 \end{bmatrix}$

$$\begin{cases} \delta_{rs} & r \neq i,j \\ {}^\sigma a^{0,1}_i & r = i,\ s = i \\ -{}^\sigma a^{1,0}_{is} & r = i,\ s \neq i \\ {}^\sigma a^{1,0}_{ji} & r = j,\ s = i \\ {}^\sigma a^{2,-1}_{jis} & r = j,\ s \neq i \end{cases}$$

if $i \neq j$, and ${}^\sigma M(a)^{1,0}_{ii} = 1$,

${}^\sigma M^{q,1-q}_{j_0\ldots j_q}$, for $q \geqslant 2$, is given by the row vector

$$\left[(-1)^{q+1}\,{}^\sigma a^{q+1,-q}_{j_0,\ldots,j_q,0},\ (-1)^{q+1}\,{}^\sigma a^{q+1,-q}_{j_0,\ldots,j_q,1},\ \ldots,\ {}^\sigma a^{q,1-q}_{j_0,\ldots,j_q},\ \ldots,\ (-1)^{q-1}\,{}^\sigma a^{q+1,-q}_{j_0,\ldots,j_q,p}\right].$$

In dimension 1:

${}^\sigma M(a)^{0,1}_i = (0,\ldots,0,\ {}^\sigma a^{i,0}_i,\ 0,\ldots,0)$, ${}^\sigma M(a)^{1,0}_{ji}$ has $r-s$ entry given by

$$\begin{cases} \delta_{rs} & r \neq i,j \\ 1 & r = i,\ s = i \\ -{}^\sigma a^{1,0}_{is} & r = i,\ s \neq i \\ {}^\sigma a^{1,0}_{ji} & r = j,\ s = i \\ {}^\sigma a^{0,1}_j\,{}^\sigma a^{2,-1}_{jis} & r = j,\ s \neq i \end{cases}$$

if $i \neq j$, and ${}^\sigma M(a)^{1,0}_{ii} = 1$, ${}^\sigma M(a)^{q,1-q} = 0$ for $q \geqslant 2$.

2020 ⇝ 2024

Definition 3.3.4. A *GTT-labelling of* $\Delta[p]$ *by* $[\![\mathcal{N}^{\mathrm{dg}}\,\mathsf{Free}(U)]\!]$ (or simply a *GTT-labelling of* $\Delta[p]$) consists of a labelling of $\Delta[p]_{\mathrm{pair}}$ (Definition 2.6.1) subject to some conditions. More precisely, it consists of the following data:

(a) To each 0-cell

$$(\sigma,\sigma) \quad\longleftrightarrow\quad \{i_0 < \ldots < i_k\} \subseteq \{i_0 < \ldots < i_k\} \subseteq [p]$$

we assign a k-simplex of $[\![\mathcal{N}^{\mathrm{dg}}\,\mathsf{Free}(U)]\!]$, i.e. bounded complexes of finite-rank free $\mathcal{O}_U$-modules

$$C_{i_0}(\sigma),\ C_{i_1}(\sigma),\ \ldots,\ C_{i_k}(\sigma) \in \mathsf{Free}(U)$$

along with, for all non-empty subsets $J = \{j_0 < \ldots < j_\ell\} \subseteq \{i_0 < \ldots < i_k\}$, morphisms

$$\varphi_J(\sigma) \in \mathrm{Hom}^{1-\ell}_{\mathsf{Free}(U)}\left(C_{j_\ell}(\sigma), C_{j_0}(\sigma)\right)$$

such that

- if $|J| = 1$, then $\varphi_J(\sigma)$ is the differential on $C_J(\sigma)$;
- if $|J| = 2$, then $\varphi_J(\sigma)\colon C_{j_1}(\sigma) \to C_{j_0}(\sigma)$ is a chain map;
- if $|J| \geqslant 3$, then

$$\partial\varphi_J(\sigma) = \sum_{m=1}^{\ell-1}(-1)^{m-1}\varphi_{J\setminus\{j_m\}}(\sigma) + \sum_{m=1}^{\ell-1}(-1)^{\ell(m-1)+1}\varphi_{\{j_m<\ldots<j_k\}}(\sigma)\circ\varphi_{\{j_0<\ldots<j_m\}}(\sigma).$$

(b) To each $(k-\ell)$-cell

$$(\tau,\sigma) \quad\longleftrightarrow\quad \{j_0 < \ldots < j_\ell\} \subset \{i_0 < \ldots < i_k\} \subseteq [p]$$

(where $0 \leqslant l < k$) we assign an $(\ell+1)$-tuple of objects

$$\left(C^{\perp\sigma}_{j_m}(\tau) \in \mathsf{Free}(U)\right)_{0\leqslant m\leqslant\ell}$$

where each $C^{\perp\sigma}_{j_m}(\tau)$ is elementary.[19] We refer to the $C^{\perp\sigma}_{j_m}(\tau)$ as the *elementary (orthogonal) complements.*

This data is subject to the conditions that, for any $(k-\ell)$-cell

$$(\tau,\sigma) \quad\longleftrightarrow\quad \{j_0 < \ldots < j_\ell\} \subset \{i_0 < \ldots < i_k\} \subseteq [p]$$

(where $0 \leqslant l < k$) the following are satisfied:

(i) There is a direct-sum decomposition

$$\theta^{\perp\sigma}_{j_m}(\tau)\colon C_{j_m}(\tau) \oplus C^{\perp\sigma}_{j_m}(\tau) \xrightarrow{\cong} C_{j_m}(\sigma)$$

for all $0 \leqslant m \leqslant \ell$. We refer to the isomorphism $\theta^{\perp\sigma}_{j_m}(\tau)$ as the *τ-trivialisation of* $C_{j_m}(\sigma)$.

(ii) For any non-empty subset $K \subseteq \{j_0 < \ldots < j_\ell\}$ with $|K| \geqslant 2$, the diagram

$$\begin{tikzcd} C_{\mathrm{ver}_{|K|}K}(\tau) \arrow[r, hook] \arrow[d, "\varphi_K(\tau)"'] & C_{\mathrm{ver}_{|K|}K}(\tau) \oplus C^{\perp\sigma}_{\mathrm{ver}_{|K|}K}(\tau) \arrow[r, two heads] \arrow[d, "\varphi_K(\sigma)_\tau"] & C^{\perp\sigma}_{\mathrm{ver}_{|K|}K}(\tau) \arrow[d, dashed] \\ C_{\mathrm{ver}_0 K}(\tau) \arrow[r, hook] & C_{\mathrm{ver}_0 K}(\tau) \oplus C^{\perp\sigma}_{\mathrm{ver}_0 K}(\tau) \arrow[r, two heads] & C^{\perp\sigma}_{\mathrm{ver}_0 K}(\tau) \end{tikzcd}$$

commutes, where the $\hookrightarrow$ are the inclusions and the $\twoheadrightarrow$ are the projections of the direct sum, the dashed arrow on the far right is the elementary morphism[20] of degree $(2-|K|)$, and we write $\varphi_K(\sigma)_\tau$ to mean the composition

$$\begin{tikzcd} C_{\mathrm{ver}_{|K|}K}(\tau) \oplus C^{\perp\sigma}_{\mathrm{ver}_{|K|}K}(\tau) \arrow[r, "\theta^{\perp\sigma}_{\mathrm{ver}_{|K|}K}(\tau)"] & C_{\mathrm{ver}_{|K|}K}(\sigma) \arrow[d, "\varphi_K(\sigma)"] \\ C_{\mathrm{ver}_0 K}(\tau) \oplus C^{\perp\sigma}_{\mathrm{ver}_0 K}(\tau) & C_{\mathrm{ver}_0 K}(\sigma) \arrow[l, "\theta^{\perp\sigma}_{\mathrm{ver}_0 K}(\tau)^{-1}"] \end{tikzcd}$$

which we refer to as the *τ-trivialisation of* $\varphi_K(\sigma)$.

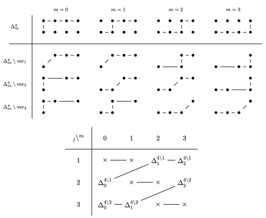
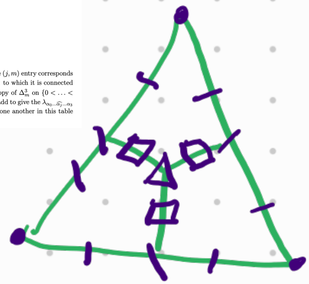
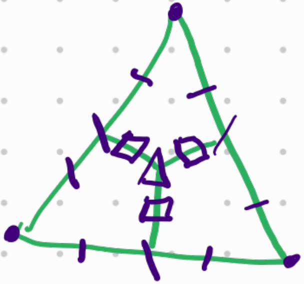
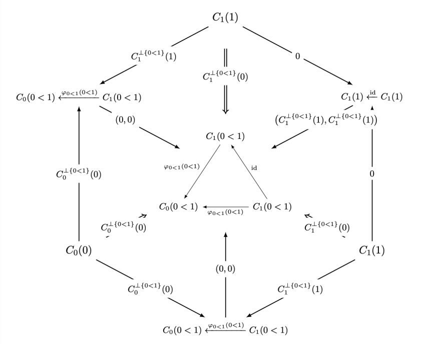


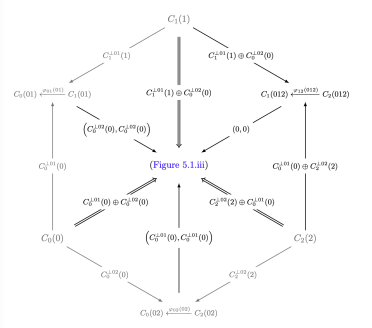
Figure B.2.v. *Top:* The 12 terms of $(\circledast_{\hat\delta})$ in the case $p = 3$. *Bottom:* The (j,m) entry corresponds to $\Delta^4_m \setminus \mathrm{ver}_j$. The entries marked with a $\times$ cancel out with the other $\times$ to which it is connected by a line; the remaining entries are labelled with $\Delta^{4\backslash j}_m$, denoting the copy of Δ^3_m on $\{0 < \ldots < \hat{j} < \ldots < 3\} \times \{0 < 1\}$, and the ones joined with lines are the ones that add to give the $\lambda_{\alpha_0\ldots\widehat{\alpha_j}\ldots\alpha_3}$ term. Note that entries that are horizontally or vertically adjacent to one another in this table have opposite signs in $(\circledast_{\hat\delta})$.

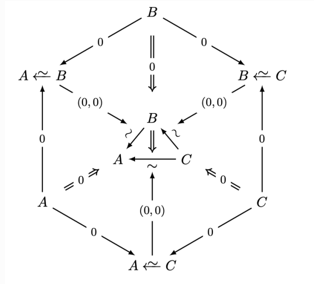


Hosgood-Zeinalian









Hosgood – Zeinalian

2024 ⇝ 2025

Glass – Hosgood

Glass-Hosgood

2025 $\rightsquigarrow$ 2026 (?)

$$N(-/\Delta^n_{inj}) \otimes_{\Delta^n_{inj}} \mathcal{L}\Pi_n$$

Brown-Glass-Hosgood

We finally have a "formal" definition!

... but let's save that for later :)

Sketch 1

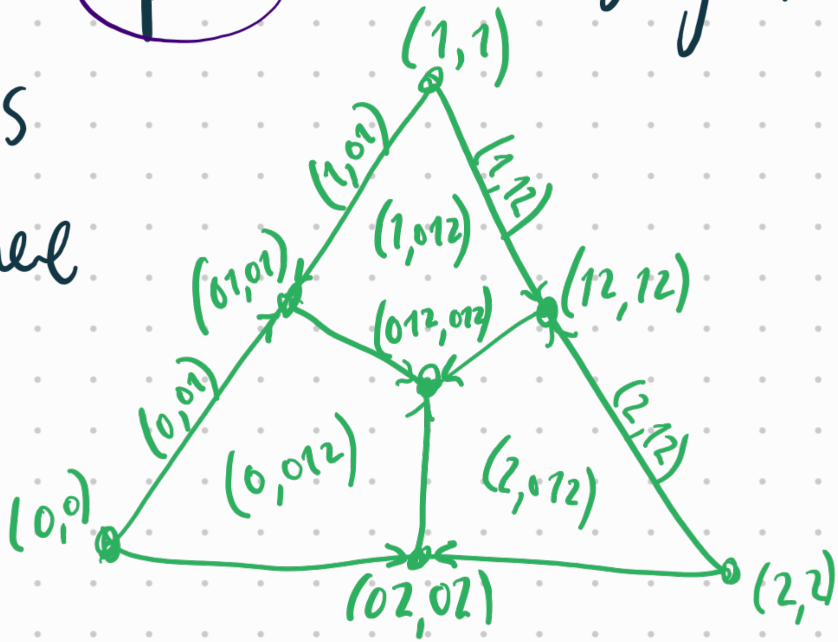
pair

barycentric

Pair(Δ^2) has <u>n-cells</u> given by (σ, τ) where

(i) $\emptyset \subsetneq \sigma \subseteq \tau \subseteq \Delta^2$

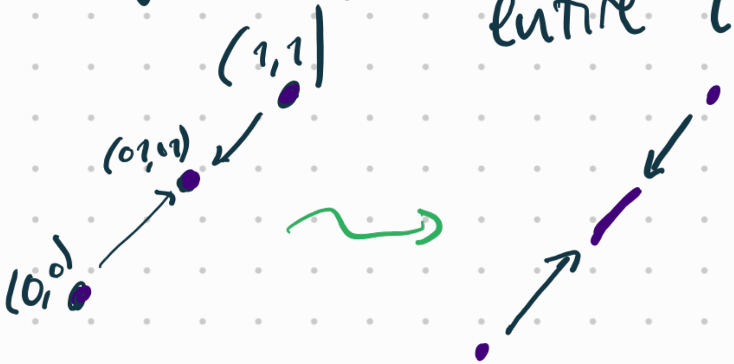
(ii) $\operatorname{codim}_\tau \sigma = n$



From (σ,τ) we can <u>almost</u> "see" all the data of σ and τ <u>but</u> we're missing either $\dim\sigma$ or $\dim\tau$

e.g. $\dim(0,0) = 0 = \dim(012, 012)$

$\Rightarrow$ Tiny change: replace the vertex (σ,σ) with an entire copy of σ



Sketch 2

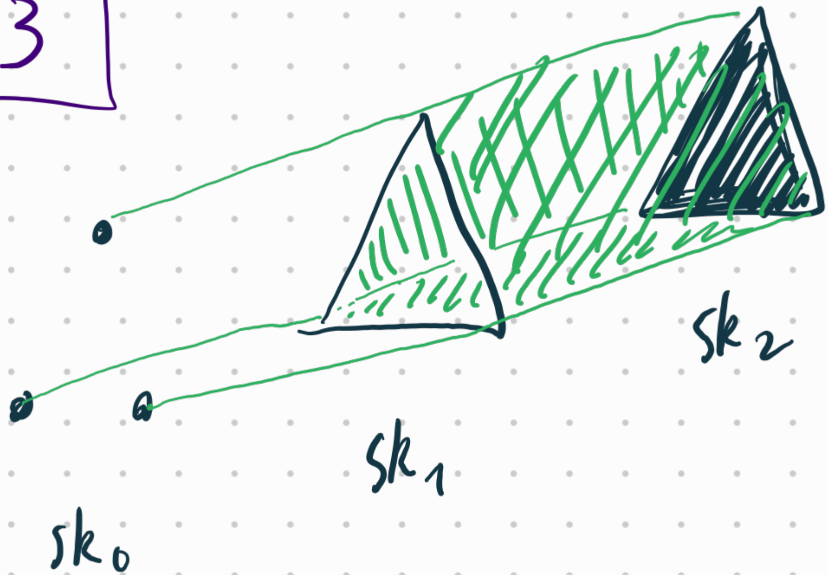
$\left(\blacktriangle = \operatorname{colim}_i \blacktriangle_i \right)$

But what if we built simplicial objects "laxly"?
i.e. instead of asking that e.g. face maps $f : X_{k+1} \to X_k$
enforce $f(\sigma) = \tau$ ask that $f(\sigma) \xrightarrow{\sim} \tau$?

Well ...

Sketch 3

1



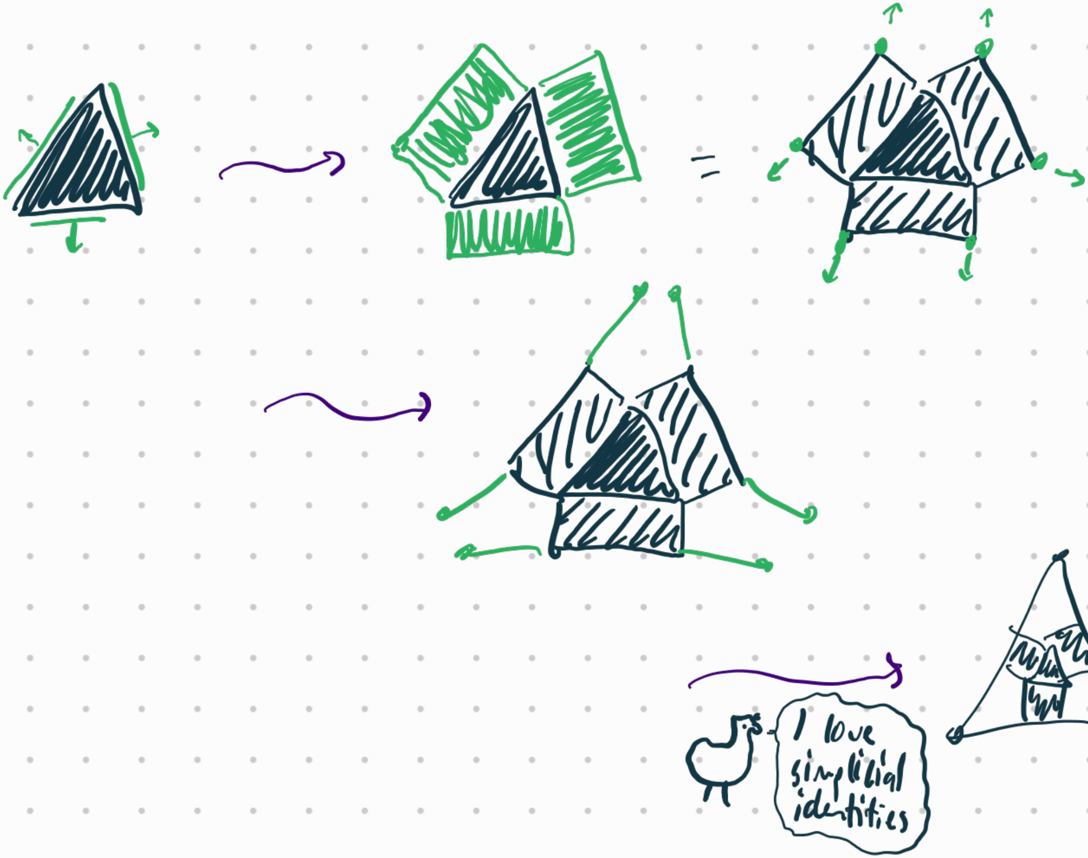
2

3

hydraulic press

Sketch 4

Iterated extrusion



2 What did Bousfield and Kan do?

2.0 The way we’ll tell this story

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1. Introduce some things about model categories
2. Introduce a thing from enriched category theory
3. Repeat some random facts about all the above and promise that it's all interesting

2.1 Some bits of model categories

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Lots of missing details here!

A **model category** is a category with three classes of morphisms:

- fibrations
- cofibrations
- weak equivalences.

These satisfy some nice properties, akin to "surjections, injections, and isomorphisms" in the category of sets, e.g. every function $f: X \to Y$ factors as $X \hookrightarrow (X \sqcup Y \setminus \operatorname{Im} f) \twoheadrightarrow Y$ (or, a bit oddly, $X \twoheadrightarrow \operatorname{Im} f \hookrightarrow Y$).

Better examples come from homological algebra (chain complexes of modules; projective, injective, and quasi-isomorphism) and cellular topology (CW complexes; fibration, inclusion, homotopy equivalence).

We say that an object F is **fibrant** if $F \to *$ is a fibration, and an object C is **cofibrant** if $\varnothing \to C$ is a cofibration.

Given X, we can always find a **fibrant replacement** $X \overset{\sim}{\hookrightarrow} F$ and a **cofibrant replacement** $C \overset{\sim}{\twoheadrightarrow} X$.

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In a model category, it turns out that the best thing is when you look at morphisms **from a cofibrant object to a fibrant object**:

- $\mathrm{Hom}(C, F)$ is a "space"
- $C' \xrightarrow{\sim} C \implies \mathrm{Hom}(C', F) \xrightarrow{\sim} \mathrm{Hom}(C, F)$
- $F \xrightarrow{\sim} F' \implies \mathrm{Hom}(C, F) \xrightarrow{\sim} \mathrm{Hom}(C, F')$.

Again, good examples come from homological algebra (Ext via projective/injective resolutions) and homotopy theory (mapping spaces and sheafification).

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The dream world: the subcategory of cofibrant-fibrant objects.

We are often interested in functor categories $[\mathcal{C}, \mathcal{D}]$, so what happens if we want this to be a model category?

- Easy: if $\mathcal{D}$ is a model category, then we can say that $\alpha\colon F \Rightarrow F'$ is e.g. a cofibration if $\alpha_c\colon F(c) \to F'(c)$ is a cofibration in $\mathcal{D}$.
  Everything is object-wise.
  Example: simplicial presheaves $[\mathcal{C}^{\mathrm{op}}, \mathsf{sSet}] = [\mathcal{C}^{\mathrm{op}}, [\Delta^{\mathrm{op}}, \mathsf{Set}]]$

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- Hard: if $\mathcal{C}$ is a model category, then... well, if it's further a **Reedy** category then we can use the **Reedy** model structure.
  Weak equivalences are object-wise, but we need to do something fancier for (co)fibrations.
  Example: simplicial objects $[\Delta^{\mathrm{op}}, \mathcal{C}]$, including simplicial sets $[\Delta^{\mathrm{op}}, \mathsf{Set}]$

2.2 “Weighted colimits”

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Don’t ever repeat any of this. These are not good definitions.

Given the data of

- $W\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{C}$
- $F\colon \mathcal{I} \to \mathcal{C}$

we can construct the **weighted limit** $\{W, F\}^{\mathcal{I}}$. This generalises the usual limit, if we take W to be the constant functor to the terminal object:

$$\{*, F\}^{\mathcal{I}} \cong \lim_{\mathcal{I}} F$$

Weighted (co)limits are very interesting, and they turn up all over the place. For the rest of this talk, I’m going to call them **functor homs**.

2.3 Some random facts and claims

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- Cosimplicial spaces $X^\star \in [\Delta, \mathsf{Space}]$ are cool and we often want to take their limit to get back a space: $\lim_\Delta X^\star$.

- There are three really interesting cosimplicial spaces:
  1. The **point** $*\colon [q] \mapsto *$
  2. The **simplex** $\Delta^\star\colon [q] \mapsto \Delta[q] = \mathsf{Hom}_\Delta(-, [q])$
  3. The **homotopical simplex** $\underline{\Delta}^\star\colon [q] \mapsto \mathcal{N}(\Delta/[n])$

  and they are interesting for a good reason that I will now actually tell you.

These three cosimplicial spaces let us calculate three interesting things:

- the limit: $\{*, X^\star\}^\Delta \simeq \lim X^\star$
  ... this is interesting because it's from category theory and we love category theory :-)
- the **totalisation**: $\{\Delta^\star, X^\star\}^\Delta \simeq \operatorname{Tot} X^\star$
  ... this is interesting because ... um ... it's easy to compute? and it does recover some interesting constructions (like principal bundles) I guess... but `:shrug:`
- the **homotopy limit**: $\{\underline{\Delta}^\star, X^\star\}^\Delta \simeq \operatorname{holim} X^\star$
  ... this is interesting because it gives us the actually well behaved version of the limit that really understands homotopical / homological things and isn't too strict (also lots of examples! e.g. loop spaces and principal bundles and representation theory) but oh heck is it difficult to compute in general

Even better, these three cosimplicial spaces fit together sort of nicely: $\Delta^\star$ and $\underline{\Delta}^\star$ are both cofibrant replacements of $*$.

$$\varnothing \hookrightarrow \Delta^\star \overset{\sim}{\twoheadrightarrow} * \overset{\sim}{\twoheadleftarrow} \underline{\Delta}^\star \hookleftarrow \varnothing$$

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Keeping in mind that $\{-,-\}$ behaves really nicely whenever the source is cofibrant, this means that $\mathrm{Tot} = \{\Delta^\star, -\}$ and $\mathrm{holim} = \{\underline{\Delta}^\star, -\}$ are both "better" versions of $\lim = \{*, -\}$ if we care about the sort of homotopical structure that our model category is describing.

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But how do Tot and holim relate to one another? If they're both "better" versions of lim then which one should we pick?

Idea. Since $\Delta^\star$ and $\underline{\underline{\Delta}}^\star$ are both weakly equivalent to the point, they're also weakly equivalent to each other, so surely $\{\Delta^\star, -\}$ and $\{\underline{\underline{\Delta}}^\star, -\}$ are "weakly equivalent" functors?

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Problem. Weak equivalences are tricky: they don't always have actual weak inverses. That is, just because we have some $f\colon X \xrightarrow{\sim} Y$, that doesn't mean that there exists any $g\colon Y \xrightarrow{\sim} X$. In fact, there doesn't even have to exist *any* non-trivial $g\colon Y \to X$ at all.

Motto. Knowing that two things are weakly equivalent is often not enough. You need a *witness*.

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Lemma. Any two cofibrant resolutions of an object automatically inherit a morphism between them (witnessing their weak equivalence)

... if one of the resolutions is also fibrant.

Problem. Neither $\Delta^\star$ nor $\underline{\underline{\Delta}}^\star$ are fibrant.

Recap

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- The point $*$ is good because it recovers the limit via $\lim = \{*, -\}$, but the limit is kind of bad because the point is not cofibrant.
- We can cofibrantly replace the point with either $\Delta^\star$ or $\underline{\Delta}^\star$, meaning that we'll get two really nice "limit" functors
  - $\mathrm{Tot} = \{\Delta^\star, -\}$ is nice because it's not much harder to compute than lim but bad because it isn't as delightful as the homotopy limit
  - $\mathrm{holim} = \{\underline{\Delta}^\star, -\}$ is nice because it's the homotopy limit (yay! limits up to homotopy! so natural!) but bad because it's a mess to compute.

Recap

- The point $*$ is good because it recovers the limit via $\lim = \{*, -\}$, but the limit is kind of bad because the point is not cofibrant.
- We can cofibrantly replace the point with either $\Delta^\star$ or $\underline{\Delta}^\star$, meaning that we'll get two really nice "limit" functors
  - $\mathrm{Tot} = \{\Delta^\star, -\}$ is nice because it's not much harder to compute than lim but bad because it isn't as delightful as the homotopy limit
  - $\mathrm{holim} = \{\underline{\Delta}^\star, -\}$ is nice because it's the homotopy limit (yay! limits up to homotopy! so natural!) but bad because it's a mess to compute.
- We *could* say that $\mathrm{Tot} \simeq \mathrm{holim}$ if we had an *explicit* weak equivalence $\underline{\Delta}^\star \to \Delta^\star$ but none of the machinery of model categories will give us one for free, because everything is just of the wrong chirality.

- **Construction.** (Bousfield–Kan) There exists a weak equivalence $\underline{\Delta}^\star \xrightarrow{\sim} \Delta^\star$ and we can write it down and it's not that complicated.
- **Corollary.** If you want to compute the homotopy limit of something (fibrant) then you can just compute the totalisation.

3 Some new results

The short summary

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There is *another* interesting cofibrant replacement of the point: the **homotopy-coherent simplex** $\triangle^\star$.

This means that we get *another* "limit" functor

$$\mathsf{hcTot} = \{\triangle^\star, -\}$$

... but why should we be interested?

The short summary

There is *another* interesting cofibrant replacement of the point: the **homotopy-coherent simplex** $\triangle^\star$.

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... but why should we be interested?

1. The way in which $\triangle^\star$ was first noticed was precisely as the coclassifier of the hcTot construction (for **simplicial twisting cochains**) back in this work by Green–Toledo–Tong.
2. It's basically just as easy to compute as Tot.
3. It's "closer" to holim than Tot is.

It turns out that $\triangle^\star$ is sort of the tensor product of $\Delta^\star$ and $\underline{\Delta}^\star$ so it comes with natural comparison maps to both of them:

$$\underbrace{* \otimes y\pi_n}_{\Delta^n} \xleftarrow{!\otimes \mathrm{id}} \underbrace{\mathcal{N}(-/\Delta^n_{\mathrm{inj}}) \otimes y\pi_n}_{\triangle^n} \xrightarrow{\mathrm{id}\otimes !} \underbrace{\mathcal{N}(-/\Delta^n_{\mathrm{inj}}) \otimes *}_{\mathrm{Sd}\,\Delta^n} \hookrightarrow \underbrace{\mathcal{N}(\Delta/[n])}_{\underline{\Delta}^n}$$

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Even better:

Lemma. (Homotopy-coherent Bousfield–Kan) The map $\triangle^\star \to \Delta^\star$ is a weak equivalence.

Corollary. The three following functors are all weakly equivalent on fibrant objects:

$$\begin{aligned} \mathrm{Tot} &= \{\Delta^\star, -\} \\ \mathrm{hcTot} &= \{\triangle^\star, -\} \\ \mathrm{holim} &= \{\underline{\Delta}^\star, -\} \end{aligned}$$

If they're all the same then why do we care?

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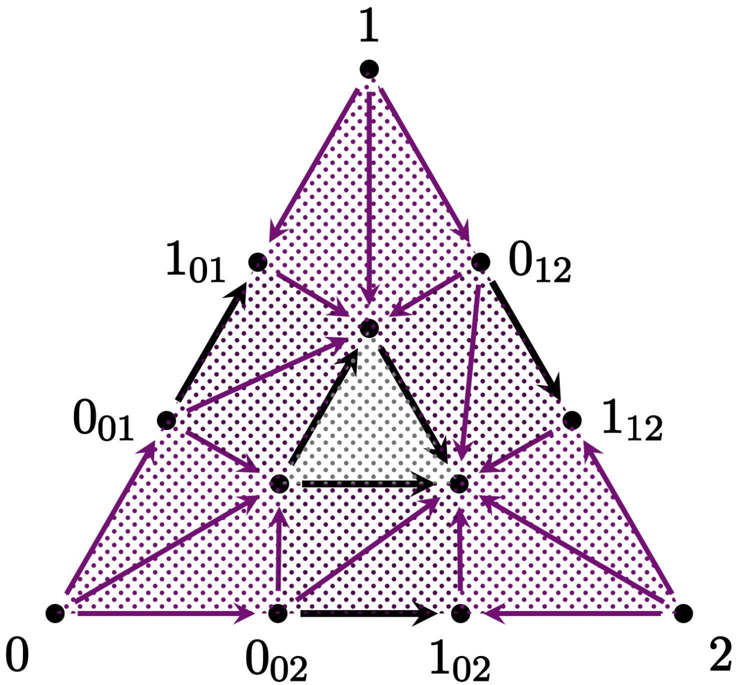
These three functors are all weakly equivalent (on fibrant objects), but sometimes it's very useful to have a specific presentation of an equivalence class. In particular, this construction of simplicial twisting cochains is cool for "geometry reasons" (I promise) and we don't see it if we only have Tot or holim.



1
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4 (Homotopy homotopy limits)

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Conjecture. $\mathrm{hcTot} \simeq \mathrm{hoTot}$