Hyperdoctrines in DOTS

presented at the Topos Institute seminar by Amitai Nachmany

First of all RIP Yayoi Kusama queen of dots whose passing was announced this morning (even though she died two weeks ago). She would have loved *Towards a double operadic theory of systems (Myers and Libkind, 2025).*

Yayoi Kusama
草間 彌生

Kusama in 2013

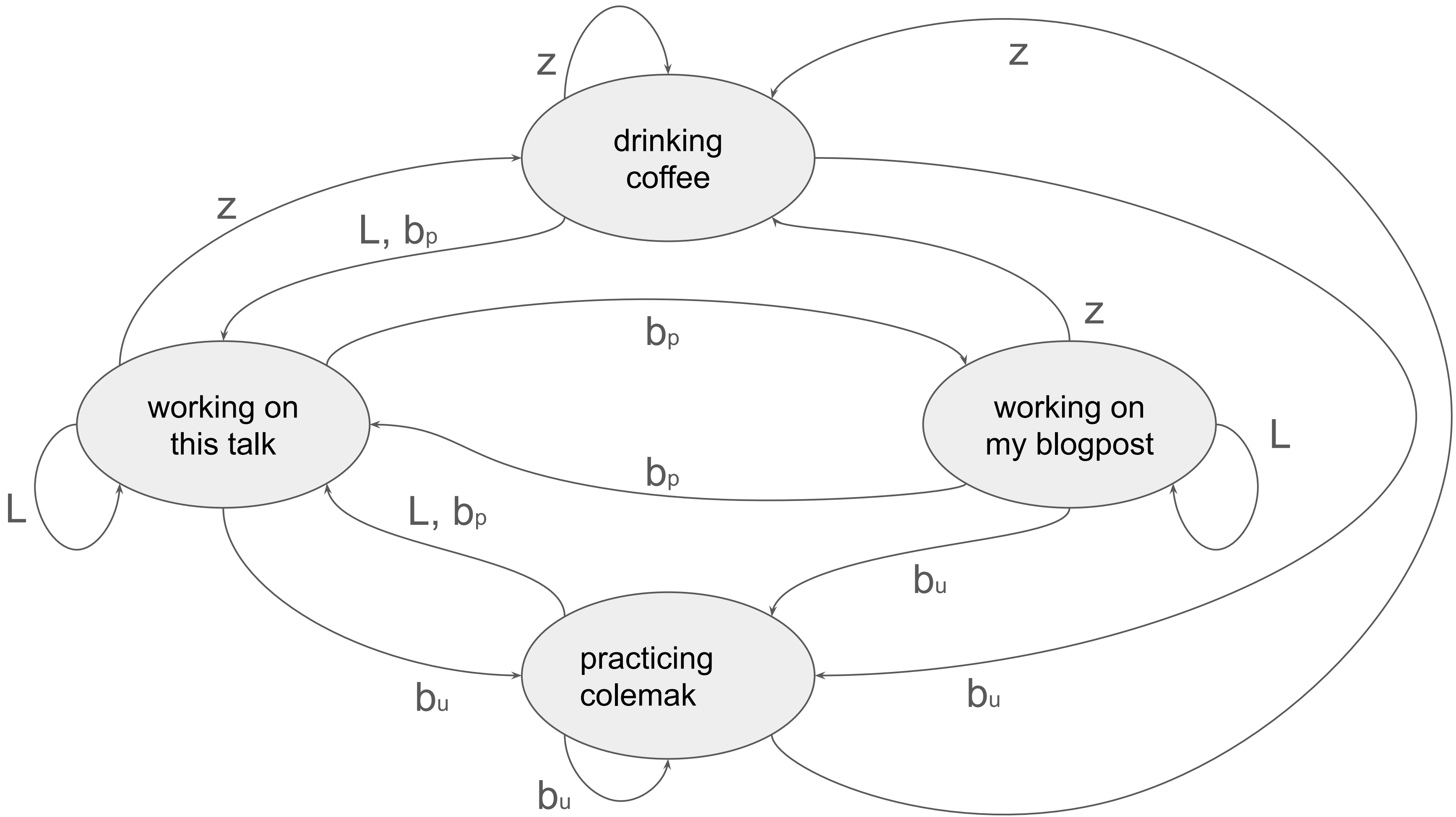
Born	Yayoi Kusama (草間 彌生) 22 March 1929 Matsumoto, Nagano, Japan
Died	14 August 2026 (aged 97) Tokyo, Japan

Prerequisites

- This talk is meant to be at 40% technical-level. I will explain the category theory behind what is happening but I hope that it's not just buzzwords.

- You should probably know what a double category is. And a monoidal category. And some properties of functors (lax monoidal, pseudo double, etc.)

- Don't worry the polka dot pattern is not going to be there for the whole talk


drinking coffee
working on this talk
working on my blogpost
practicing colemak
z
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L, b_p
b_p
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L
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L, b_p
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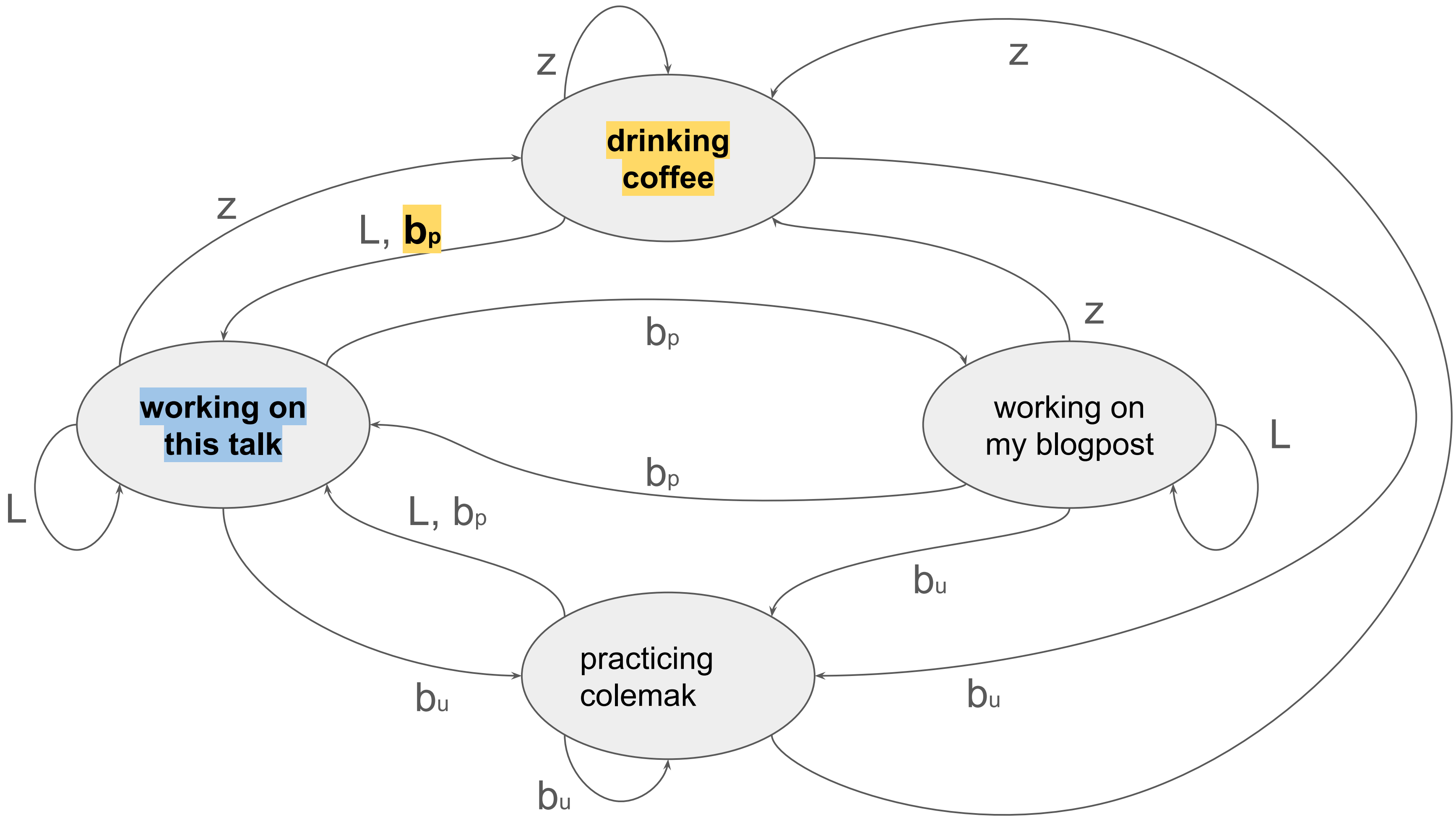

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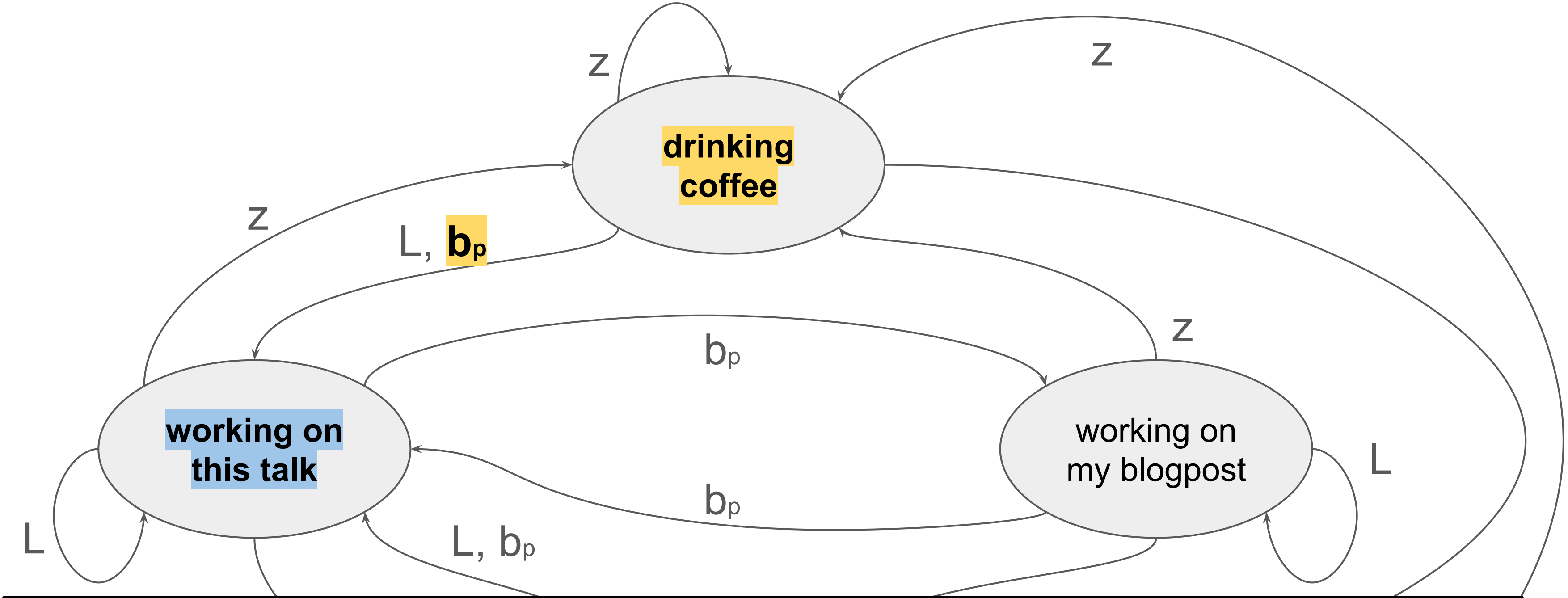
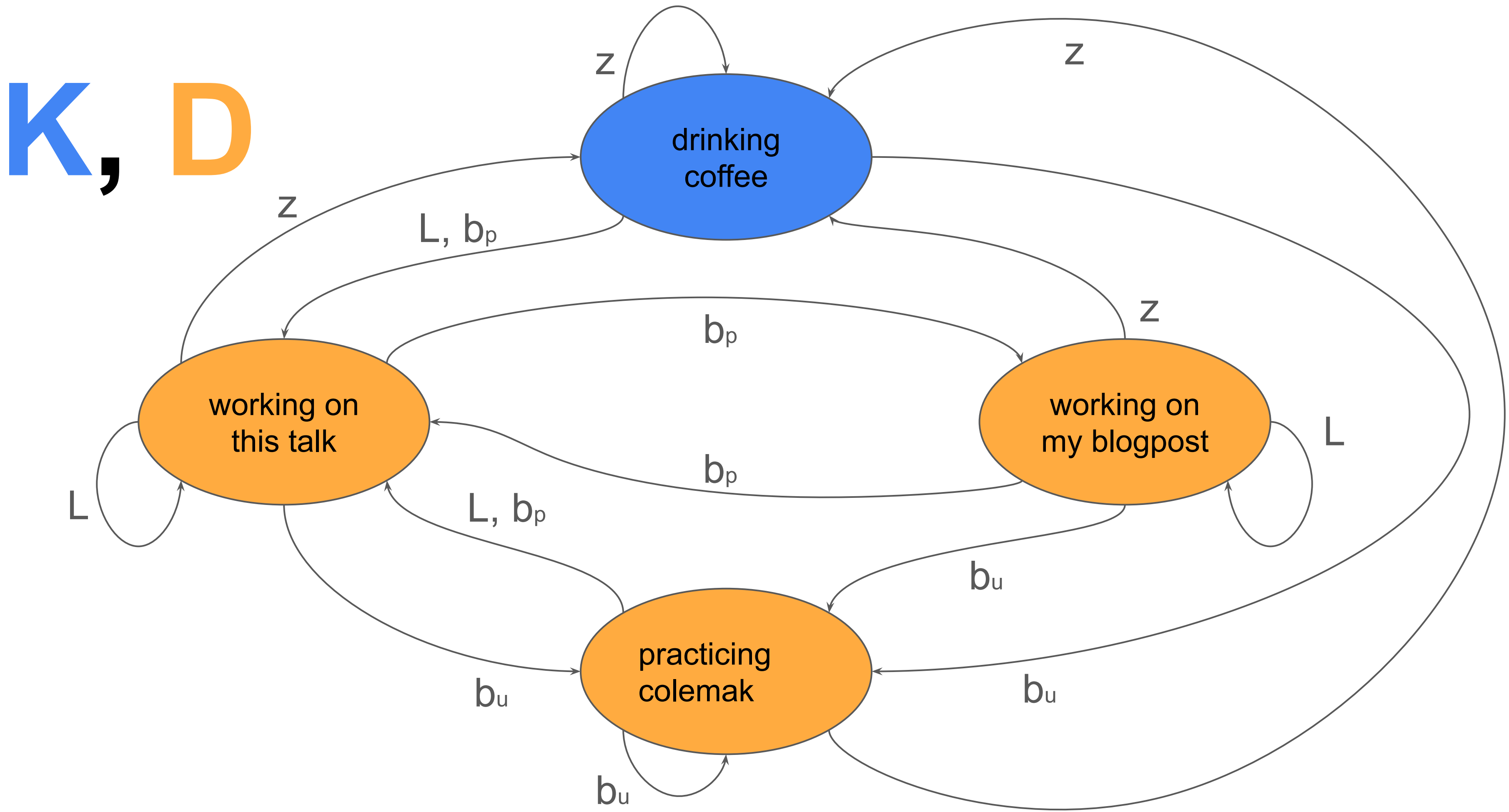


$$I \times S \to S$$

$$b_p \cdot \text{drinking coffee} = \text{working on this talk}$$


K, D
drinking coffee
working on this talk
working on my blogpost
practicing colemak
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L, b_p
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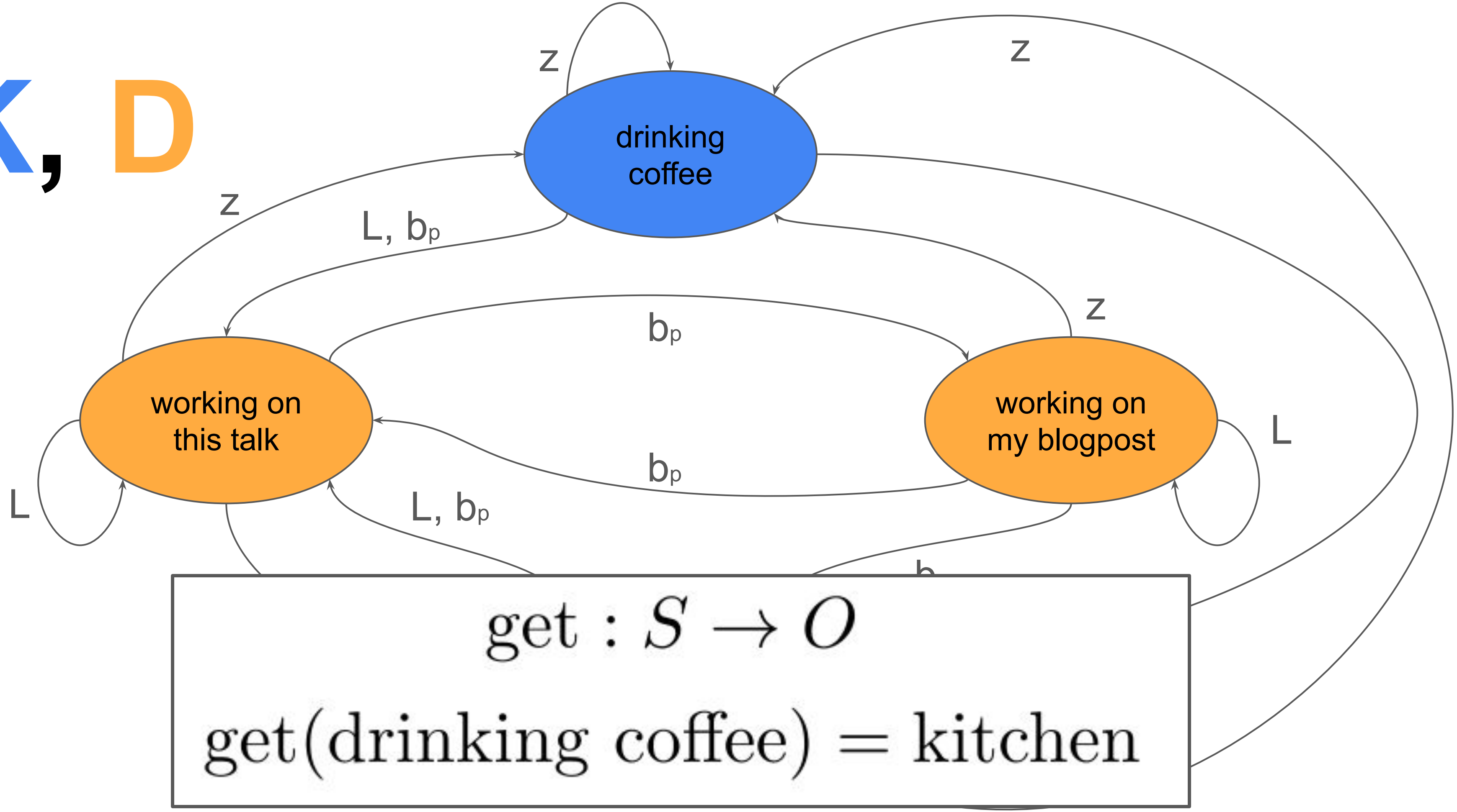
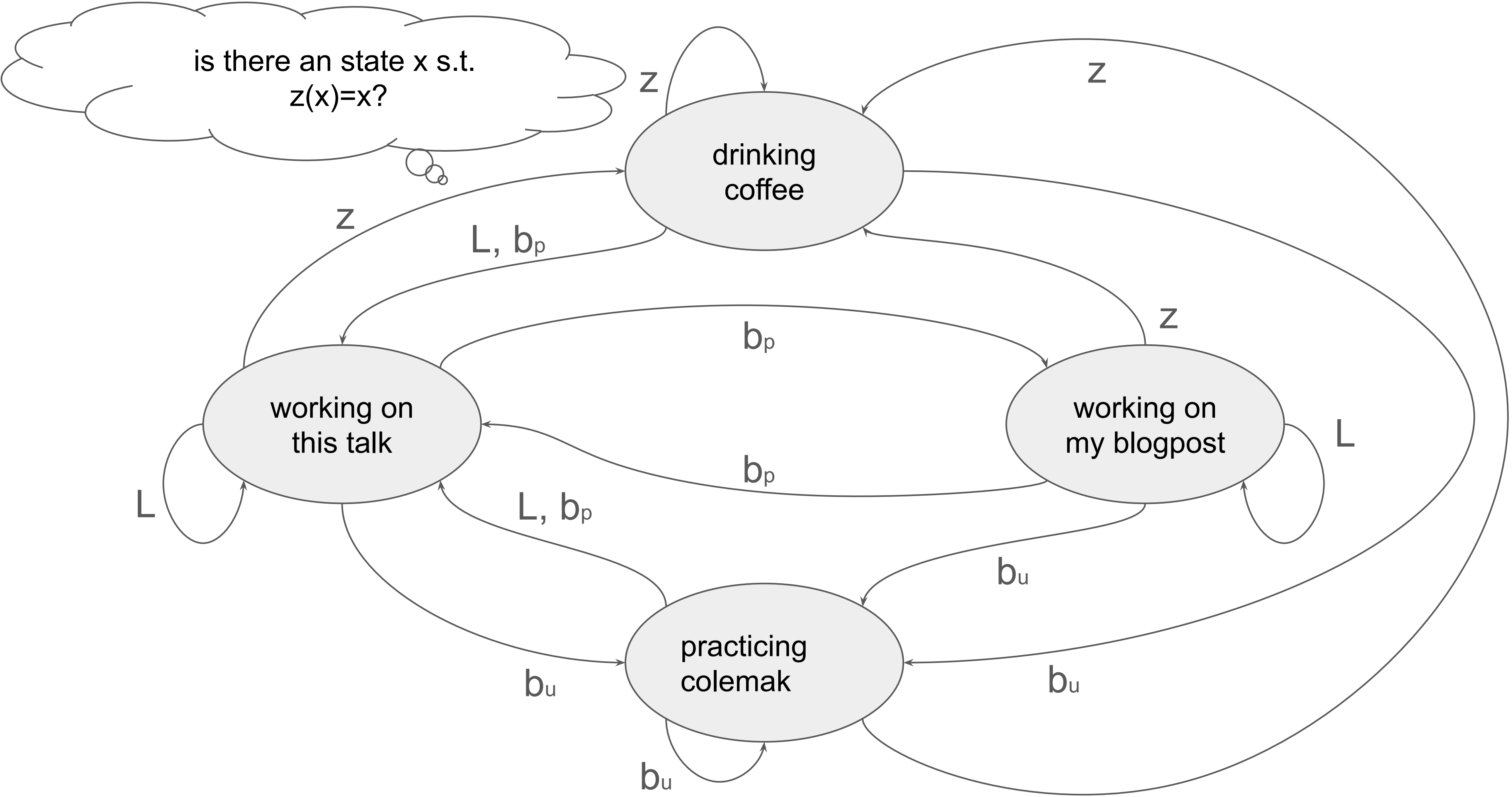
K, D



$$\text{get} : S \to O$$

$$\text{get}(\text{drinking coffee}) = \text{kitchen}$$


is there an state x s.t.
z(x)=x?
drinking
coffee
working on
this talk
working on
my blogpost
practicing
colemak
z
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z
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L, b_p
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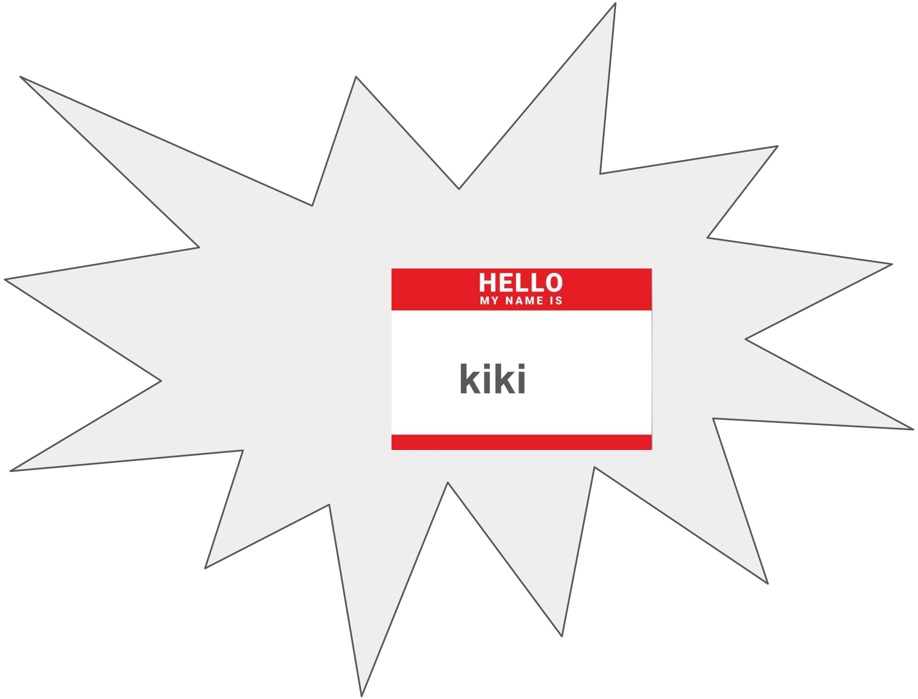
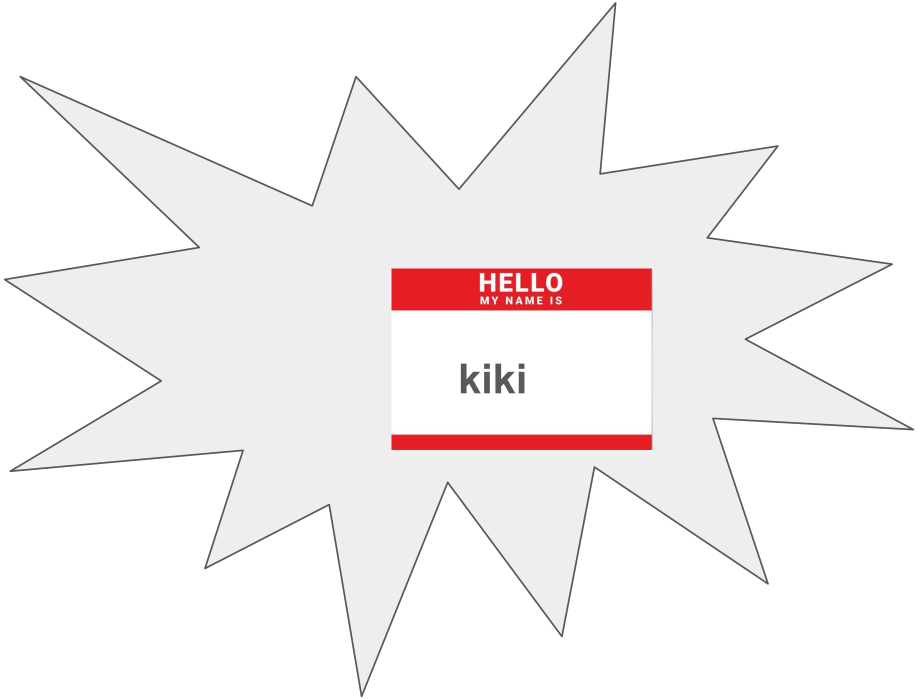
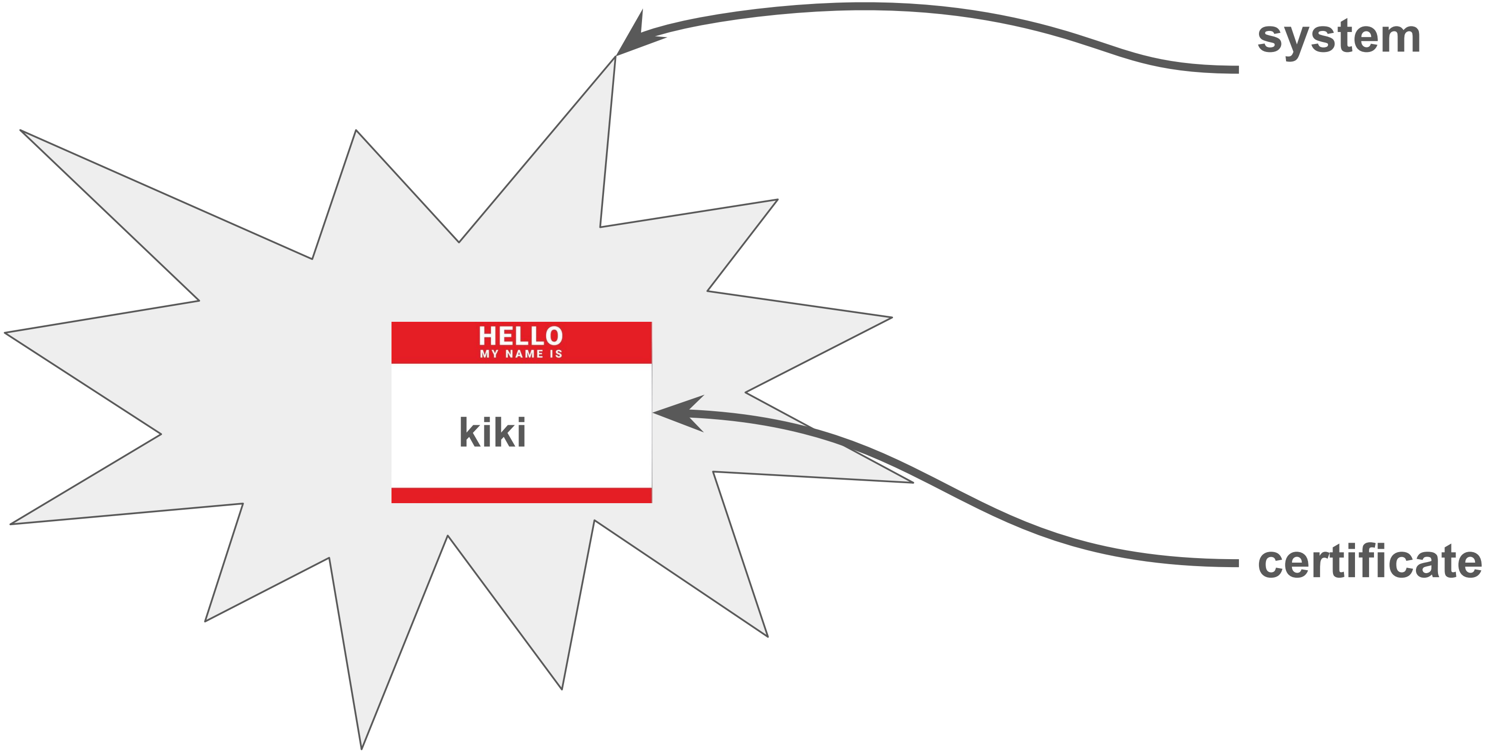

HELLO
MY NAME IS
kiki


thinking cap
this is called
kiki actually
(literal kikiest thing ever)

awesome, “formal”, internal, hard



sloppy, extralogical, informal, easy


system
HELLO
MY NAME IS
kiki
certificate

What are hyperdoctrines?

$$\frac{}{\Gamma \vdash \top \text{ prop}}$$

$$\frac{\Gamma \vdash A \text{ prop} \qquad \Gamma \vdash B \text{ prop}}{\Gamma \vdash A \wedge B \text{ prop}}$$

$$\frac{\Gamma,\ a : A \vdash \phi \text{ prop}}{\Gamma \vdash \exists_{a:A}\phi \text{ prop}}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash a = b \text{ prop}}$$

i’m not going to tell you how the semantics work but you can probably guess 😛

$$
\begin{tikzcd}
& \top & \\
s \text{ involves me working} \arrow[ur] \arrow[rr] & & s \text{ involves me in the office} \arrow[ul]
\end{tikzcd}
$$

$$\begin{tikzcd} S \arrow[r, "z \cdot -"'] & S \end{tikzcd}$$

$$\begin{tikzcd} PS & PS \arrow[l, "P(z \cdot -)"] \end{tikzcd}$$

$$\begin{tikzcd} & \top & \\ z \cdot s \text{ involves me working} \arrow[ur] \arrow[rr] & & z \cdot s \text{ involves me in the office} \arrow[ul] \end{tikzcd}$$

$$
\begin{tikzcd}
S \arrow[rr, "\mathrm{get}"] & & O \\
\text{drinking coffee} & & \text{in the kitchen} \\
\\
P(S) & & P(O) \arrow[ll, "P(\mathrm{get})"'] \\
\exists_{s:S}\big(\mathrm{get}(b_p \cdot s) = \mathrm{get}(s')\big) & & \exists_{s:S}\big(\mathrm{get}(b_p \cdot s) = o\big)
\end{tikzcd}
$$

This guy lives in the category of contexts

$$S \xrightarrow{\text{get}} O$$

drinking coffee

in the kitchen

This guy lives in the category of posets

$$P(S) \xleftarrow{P(\text{get})} P(O)$$

$$\exists_{s:S}\big(\text{get}(b_p \cdot s) = \text{get}(s')\big)$$

$$\exists_{s:S}\big(\text{get}(b_p \cdot s) = o\big)$$

This guy lives in the category of contexts

$$
\begin{tikzcd}
S \arrow[rr, "\mathrm{get}"] & & O
\end{tikzcd}
$$

drinking coffee | in the kitchen

This guy lives in the category of meet semi-lattices or maybe SM(Cat)

$$
\begin{tikzcd}
P(S) & & P(O) \arrow[ll, "P(\mathrm{get})"']
\end{tikzcd}
$$

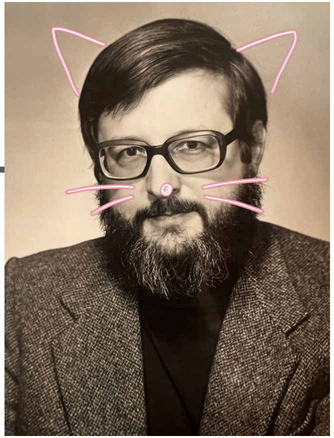
$\exists_{s:S}\big(\mathrm{get}(b_p \cdot s) = \mathrm{get}(s')\big)$ | $\exists_{s:S}\big(\mathrm{get}(b_p \cdot s) = o\big)$

$$\begin{array}{ccc}
S & \xrightarrow{\text{get}} & O \\
\text{drinking coffee} & & \text{in the kitchen} \\
 & & \\
P(S) & \xrightarrow{\exists_{\text{get}}} & P(O) \\
s = \text{drinking coffee} & & \exists_{s:S}(\text{get}(s) = o \wedge s = \text{drinking coffee})
\end{array}$$

$$\exists_{s:S}(\mathrm{get}(\mathrm{s}) = o \ \wedge \phi(s)) \longrightarrow \psi(o)$$

$$\phi(s) \longrightarrow \psi(\mathrm{get}(s))$$

$$\boxed{\exists_f \dashv P(f)}$$



This guy lives in the category of contexts

$$S \xrightarrow{\text{get}} O$$

drinking coffee | in the kitchen

This is good because we can do adjunctions :)

This guy lives in the category of meet semi-lattices or maybe SM(Cat)

$$P(S) \xleftarrow{P(\text{get})} P(O)$$

$$\exists_{s:S}\big(\text{get}(b_p \cdot s) = \text{get}(s')\big) \qquad \exists_{s:S}\big(\text{get}(b_p \cdot s) = o\big)$$

category of contexts; in our special case, this was generated by S, I and O

$$\mathcal{C}^{op} \to SM(\mathrm{Cat})$$

we also need $\wedge, \ \exists$ to play nicely, so we add conditions to this functor P

The Beck-Chevalley condition ensures $\exists_f(\phi \wedge \psi) \dashv\vDash \exists_f \phi \wedge \exists_f \psi$

Frobenius reciprocity ensures $\exists_f(Pf \wedge g) \dashv\vDash f \wedge \exists_f g$

This is a photo I found when I searched ‘hyperdoctrine’. Apparently ‘hyperdoctrine’ also means something to one small sect of terminally online right-wing Calvinists. The image is very gauche; I wanted to include it.

Any such functor P satisfying these properties:

- left adjoint exists for all Pf
- Beck-Chevalley condition
- Frobenius reciprocity

is thus called a

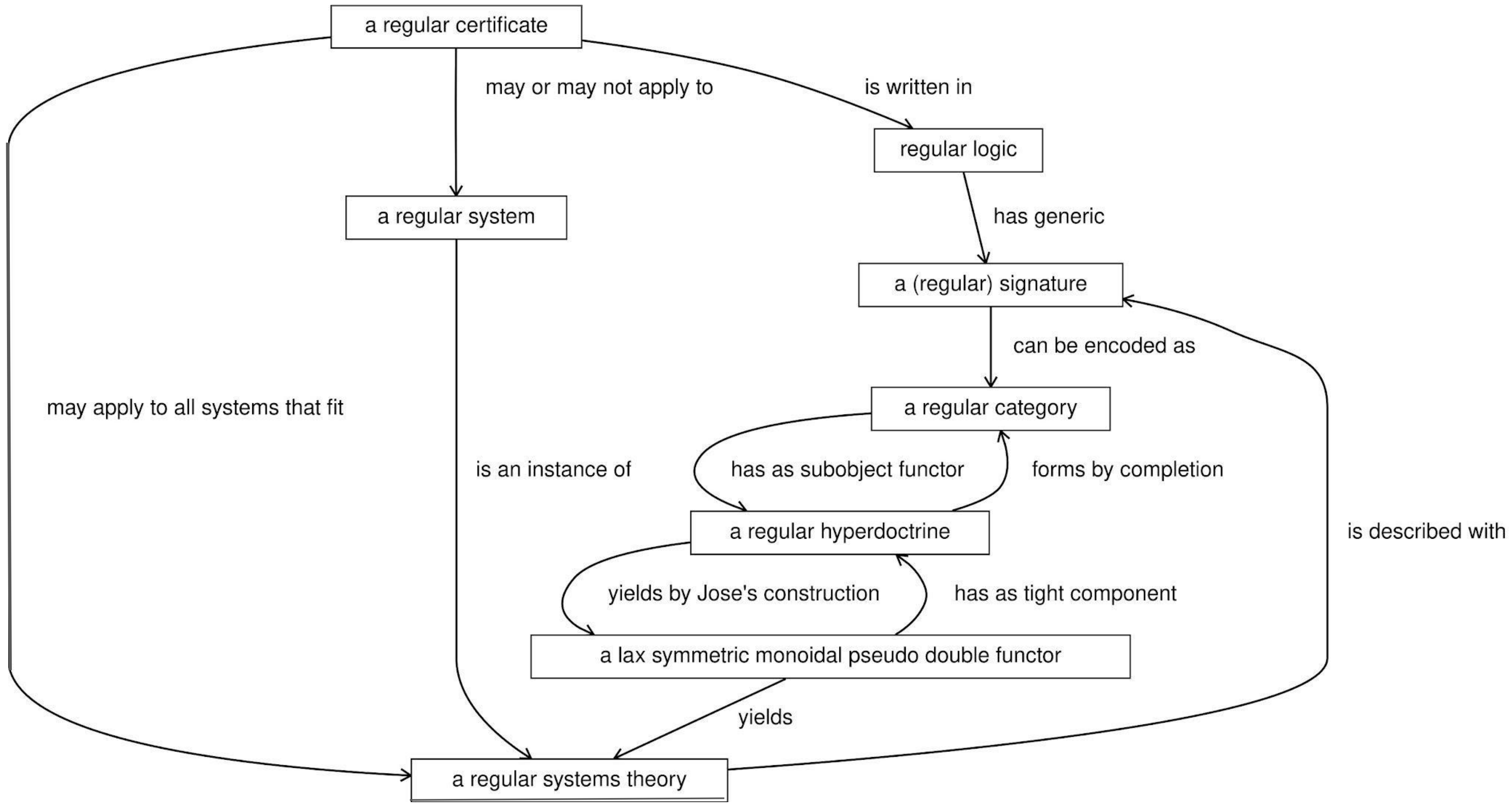
regular hyperdoctrine

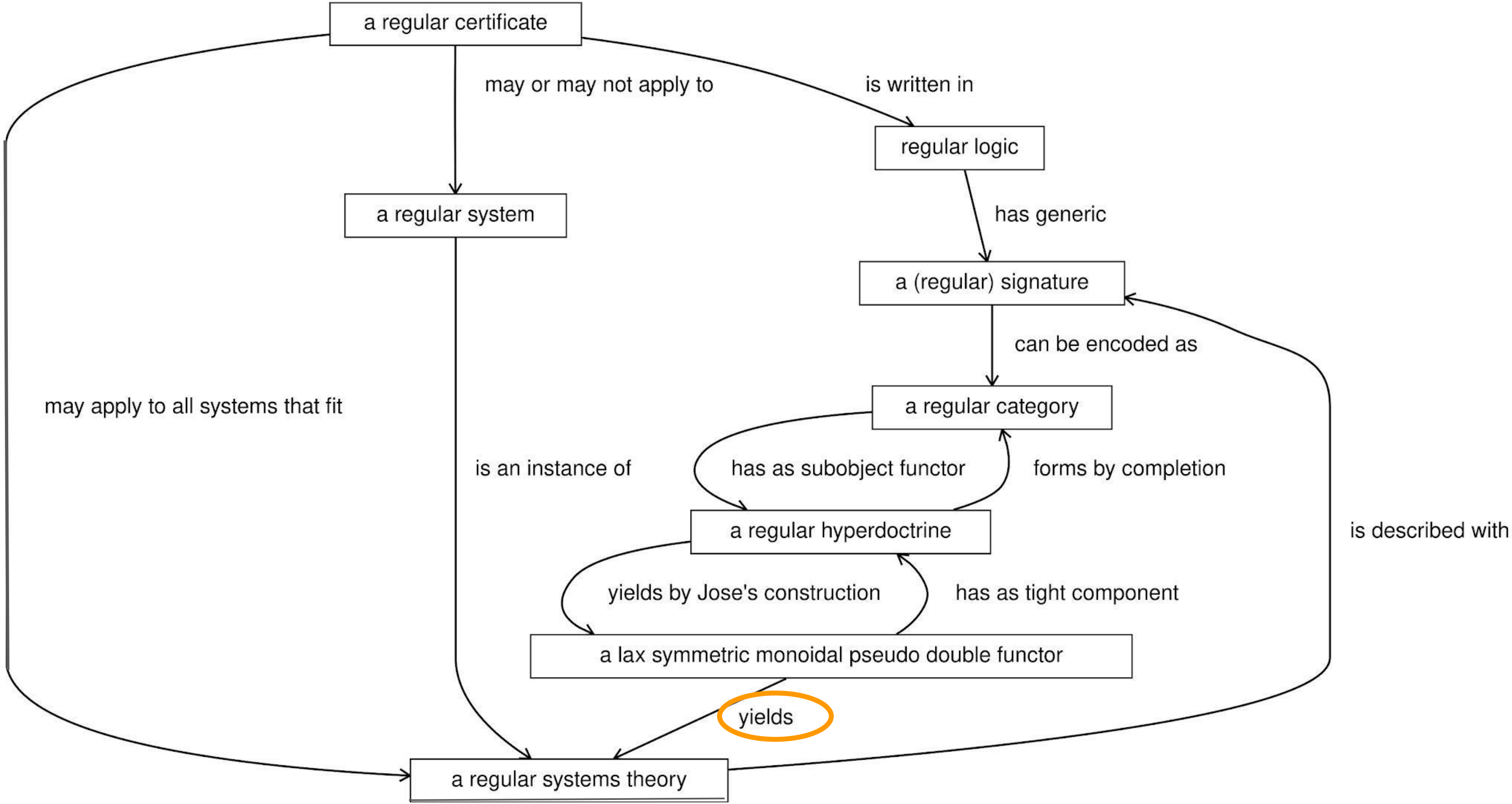
you can make stronger hyperdoctrines (coherent, first-order, Boolean) if you pay for extra bullet points

Theorem: *Let $\mathfrak{C}$ be a cartesian adequate triple and $\mathcal{K}$ be a symmetric monoidal 2-category. Every $\mathfrak{C}$-regular $\mathcal{K}$-hyperdoctrine P can be extended to a lax symmetric monoidal pseudo double functor $P^\bullet : \mathbb{S}pan(\mathfrak{C})^{\mathrm{op}} \to \mathcal{K}$ whose monoidal laxators are companion commuter transformations. Conversely, if the class $\mathfrak{C}^l$ includes all diagonals, then the tight component of every such double functor which is strict is a $\mathfrak{C}$-regular $\mathcal{K}$-hyperdoctrine.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F", "|" marking] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[d, "H"] \\ \mathcal{E} \arrow[r, "I"', "|" marking] & \mathcal{F} \end{tikzcd}$$

$$\hat{\alpha} : IG \Longrightarrow HF$$


a regular certificate
may or may not apply to
is written in
regular logic
a regular system
has generic
a (regular) signature
can be encoded as
may apply to all systems that fit
a regular category
is an instance of
has as subobject functor
forms by completion
a regular hyperdoctrine
is described with
yields by Jose's construction
has as tight component
a lax symmetric monoidal pseudo double functor
yields
a regular systems theory


a regular certificate
may or may not apply to
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is an instance of
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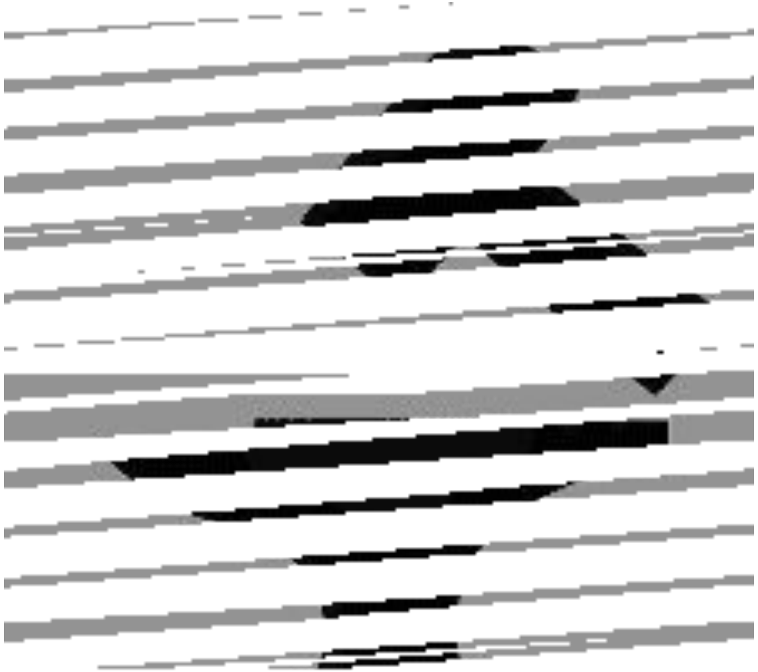
a regular hyperdoctrine
is described with
yields by Jose's construction
has as tight component
a lax symmetric monoidal pseudo double functor
yields
a regular systems theory

What is DOTS?

system $\longrightarrow$ interface

system $\curvearrowleft$ interface interaction

Mix in the other desirable properties of systems (e.g. systems can be compared, so should form a category; systems can be put in parallel, so this category should be monoidal) and you get…



the Double Operadic Theory of Systems

(dots)

$$
\begin{tikzcd}
A \arrow[r, "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & E \arrow[r, "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\aleph"] & B \arrow[r, "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\beta"] & C \arrow[d] \\
I \arrow[r, "\shortmid" marking] \arrow[dr, phantom, "\Downarrow"] & O \arrow[r, "\shortmid" marking] \arrow[dr, phantom, "\Downarrow"] & D \arrow[r, "\shortmid" marking] \arrow[dr, phantom, "\Downarrow"] & F \\
0 \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] & 0 \arrow[r, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "a"] & 1 \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] & 1 \arrow[d, equal] \\
0 \arrow[r, equal, "\shortmid" marking] & 0 \arrow[r, "\shortmid" marking] & 1 \arrow[r, equal, "\shortmid" marking] & 1
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=large, row sep=large]
A \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & E \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\aleph"] & B \arrow[r, "|" marking] \arrow[d] & C \\
I \arrow[r, "|" marking] \arrow[dr, phantom, "\Downarrow"] & O \arrow[r, "|" marking] \arrow[dr, phantom, "\Downarrow"] & D \arrow[dr, phantom, "\Downarrow"] & \\
0 \arrow[r, equals, "|" marking] \arrow[d, equals] & 0 \arrow[r, "|" marking] \arrow[d, equals] \arrow[dr, phantom, "a"] & 1 \arrow[r, equals, "|" marking] \arrow[d, equals] & 1 \arrow[d, equals] \\
0 \arrow[r, equals, "|" marking] & 0 \arrow[r, "|" marking] & 1 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & E \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\aleph"] & B \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\beta"] & C \arrow[d] \\
I \arrow[r, "|" marking] \arrow[dr, phantom, "\Downarrow"] & O \arrow[r, "|" marking] \arrow[dr, phantom, "\Downarrow"] & D \arrow[r, "|" marking] \arrow[dr, phantom, "\Downarrow"] & F \\
0 \arrow[r, equal, "|" marking] \arrow[d, equal] & 0 \arrow[r, "|" marking] \arrow[d, equal] \arrow[dr, phantom, "a"] & 1 \arrow[r, equal, "|" marking] \arrow[d, equal] & 1 \arrow[d, equal] \\
0 \arrow[r, equal, "|" marking] & 0 \arrow[r, "|" marking] & 1 \arrow[r, equal, "|" marking] & 1
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & E \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\aleph"] & B \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\beta"] & C \arrow[d] \\
I \arrow[r, "|" marking] \arrow[dr, phantom, "\Downarrow"] & O \arrow[r, "|" marking] \arrow[dr, phantom, "\Downarrow"] & D \arrow[r, "|" marking] \arrow[dr, phantom, "\Downarrow"] & F \\
0 \arrow[r, equals, "|" marking] \arrow[d, equals] & 0 \arrow[r, "|" marking] \arrow[d, equals] \arrow[dr, phantom, "a"] & 1 \arrow[r, equals, "|" marking] \arrow[d, equals] & 1 \arrow[d, equals] \\
0 \arrow[r, equals, "|" marking] & 0 \arrow[r, "|" marking] & 1 \arrow[r, equals, "|" marking] & 1
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & E \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\aleph"] & B \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\beta"] & C \arrow[d] \\
I \arrow[r, "|" marking] \arrow[dr, phantom, "\Downarrow"] & O \arrow[r, "|" marking] \arrow[dr, phantom, "\Downarrow"] & D \arrow[r, "|" marking] \arrow[dr, phantom, "\Downarrow"] & F \\
0 \arrow[r, equal, "|" marking] \arrow[d, equal] & 0 \arrow[r, "|" marking] \arrow[d, equal] \arrow[dr, phantom, "a"] & 1 \arrow[r, equal, "|" marking] \arrow[d, equal] & 1 \arrow[d, equal] \\
0 \arrow[r, equal, "|" marking] & 0 \arrow[r, "|" marking] & 1 \arrow[r, equal, "|" marking] & 1
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\alpha"] & E \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\aleph"] & B \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\beta"] & C \arrow[d] \\
I \arrow[r, "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow"] & O \arrow[r, "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow"] & D \arrow[r, "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow"] & F \\
0 \arrow[r, equal, "\shortmid"{marking}] \arrow[d, equal] & 0 \arrow[r, "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "a"] & 1 \arrow[r, equal, "\shortmid"{marking}] \arrow[d, equal] & 1 \arrow[d, equal] \\
0 \arrow[r, equal, "\shortmid"{marking}] & 0 \arrow[r, "\shortmid"{marking}] & 1 \arrow[r, equal, "\shortmid"{marking}] & 1
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\alpha"] & E \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\aleph"] & B \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\beta"] & C \arrow[d] \\
I \arrow[r, "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow"] & O \arrow[r, "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow"] & D \arrow[r, "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow"] & F \\
0 \arrow[r, equals, "\shortmid"{marking}] \arrow[d, equals] & 0 \arrow[r, "\shortmid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "a"] & 1 \arrow[r, equals, "\shortmid"{marking}] \arrow[d, equals] & 1 \arrow[d, equals] \\
0 \arrow[r, equals, "\shortmid"{marking}] & 0 \arrow[r, "\shortmid"{marking}] & 1 \arrow[r, equals, "\shortmid"{marking}] & 1
\end{tikzcd}
$$

We can formalise this notion of 'pre-image of the identity square' using the pullback (which exists in Dbl) as

$$
\begin{tikzcd}
\mathbb{D}_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
\mathbb{D} \arrow[r] & \mathbb{L}\mathrm{oose}
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & E \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\aleph"] & B \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\beta"] & C \arrow[d] \\
I \arrow[r, "|" marking] \arrow[dr, phantom, "\Downarrow"] & O \arrow[r, "|" marking] \arrow[dr, phantom, "\Downarrow"] & D \arrow[r, "|" marking] \arrow[dr, phantom, "\Downarrow"] & F \\
0 \arrow[r, equal, "|" marking] \arrow[d, equal] & 0 \arrow[r, "|" marking] \arrow[d, equal] \arrow[dr, phantom, "a"] & 1 \arrow[r, equal, "|" marking] \arrow[d, equal] & 1 \arrow[d, equal] \\
0 \arrow[r, equal, "|" marking] & 0 \arrow[r, "|" marking] & 1 \arrow[r, equal, "|" marking] & 1
\end{tikzcd}
$$

$$\mathbb{D}_0 \overset{\mathbb{M}}{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathbb{D}_1$$

Definition 3.13 (Carrier of a loose bimodule)**.** Let $\ell : \mathbb{M} \to \mathbb{L}\text{oose}$ be a loose bimodule. Its carrier $\mathsf{Car}(\mathbb{M})$ is the category whose:

- Objects are loose morphisms of $\mathbb{M}$ living over the walking loose arrow. We refer to such loose morphisms as **heteromorphisms**.
- Morphisms are squares of $\mathbb{M}$ living over the identity square of the walking loose arrow.

$$
\begin{tikzcd}
A \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\alpha"] & E \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\aleph"] & B \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\beta"] & C \arrow[d] \\
I \arrow[r, "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow"] & O \arrow[r, "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow"] & D \arrow[r, "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow"] & F \\
0 \arrow[r, equal, "\shortmid"{marking}] \arrow[d, equal] & 0 \arrow[r, "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "a"] & 1 \arrow[r, equal, "\shortmid"{marking}] \arrow[d, equal] & 1 \arrow[d, equal] \\
0 \arrow[r, equal, "\shortmid"{marking}] & 0 \arrow[r, "\shortmid"{marking}] & 1 \arrow[r, equal, "\shortmid"{marking}] & 1
\end{tikzcd}
$$

$$\mathrm{Mor}(\mathbb{D}_0) \rightharpoondown \mathrm{Car}(\mathbb{M}) \leftharpoondown \mathrm{Mor}(\mathbb{D}_1)$$

Where the left action is given by precomposition and the right action is given by postcomposition :P this is the LOOSE HOM BIMODULE $\mathbb{D}(-,-)$

$$
\begin{tikzcd}
A \arrow[r, "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & B \arrow[d] \\
C \arrow[r, "\shortmid" marking] & D
\end{tikzcd}
\quad \times \quad
\begin{tikzcd}
0 \arrow[r, "f", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & 1 \arrow[d, equal] \\
0 \arrow[r, "f"', "\shortmid" marking] & 1
\end{tikzcd}
\quad =_{\text{kinda}} \quad
\begin{tikzcd}
& A \times 1 \arrow[rr, "\shortmid" marking] \arrow[dd] & & B \times 1 \arrow[dd] \\
A \times 0 \arrow[ur] \arrow[rr, "\shortmid" marking] \arrow[dd] & & B \times 0 \arrow[ur] \arrow[dd] & \\
& C \times 1 \arrow[rr, "\shortmid" marking] & & D \times 1 \\
C \times 0 \arrow[ur] \arrow[rr, "\shortmid" marking] & & D \times 0 \arrow[ur] &
\end{tikzcd}
$$

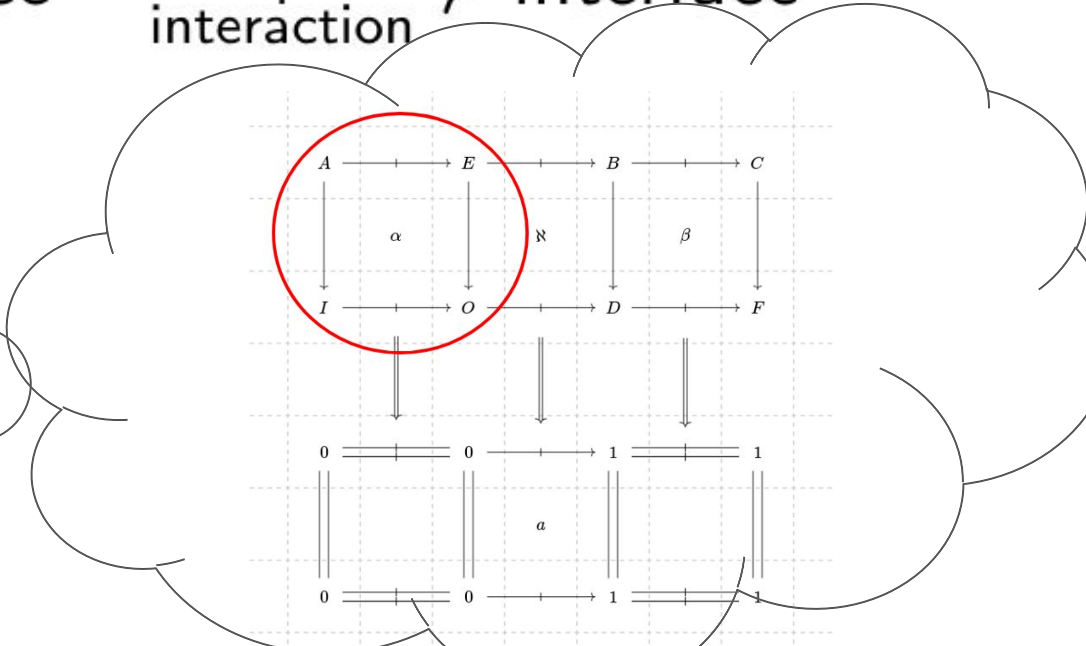
$$\mathbb{D}(-,-) : \mathbb{D} \nrightarrow \mathbb{D}$$

$$\begin{tikzcd}
\bullet \arrow[r, "\text{system}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\text{system map}"] & \text{interface} \arrow[r, "\text{interaction}", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\text{interaction map}"] & \text{interface} \arrow[d, "\text{interface map}"] \\
\bullet \arrow[r, "\text{system}"', "\shortmid"{marking}] & \text{interface} \arrow[r, "\text{interaction}"', "\shortmid"{marking}] & \text{interface}
\end{tikzcd}$$

nothingburger

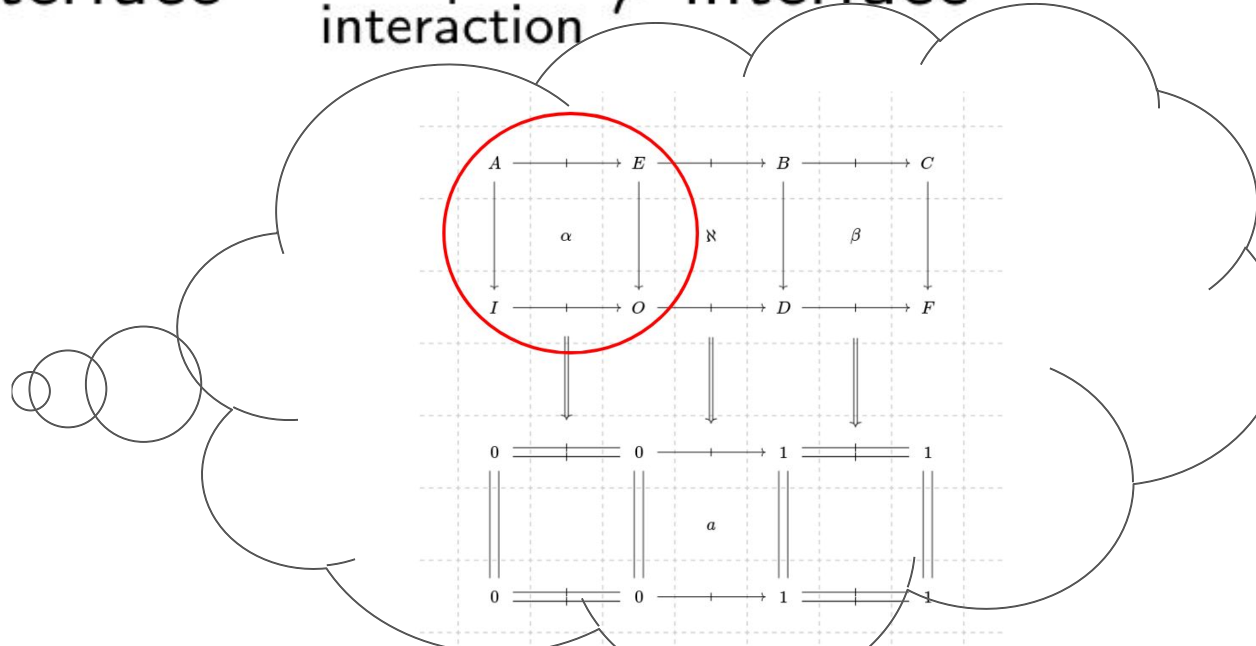
$$
\begin{tikzcd}
\bullet \arrow[r, "\text{system}"] \arrow[d, equal] \arrow[dr, phantom, "\text{system map}"] & \text{interface} \arrow[r, "\text{interaction}"] \arrow[d] \arrow[dr, phantom, "\text{interaction map}"] & \text{interface} \arrow[d, "\text{interface map}"] \\
\bullet \arrow[r, "\text{system}"'] & \text{interface} \arrow[r, "\text{interaction}"'] & \text{interface}
\end{tikzcd}
$$

nothingburger



$$
\begin{tikzcd}
\bullet \arrow[r, "\text{system}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\text{system map}"] & \text{interface} \arrow[r, "\text{interaction}", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\text{interaction map}"] & \text{interface} \arrow[d, "\text{interface map}"] \\
\bullet \arrow[r, "\text{system}"', "\shortmid"{marking}] & \text{interface} \arrow[r, "\text{interaction}"', "\shortmid"{marking}] & \text{interface}
\end{tikzcd}
$$

$$\bullet \overset{\mathbb{M}}{\nrightarrow} \mathbb{D}_1$$



$$
\begin{tikzcd}
\mathbb{E}_0 \arrow[r, dashed, "\mathbb{M}(F_0, F_1)"] \arrow[d, "F_0"'] & \mathbb{E}_1 \arrow[d, "F_1"] \\
\mathbb{D}_0 \arrow[r, "\mathbb{M}", "|"{marking}] & \mathbb{D}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, dashed, "\mathbb{M}(\mathrm{id}_\bullet, F)"] \arrow[d, "\mathrm{id}_\bullet"'] & \mathbb{E} \arrow[d, "F"] \\
\bullet \arrow[r, "\mathbb{M}", "|"{marking}] & \mathbb{D}_1
\end{tikzcd}
$$

So, we should aim to make a systems theory $\mathbb{M}$ out of regular logic in this way. What could our functor be?

Theorem: *Let $\mathfrak{C}$ be a cartesian adequate triple and $\mathcal{K}$ be a symmetric monoidal 2-category. Every $\mathfrak{C}$-regular $\mathcal{K}$-hyperdoctrine P can be extended to a lax symmetric monoidal pseudo double functor $P^{\bullet} : \mathbb{S}pan(\mathfrak{C})^{\mathrm{op}} \to \mathcal{K}$ whose monoidal laxators are companion commuter transformations. Conversely, if the class $\mathfrak{C}^{l}$ includes all diagonals, then the tight component of every such double functor which is strict is a $\mathfrak{C}$-regular $\mathcal{K}$-hyperdoctrine.*

(1) We have a loose right module $\bullet \nrightarrow \mathbb{Q}\mathrm{t}(\mathrm{Cat})$ given by flattening the loose hom bimodule $\mathbb{Q}\mathrm{t}(\mathrm{Cat}) \nrightarrow \mathbb{Q}\mathrm{t}(\mathrm{Cat})$.

(2) We restrict this loose right module along Jose's special functor $P^{\bullet}$.

(3) This gives us a new loose right module $\bullet \nrightarrow \mathbb{S}\mathrm{pan}(\mathcal{C}^{op})$.

(4) We just need to verify that the carrier of this module is symmetric monoidal.

2. Similarly, restriction of a symmetric monoidal loose bimodule by *lax* symmetric monoidal pseudofunctors (whose unitors and laxators are conjoint or companion commuter transformations, respectively) will yield a symmetric monoidal loose bimodule.

$$\begin{tikzcd}
\bullet \arrow[r, dashed, "\mathbb{M}(\mathrm{id}_\bullet, P^\bullet)"] \arrow[d, "\mathrm{id}_\bullet"'] & \mathbb{S}\mathrm{pan}(\mathcal{C})^{op} \arrow[d, "P^\bullet"] \\
\bullet \arrow[r, "\mathbb{M}", "|"] & \mathbb{Q}\mathbf{t}(\mathbf{Cat})
\end{tikzcd}$$

Explicitly, we will have $\mathsf{Car}(\mathbb{M}(F_0, F_1))$ defined as follows:

- Objects are triples $(e_0 : \mathsf{ob}(\mathbb{E}_0), e_1 : \mathsf{ob}(\mathbb{E}_1), m : F_0e_0 \nrightarrow F_1e_1)$.
- A morphism $(e_0, e_1, m) \to (e_0', e_1', m')$ is a triple $(f_0 : e_0 \to e_0', f_1 : e_1 \to e_1', \alpha)$ where α is a square in $\mathbb{M}$ of the following form:

$$\begin{tikzcd}
F_0e_0 \arrow[r, "m", "|"'] \arrow[d, "F_0f_0"'] \arrow[dr, phantom, "\Downarrow\alpha"] & F_1e_1 \arrow[d, "F_1f_1"] \\
F_0e_0' \arrow[r, "m'"', "|"] & F_1e_1
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r, dashed, "\mathbb{M}(\mathrm{id}_\bullet, P^\bullet)"] \arrow[d, "\mathrm{id}_\bullet"'] & \mathbb{S}\mathrm{pan}(\mathcal{C})^{op} \arrow[d, "P^\bullet"] \\
\bullet \arrow[r, "\mathbb{M}", "|" marking] & \mathbb{Q}\mathrm{t}(\mathrm{Cat})
\end{tikzcd}$$

Combining this, we find that objects are pairs

$$(X : \mathrm{ob}(\mathcal{C}),\ \phi \in PX)$$

and morphisms are substitution:

$$(X, \phi) \xrightarrow{f} (Y, \psi) \text{ when } \psi(y) \implies \phi(f(y))$$

Explicitly, we will have $\mathsf{Car}(\mathbb{M}(F_0, F_1))$ defined as follows:

- Objects are triples $(e_0 : \mathsf{ob}(\mathbb{E}_0), e_1 : \mathsf{ob}(\mathbb{E}_1), m : F_0 e_0 \nrightarrow F_1 e_1)$.
- A morphism $(e_0, e_1, m) \to (e'_0, e'_1, m')$ is a triple $(f_0 : e_0 \to e'_0, f_1 : e_1 \to e'_1, \alpha)$ where α is a square in $\mathbb{M}$ of the following form:

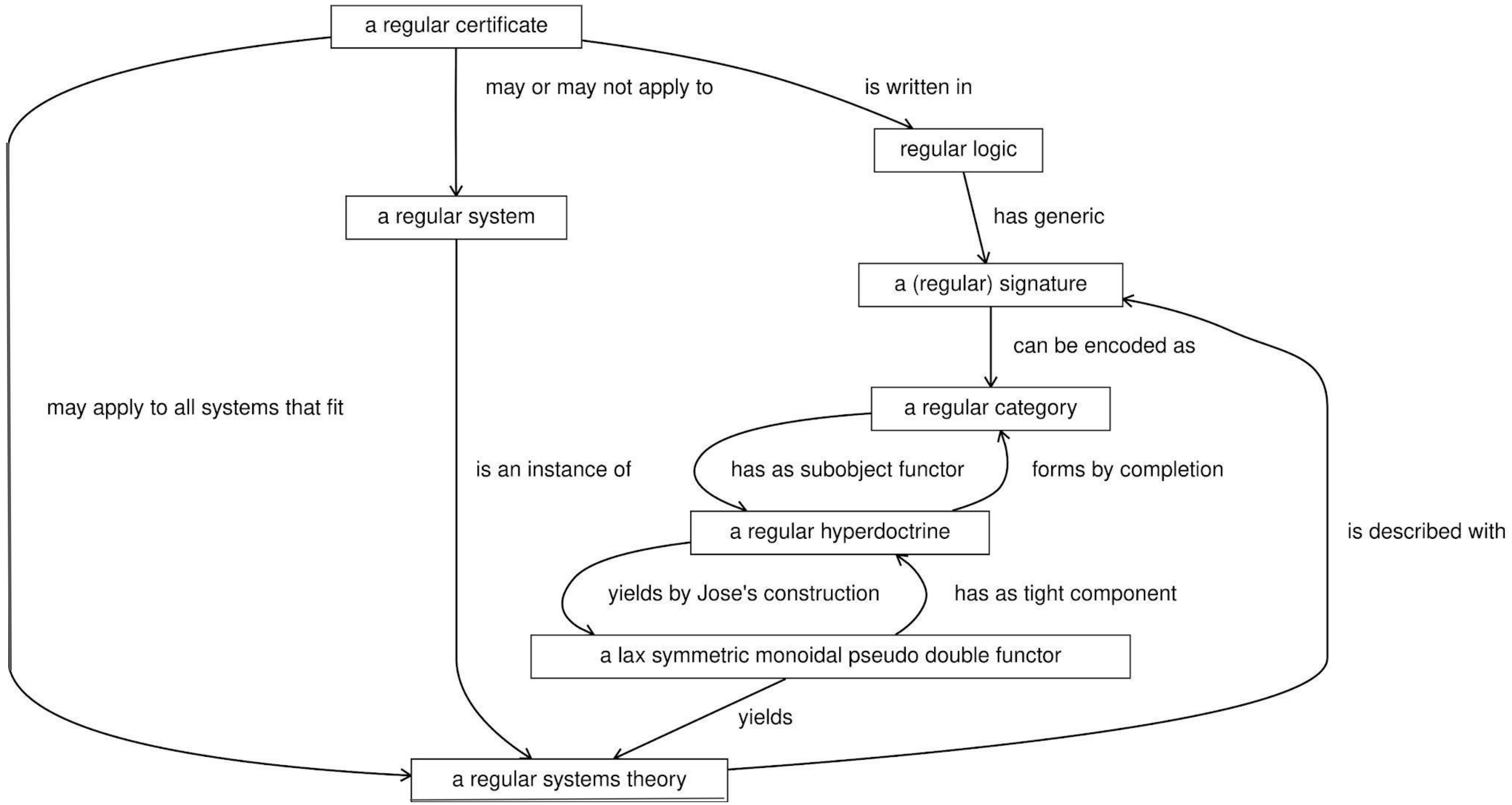
$$\begin{tikzcd}
F_0 e_0 \arrow[r, "m", "|" marking] \arrow[d, "F_0 f_0"'] \arrow[dr, phantom, "\Downarrow\alpha"] & F_1 e_1 \arrow[d, "F_1 f_1"] \\
F_0 e'_0 \arrow[r, "m'"', "|" marking] & F_1 e_1
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "T"] \arrow[dr, "\mathbb{M}"'] & \mathbb{S}\mathrm{pan}(\mathcal{C})^{op} \arrow[d, "\mathbb{C}\mathrm{ert}(P)"] \\
& \mathbb{L}\mathrm{oose}
\end{tikzcd}
$$

As double categories

$$
\mathbb{M} \xrightarrow{T} \mathbb{C}\mathrm{ert}(P)
$$

As systems theories


a regular certificate
may or may not apply to
is written in
regular logic
a regular system
has generic
a (regular) signature
can be encoded as
may apply to all systems that fit
a regular category
is an instance of
has as subobject functor
forms by completion
a regular hyperdoctrine
is described with
yields by Jose's construction
has as tight component
a lax symmetric monoidal pseudo double functor
yields
a regular systems theory

Next steps

THEORY

- Generalise this procedure to logics stronger than regular logic (Jose is working on this right now)
- Understand+explain what's happening (on the level of type assignment) when we have a map of systems theories into $\mathbb{C}\mathrm{ert}(P)$

PEDAGOGY

- Make a bunch of examples of specific certificates which hold of specific Moore machines, and demonstrate that they formally hold iff the desired property actually holds (strong parallel with proof-by-internal-language in topos theory). And get some literature written explaining this procedure in full depth/detail!

Thank you for listening!