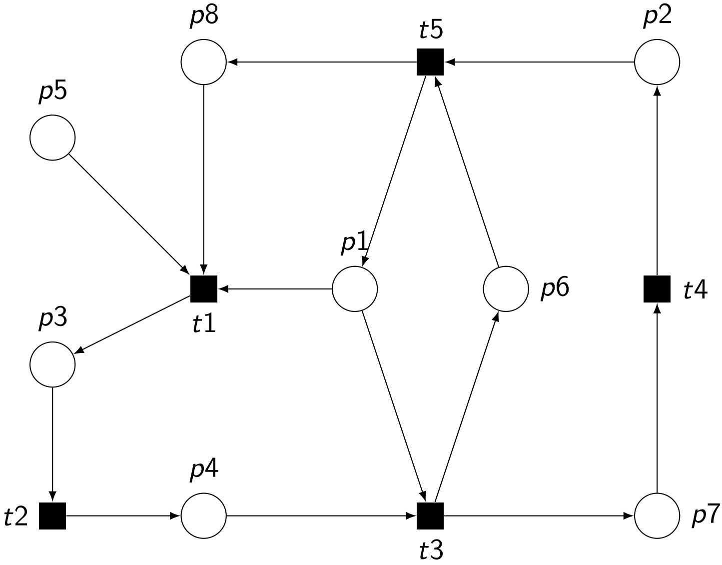
An Overview of Petri Net Theory

Adrián Puerto Aubel

with excerpts from
Eike Best and Raymond Devillers. *Petri Net Primer - A Compendium on the Core Model, Analysis, and Synthesis*. Springer, 2024. URL: https://doi.org/10.1007/978-3-031-48278-6

"An excellent undergraduate textbook!!"

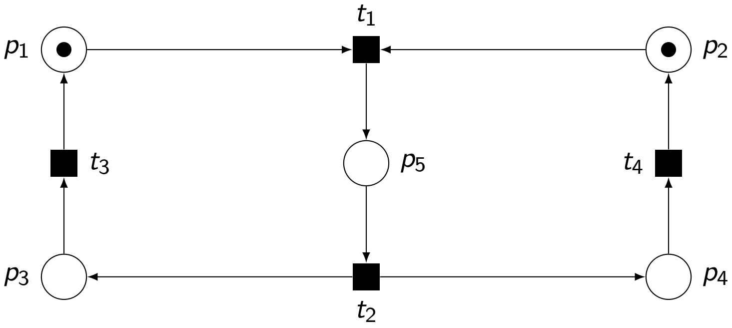

p8
t5
p2
p5
p1
p6
t4
t1
p3
p4
t2
t3
p7

Most Basic Model

Elementary Net Systems

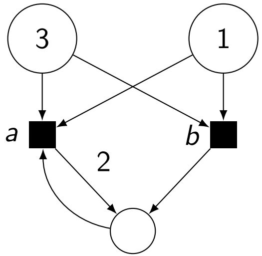
An E.N.S. is a tuple $\Sigma = (B, E, \mathcal{F}, m_0)$

- B: Places $\rightarrow$ Conditions
- E: Transitions $\rightarrow$ Events
- $\mathcal{F} \subseteq (B \times E \cup E \times B)$: Arrows $\rightarrow$ Flow relation
- $m_0 : B \rightarrow \{0, 1\}$: Initial Marking (Global State)



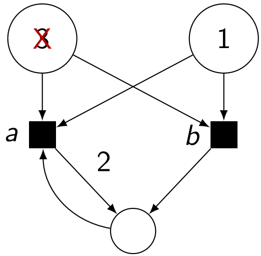
Structural Properties

Restricted Classes:



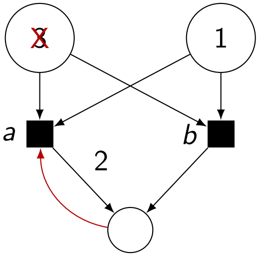
Restricted Classes:

- Safe:
  Only one token per place



Restricted Classes:

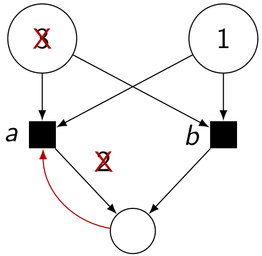
- Safe:
  Only one token per place
- Pure:
  Only consume or produce but not both



Structural Properties

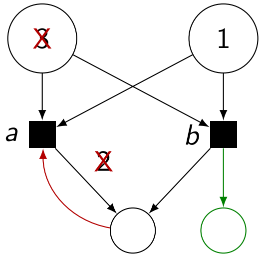
Restricted Classes:

- Safe:
  Only one token per place
- Pure:
  Only consume or produce but not both
- Plain:
  All weights = 1



Restricted Classes:

- Safe:
  Only one token per place
- Pure:
  Only consume or produce but not both
- Plain:
  All weights $= 1$
- Simple:
  All transitions have different effects



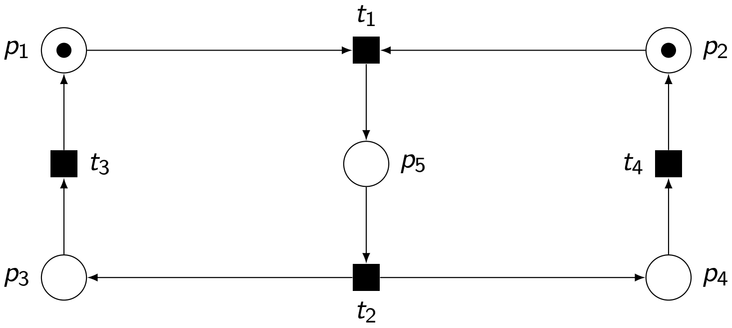
E.N.S. Dynamics

Enabled Events

Event e is enabled at marking m: $m[e\rangle$ iff:

- each $b \in {}^\bullet e$ has $m(b) = 1$ (all pre-conditions are true), AND
- each $b \in e^\bullet$ has $m(b) = 0$ (all post-conditions are false)

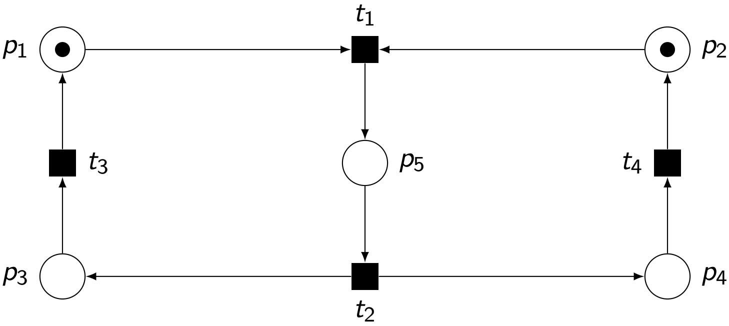
Two events e_1, e_2 are *independent* iff $({}^\bullet e_1 \cup e_1^\bullet) \cap ({}^\bullet e_2 \cup e_2^\bullet) = \emptyset$ Note: several events can be enabled at the same marking.



E.N.S. Dynamics

Firing Rule

- When an event is enabled, it may *fire*:
- $m_1[e\rangle m_2$ means that:
  - e is enabled at m_1, AND
  - $m_2 = (m_1 \setminus {}^{\bullet}e) \cup e^{\bullet}$



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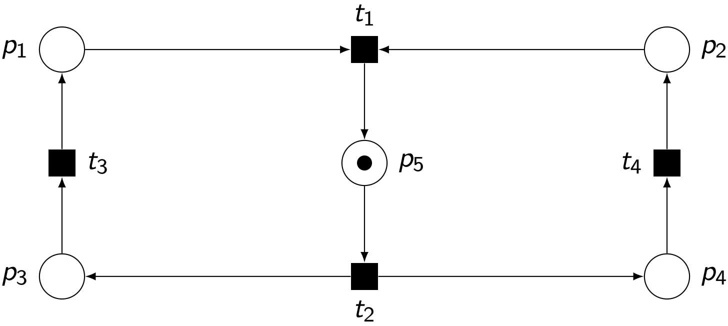

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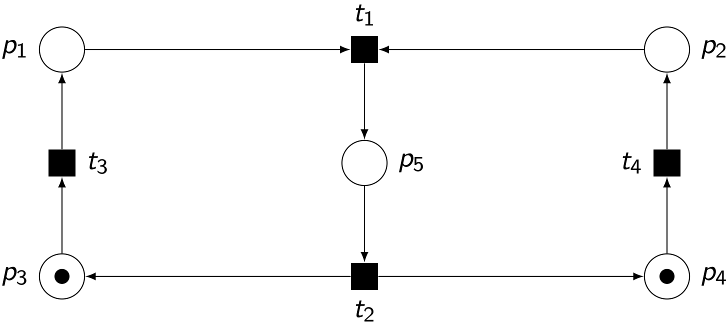


Structural Properties: Confusion

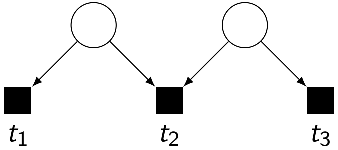
Analysis fails or is more difficult if Conflict interacts with Concurrency:

Confusion

- at m_1, t_1 and t_3 are concurrent, AND
- t_1 is in choice with t_2, BUT
- firing t_3 disables t_2.
- $m_1[t_3\rangle m_2$
- at m_2 the choice has disappeared.

"One component had a choice, but a remote action decided instead"

Confusion makes analysis more difficult.

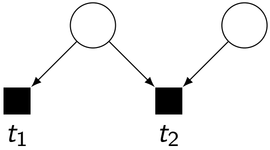


Structural Properties: Free Choice

Free-Choice Nets

To avoid confusion, we can build our model so that:

- $\forall t_1, t_2 \in T : {}^\bullet t_1 \cap {}^\bullet t_2 \neq \emptyset \Rightarrow {}^\bullet t_1 = {}^\bullet t_2$
- If two transitions share a pre-condition, they must share all their pre-conditions



Better Analysis

For most verification problems, better algorithms exist for Free-Choice nets.

Structural Properties: Free Choice

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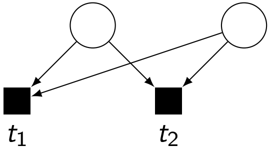


Better Analysis

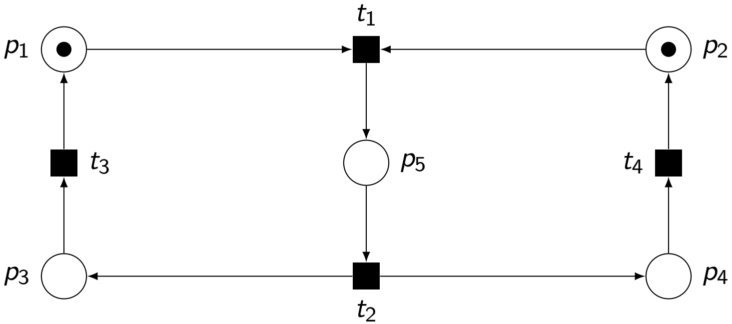
For most verification problems, better algorithms exist for Free-Choice nets.

E.N.S. Dynamics

Neighbourhoods of Events

Given an Event $e \in E$

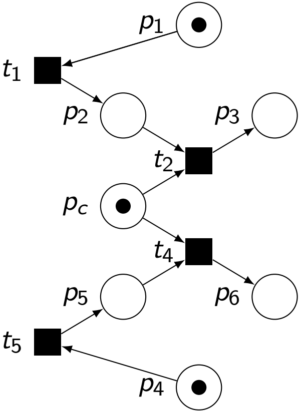
- Pre-conditions: ${}^{\bullet}e = \{b \in B \mid (b, e) \in \mathcal{F}\}$
  conditions that "feed" the event
- Post-conditions: $e^{\bullet} = \{b \in B \mid (e, b) \in \mathcal{F}\}$
  conditions which are "fed" by the event.
- Two events e_1, e_2 are *independent* iff $({}^{\bullet}e_1 \cup e_1^{\bullet}) \cap ({}^{\bullet}e_2 \cup e_2^{\bullet}) = \emptyset$

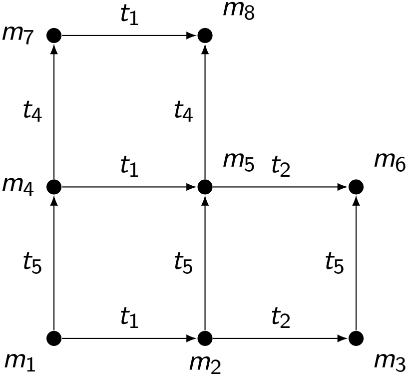


Semantics: Marking Graph (Reachability Graph)

Marking Graph

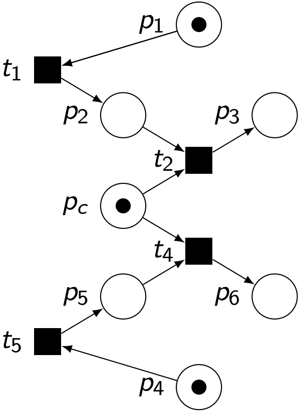
- Each node is a marking m
- We add an arc (m_1, m_2) with label e, if $m_1[e\rangle m_2$.

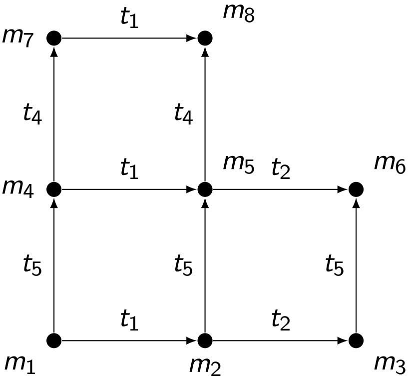




Notions

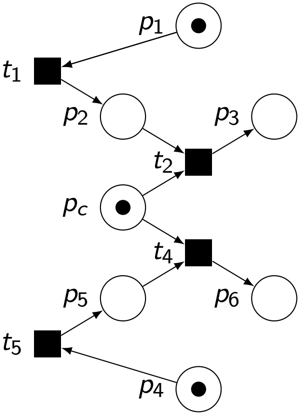
- concurrent = enabled at m + independent
- choice = enabled at m + NOT independent

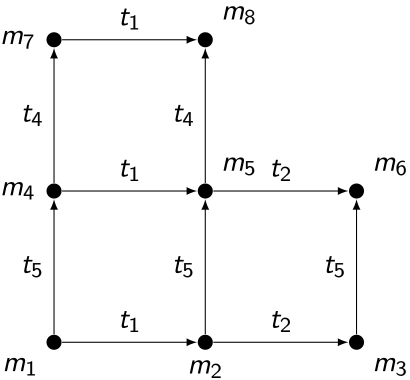




Notions

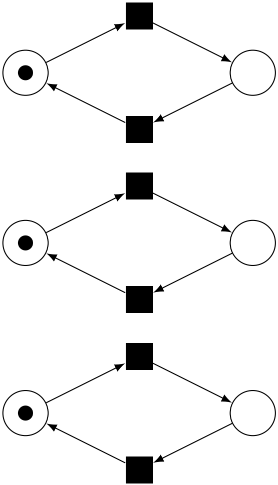
- deadlock
- liveness



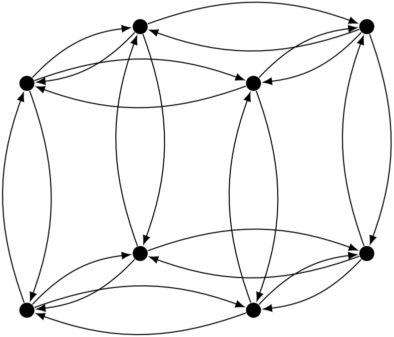


State Space Explosion

K components of size N



Size $= N * K$



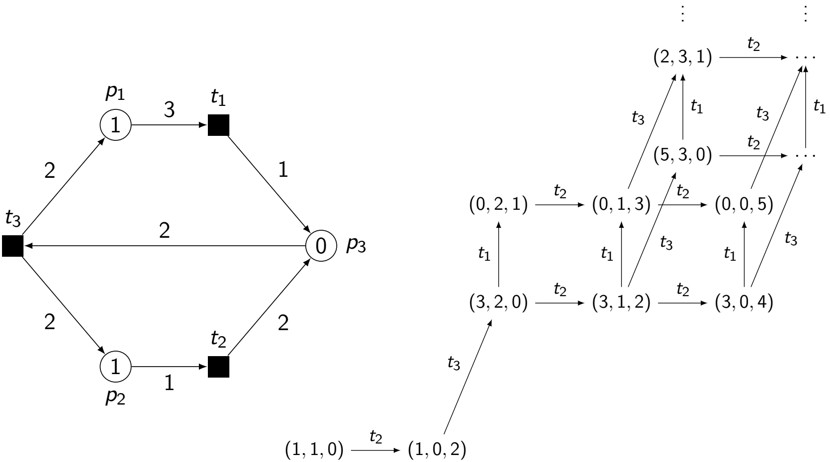
Size$= N^K$

Place Transition Systems: Definitions

Place Transition Systems

- P: Places $\rightarrow$ Counters
- T: Transitions $\rightarrow$ Consume and Produce
- $\mathcal{F} : (B \times E \cup E \times B) \rightarrow \mathbb{N}$: Arcs are now weighted
- $m : P \rightarrow \mathbb{N}$: Marking assigns a number to each place
- Firing rule: $m_1[t\rangle m_2$
  - $\forall p \in P : \mathcal{F}(p, t) \leq m_1(p)$
    Places have enough tokens for the transition to fire, AND
  - $\forall p \in P : m_2(p) = m_1(p) - \mathcal{F}(p, t) + \mathcal{F}(t, p)$
    The weights on the arcs indicate how much is consumed and produced.

Place Transition Systems are VAS

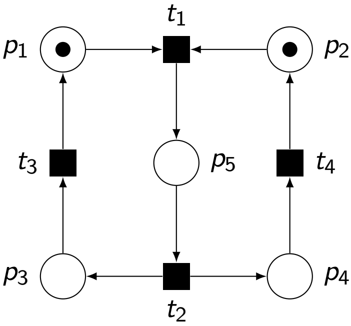


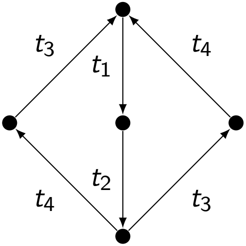
$t_1 = (-3, 0, 1)$; $t_2 = (0, -1, 2)$; $t_3 = (2, -2, 0)$

Note: Unbounded Behaviour!

Notions

- firing sequence, reachability
- interleaving (steps)





firing sequence:

Semantics: Mazurkiewicz Traces

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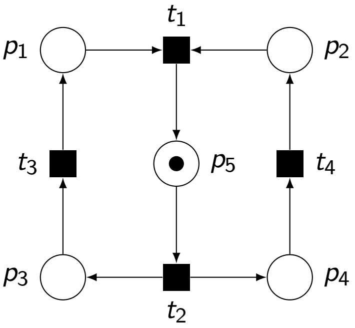
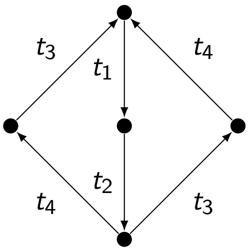


firing sequence: t_1



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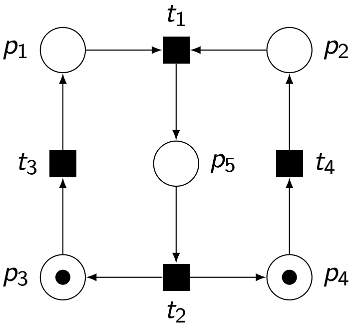
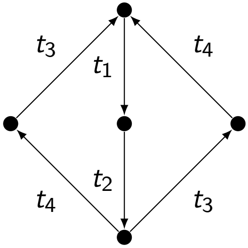


firing sequence: $t_1 t_2$



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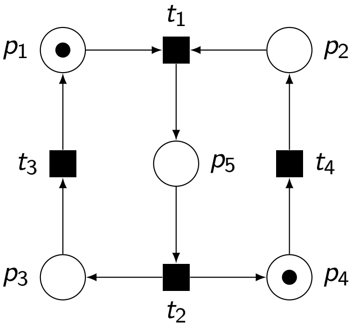
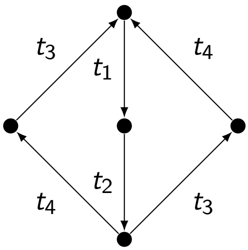


firing sequence: $t_1 \, t_2 \, t_3$



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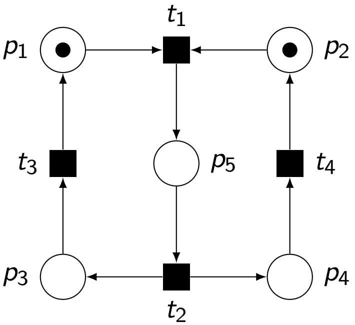

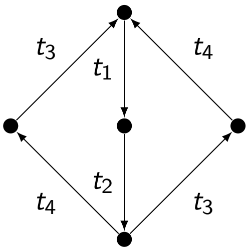

firing sequence: $t_1\, t_2\, t_3\, t_4 \ldots$

Semantics: Mazurkiewicz Traces

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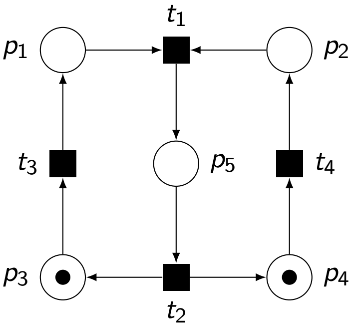

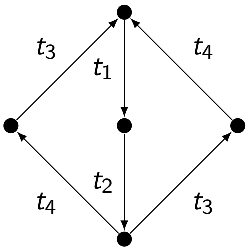

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$t_1 t_2$

Semantics: Mazurkiewicz Traces

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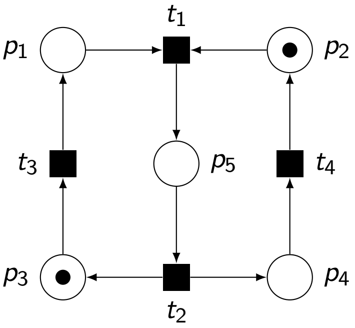

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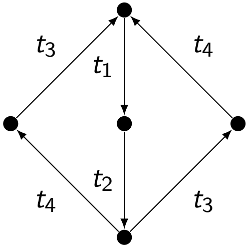
$t_1\, t_2\, t_4$



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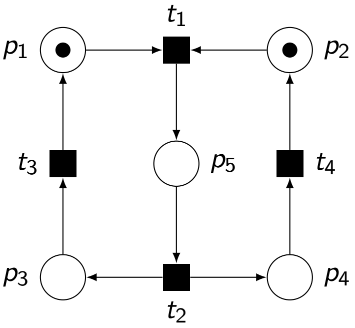

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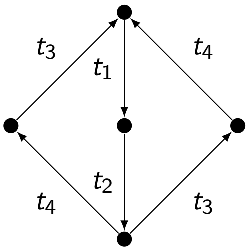

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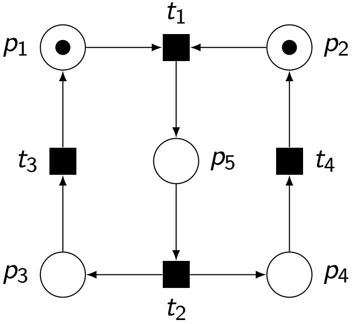

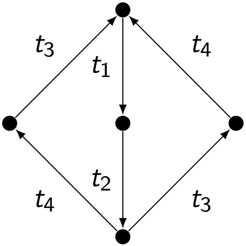

firing sequence: $t_1\, t_2\, t_3\, t_4 \ldots$

$t_1\, t_2\, t_4\, t_3 \ldots$

trace: $t_1\, t_2 \{ t_3, t_4 \} \ldots$

Semantics: Mazurkiewicz Traces

Notions

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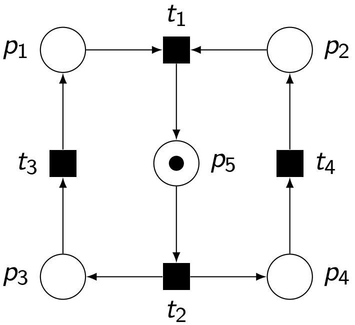
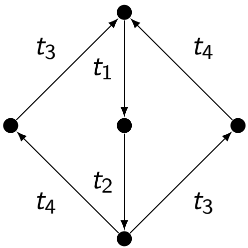
- interleaving (steps)





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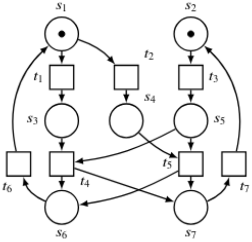
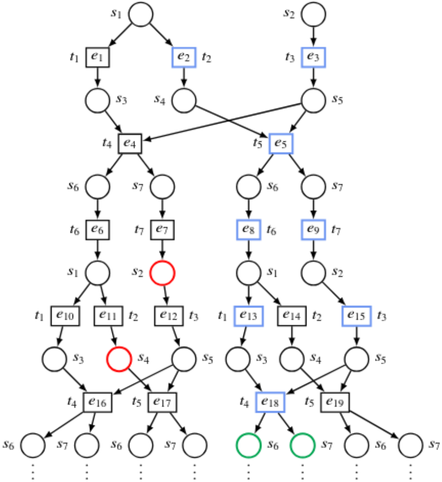
$t_1 t_2 t_4 t_3 t_1 \ldots$

trace: $t_1 t_2 \{t_3, t_4\} t_1 \ldots$

Semantics: Unfoldings





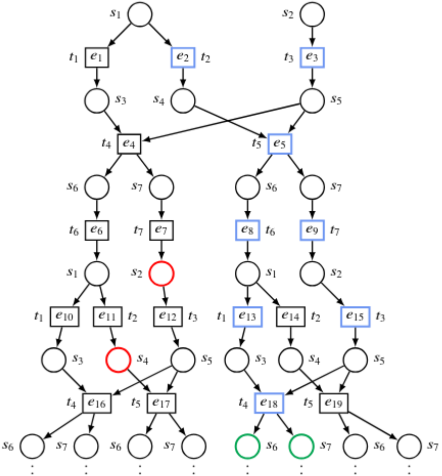
Semantics: Event Structures

Definition CAUSALITY, CONFLICT, AND PARALLELISM RELATIONS

Let (S, T, F) be a (possibly infinite) plain net and let $x_1, x_2 \in S \cup T$.

- *Causality:* $x_1 < x_2$ if there is a nonempty directed path from x_1 to x_2.
- *Conflict:* $x_1 \# x_2$, if there are a place s and two directed paths $s\, t_1 \ldots x_1$ and $s\, t_2 \ldots x_2$ with $t_1 \neq t_2$.
- *Parallelism:* $x_1\, co\, x_2$ if neither $x_1 < x_2$ nor $x_2 < x_1$ nor $x_1 \# x_2$.

A net is *acyclic* if $<$ is a partial order.



Problems

Boundedness and Coverability

A P/T net system is *bounded* iff its set of reachable markings is finite.

- General case: EXPSPACE (in $|P|$) and EXPSPACE-complete ,
- PTIME for conflict-free nets (no choices)

Decision Problems

Reachability

Given a P/T net system with initial marking m_0 and target marking m_t, decide whether m_t is reachable from m_0.

- In general, decidable but primitive recursive space!
- Undecidable if we allow for (at least) two zero-test arcs.
- 2EXPTIME if $|P| \leq 5$
- PSPACE-complete for 1-safe nets ($\simeq$ E.N.S)
- NP-complete for nets without cycles, and also for conflict-free nets (no choices).
- PTIME for bounded conflict-free nets
- PTIME for marked graphs
- PTIME for nets that are live, bounded, cyclic and free-choice.

Decision Problems

Liveness

Deciding whether for any transition t, and from any reachable marking, there is another reachable marking that enables t.

- In general, primitive recursive equivalent to reachability, hence decidable.
- General complexity is an open problem.
- PSPACE-complete for 1-safe nets.
- co-NP-complete free-choice nets.
- PTIME for bounded bounded free-choice nets
- PTIME for conflict-free nets

Decision Problems

Deadlock-freedom

A net is *deadlock-free* iff every reachable marking enables some transition.

- In general, reduction to reachability in PTIME
- PSPACE-complete for 1-safe nets
- NP-complete 1-safe free-choice nets
- PTIME for conflict-free nets.

Definition: Incidence Matrix

Let $N = (B, T, F)$ be a net. The *incidence matrix* of N is defined as the function $C : B \times T \to \mathbb{Z}$ such that $C(b,t) = F(t,b) - F(b,t)$.

Structural Boundedness, Structural Liveness

A net $N = (B, T, F)$ is called

- *structurally bounded*, if for all marking $M \in \mathbb{N}^B$, the net system (B, T, F, M) is bounded.
- *structurally live*, if for all marking $M \in \mathbb{N}^B$, the net system (B, T, F, M) is live.

A taste of Structural Analysis

Characterisation of Structural Boundedness

N is structurally bounded iff there is a vector $x \in \mathbb{N}^B$ with $\forall b \in B \, . \, x(b) > 0$ and $C^t \cdot x \leq 0$.

Existence of an Infinite Firing Sequence

Let $N = (B, T, F)$ be a net. Then there is some marking M such that (B, T, F, M) has an infinite firing sequence iff there is a vector $y \in \mathbb{N}^B$ with $y > 0$ and $C \cdot y \geq 0$.

Structural Analysis: A rich tractable theory

Chapters 4,5,6,7

Best and Devillers, *Petri Net Primer - A Compendium on the Core Model, Analysis, and Synthesis*