A Second Taste of Quantitative Logic

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What is quantitative logic?

In the words of Lawvere '73 paper on *generalised logics*:

> *"Logic signifies formal relationships which are general in character, we may more precisely identify logic with this scheme of interlocking adjoints and then observe that all of logic applies directly to structures valued in arbitrary closed categories V[, f]or example in quantitative logic (V = R)."*

Thus **quantitative logic is logic whose *judgments* are valued in R.**

Multiplicative Reals

Multiplicative structure of the positive reals

polarity		additive			multiplicative
duality $a^* := 1/a$	**positive**	**false** $:= 0$ $a \vee b$	$\xleftarrow{p\to\infty}$	$\mathbf{0} := 0$ $a \oplus^p b$	$\mathbf{1} := 1$ $a \otimes b$
	negative	**true** $:= \infty$ $a \wedge b$	$\xleftarrow{p\to\infty}$	$\top := \infty$ $a \oplus^{-p} b$	$\bot := 1$ $a \otimes^* b$

$a \otimes b$	0	$a \in (0, \infty)$	∞
0	0	0	0
$b \in (0, \infty)$	0	ab	∞
∞	0	∞	∞

$a \otimes^* b$	0	$a \in (0, \infty)$	∞
0	0	0	∞
$b \in (0, \infty)$	0	ab	∞
∞	∞	∞	∞

First-order quantitative logic

The spectrum of p-means

Definition. For any $p \in (-\infty, \infty)$, $p \neq 0$, the ***p*-mean** of a finite set of numbers $(a_i)_{i\in I}$ is

$$\int_{i\in I}^{p} a_i := \left(\bigoplus_{i\in I} \frac{a_i^p}{|I|} \right)^{1/p}$$

$$\int_i^{-0} a_i = \bigotimes_{i\in I} a_i^{1/|I|}$$

$$\int_i^{0} a_i = \bigotimes^*_{i\in I} a_i^{1/|I|}$$

when $p < 0$ we call p-means **harmonic**.

and $p = \pm 0$

One extends the above definition to $p = \pm\infty$ by taking suitable limits.

$$\begin{array}{ccccccccccccc}
\bigwedge & \cdots & \int^{-p} & \cdots & \int^{*} & & GM & & \int & \cdots & \int^{p} & \cdots & \bigvee \\
-\infty & \longleftarrow & -p & \longrightarrow & -1 & \longrightarrow & 0 & \longleftarrow & 1 & \longrightarrow & p & \longrightarrow & \infty
\end{array}$$

$$\int^{-0} \qquad \int^{0}$$

$$0 \otimes \infty = 0 \leftarrow \otimes \qquad \otimes^* \rightsquigarrow 0 \otimes^* \infty = \infty$$

rad to $\otimes$

The amazing p-means

$$\int^{-p}(\psi(i) \multimap a) = \left(\int^{p}\psi(i)\right) \multimap a$$

$$\xrightarrow{p\to\infty} \quad \forall_i(\psi(i) \Rightarrow a) = (\exists_i \psi(i)) \Rightarrow a$$

Lemma. Harmonic sum

1. satisfies the identity:

$$\int_{i\in I}^{-p} \frac{a}{\psi(i)} = \frac{a}{\int_{i\in I}^{p} \psi(i)}$$

2. satisfies the Fubini property:

$$\forall_i \forall_j = \forall_j \forall_i$$

$$\int_{i\in I}^{-p}\int_{j\in J}^{-p} \varphi(i,j) = \int_{(i,j)\in I\times J}^{-p} \varphi(i,j) = \int_{j\in J}^{-p}\int_{i\in I}^{-p} \varphi(i,j),$$

3. is homogeneous (i.e. multiplication distributes over it):

$$k\int f(x)\,dx = \int k f(x)\,dx$$

$$k \otimes^* \int_{i\in I}^{-p} \varphi(i) = \int_{i\in I}^{-p} k \otimes^* \varphi(i),$$

$\otimes$ $\otimes$

$$k \otimes \forall i.\ \varphi(i) = \forall i.\ k \otimes \varphi(i)$$

Frobenius property

Lemma. Harmonic sum

1. is monotonic in the argument: if, for each $i \in I$, $\varphi(i) \leq \psi(i)$, then

$$\int_{i \in I}^{-p} \varphi(i) \leq \int_{i \in I}^{-p} \psi(i),$$

2. is antitonic in the index: when $J \subseteq I$, one has

$$\int_{i \in I}^{-p} \varphi(i) \leq \int_{j \in J}^{-p} \varphi(j).$$

arithmetic mean $\quad AM(a_i)$ s.t.

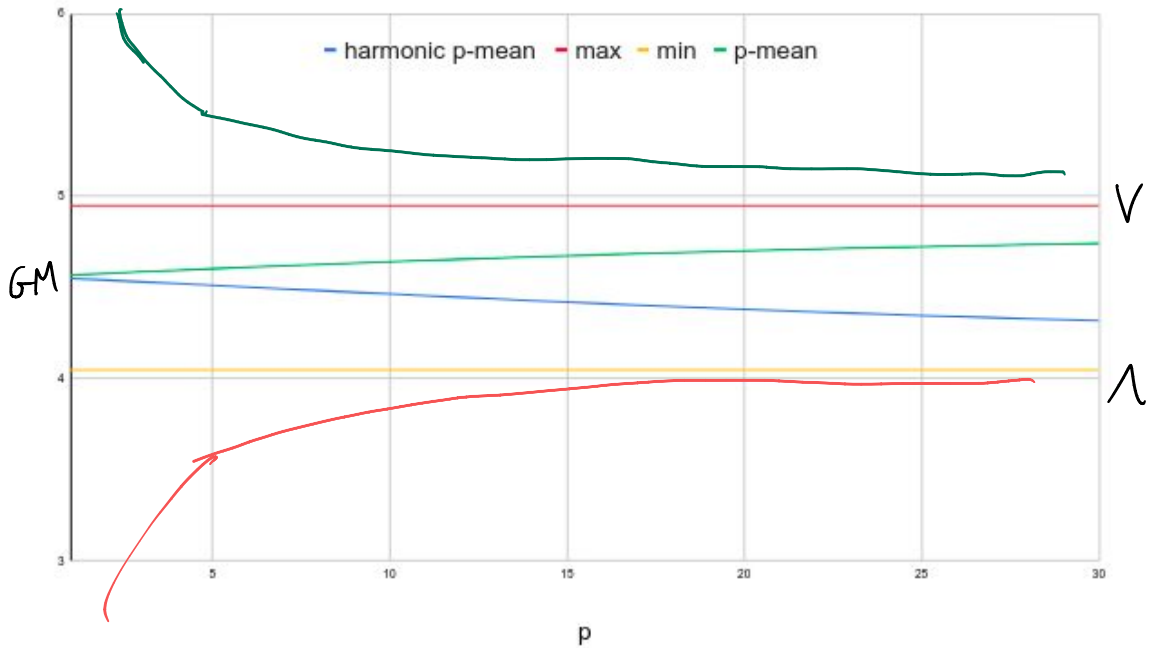
$$\sum_{i \in I} AM(a_i) = \sum_i a_i$$

$$(\mathrm{colim}_i\, a_i) \to b = \lim_i (a_i \to b)$$

$$(\textstyle\sum_i a_i) \multimap b = \sum_i^{*} (a_i \multimap b)$$

The spectrum of p-means

We have $\bigwedge \leq \int^{-p} \leq \int^{p} \leq \bigvee$, with the (exterior) gap narrowing as p increases:



Unlike p-sums, p-means **compensate for cumulative effects**.

Means as bounded quantifiers

Idea. Mean and harmonic mean correspond to **bounded quantification**:

$$\exists i.\ (i \in I) \wedge a(i) \quad \leftsquigarrow\!\!\!\rightsquigarrow \quad \bigoplus_{i \in I} \frac{1}{|I|} \otimes a(i)$$

A mean is a just an integral over a probability space so we can directly generalize:

$$\exists i \in I.\ a(i) \quad \leftsquigarrow\!\!\!\rightsquigarrow \quad \int_{i \in I} a(i)\, di.$$

(I, di)

$a : I \xrightarrow[\text{msbl}]{} [0, \infty]$

for I any probability space. Duality also suggests this interpretation since

$$\forall i \in I.\ a(i) = \neg \exists i \in I.\ \neg a(i) \quad \leftsquigarrow\!\!\!\rightsquigarrow \quad \int_{i \in I}^{-p} a(i) = \left(\int_{i \in I}^{p} a(i)^{*}\, di \right)^{*},$$

i.e. we dualize $a(i)$ but not the domain of quantification.

The mirage of an enriched hyperdoctrine

To each probability space I we can try to associate an $[0,\infty]_\otimes$-enriched p-**Lindenbaum–Tarski algebra** of real-valued predicates:

$[0,\infty]_\otimes\text{-Cat}$

$$\mathbf{LT}^p(I,[0,\infty]) = \begin{cases} \text{elements} & \varphi : I \to [0,\infty]_\otimes \\ \text{entailment} & \varphi \vdash_I \psi := \int_{i\in I}^{-p} \varphi(i) \multimap \psi(i) = \left(\int_{i\in I} \frac{\varphi(i)^p}{\psi(i)^p}\, di\right)^{-1/p} \end{cases}$$

If it worked, it would embody the enriched generalized logic as a functor

$$\mathbf{LT}^p : \mathbf{Prob}^{\mathrm{op}} \longrightarrow [0,\infty]_\otimes\text{-}\mathbf{Cat}$$

and quantifiers would be given as adjoints to reindexing, justifying their definition.

$f : I \longrightarrow J$

$f_* d_i \leq d_j$

The mirage of an enriched hyperdoctrine

Fix $\pi_I : I \times J \to I$, an enriched left adjoint $\int_i^p \dashv \pi_I^*$ would satisfy

for all $\varphi \in \mathbf{LT}^p(I \times J), \psi \in \mathbf{LT}^p(I)$, $\quad \int_i^p \varphi \vdash_I \psi = \varphi \vdash_{I \times J} \pi_I^* \psi$

which unpacks to

$$\int_{i \in I}^{-p} \left(\int_{j \in I}^{p} \varphi(i,j) \right) \multimap \psi(i) = \int_{\substack{i \in I \\ j \in J}}^{-p} \varphi(i,j) \multimap \psi(i)$$

which is true by the fundamental relation of harmonic sum.

Similarly, we would have $\pi_I^* \dashv \int^{-p}$.

Notice: it's absolutely crucial that $\vdash_I$ is enriched!

The mirage of an enriched hyperdoctrine

Why doesn't it *just* work?

Entailment isn't transitive. It's worth to see in detail why (for $p = 1$ for simplicity):

$$(\varphi \vdash_I \psi) \otimes (\psi \vdash_I \sigma) \leq (\varphi \vdash_I \sigma) \iff (\varphi \vdash_I \psi)^{-1} \otimes (\psi \vdash_I \sigma)^{-1} \geq (\varphi \vdash_I \sigma)^{-1}$$

Thus

$$\Delta \longrightarrow \Delta \times \Delta$$

$$\int_{i \in I} \int_{i' \in I} \frac{\varphi(i)}{\psi(i)} \frac{\psi(i')}{\sigma(i')} \, di \, di' \geq \int_{i \in \Delta_I} \frac{\varphi(i)}{\cancel{\psi(i)}} \frac{\cancel{\psi(i)}}{\sigma(i)} \, di \, di \not\geq \int_{i \in I} \frac{\varphi(i)}{\sigma(i)} \, di$$

$P = 1$

$$(\varphi \vdash_I \psi)^{-1} \otimes (\psi \vdash_I \chi)^{-1} \geq (\varphi \vdash_I \chi)^{-1}$$

$$\boxed{\|f\|_1 \otimes \|g\|_1 \geq \|f \otimes g\|_1}$$

Hölder's inequality $\quad p, q > 1 \quad$ "conjugate" $\quad \frac{1}{p} + \frac{1}{q} = 1$

$$\|f\|_p \, \|g\|_q \geq \|fg\|_{p \oplus^* q}$$

$p = 1, q = 1 \quad \frac{1}{1} + \frac{1}{1} = 2$

$$p \oplus^* q = 1$$

Grading to the rescue!

Lemma 1.13 (Generalized Hölder's inequality). *Let $p, q \in [0,\infty]_{\oplus^*}$, let $f, g : X \to [0,\infty]$ be measurable functions. Then*

$$\underbrace{\int_{i\in I}^{p\oplus^* q} (f(i) \otimes^* g(i))\, \mathrm{d}i}_{\|f\otimes^* g\|_{p\oplus^* q}} \le \underbrace{\left(\int_{i\in I}^{p} f(i)\, \mathrm{d}i\right)}_{\|f\|_p} \otimes^* \underbrace{\left(\int_{i\in I}^{q} g(i)\, \mathrm{d}i\right)}_{\|g\|_q}. \tag{1.32}$$

Lemma 1.14 (Dual Generalized Hölder's inequality). *Let $p, q \in [0,\infty]_{\oplus^*}$, let $f, g : I \to [0,\infty]$ be measurable functions. Then*

$$\left(\int_{i\in I}^{-p} f(i)\, \mathrm{d}i\right) \otimes \left(\int_{i\in I}^{-q} g(i)\, \mathrm{d}i\right) \le \int_{i\in I}^{-p\oplus^* q} (f(i) \otimes g(i))\, \mathrm{d}i. \tag{1.33}$$

Consider now the sets $L(I)$ of measurable functions valued in the extended positive reals $[0,\infty]$. For each $p \in [0,+\infty)$ define

$$\varphi \vDash_p \psi := \int_{i\in I}^{-p} \varphi(i) \multimap \psi(i)\,\mathrm{d}i. \tag{4.1}$$

$\rightsquigarrow \left(\vdash_p : LI \times LI \longrightarrow [0,\infty]_{\otimes}\right)_{p \in [0,\infty]_{\oplus^*}}$

This satisfies

1. **graded reflexivity**:

$$1 = \int_{i\in I}^{-\infty} 1\,\mathrm{d}i \le \int_{i\in I}^{-\infty} \frac{\varphi}{\varphi}\,\mathrm{d}i \;=\; (\varphi \vdash_\infty \varphi) \tag{4.2}$$

$\mathcal{H}$

$\left([0,\infty], \text{"}\infty\text{"}, \oplus^*, \le\right)$

2. **graded transitivity**:

$$(\varphi \vDash_p \psi) \otimes (\psi \vDash_q \chi) \le (\varphi \vDash_{p\oplus^* q} \chi) \tag{4.3}$$

which amounts to dual generalized Hölder inequality:

$$\left(\int_{i\in I}^{-p} \frac{\psi(i)}{\varphi(i)}\,\mathrm{d}i\right) \otimes \left(\int_{i\in I}^{-q} \frac{\chi(i)}{\psi(i)}\,\mathrm{d}i\right) \le \int_{i\in I}^{-p\oplus^* q} \frac{\chi(i)}{\varphi(i)}\,\mathrm{d}i \tag{4.4}$$

3. **relaxation**: for $p \le q$,

$L^q I \hookrightarrow L^p I$ for I prob

$$(\varphi \vDash_q \psi) \le (\varphi \vDash_p \psi) \qquad \|f\|_q \le \|f\|_p \tag{4.5}$$

by general properties of L^p norms over probability spaces.

$$L : \mathbf{Prob}^{op} \longrightarrow (\oplus^*, \otimes)\text{-}\mathbf{Prd} \simeq \mathcal{V}^{\mathcal{H}^{op}}\text{-}\mathbf{Cat}$$

$$\begin{array}{ccc} I & & LI \\ f \downarrow & \longmapsto & \uparrow f^* \\ J & & LJ \end{array}$$

LI ⟶ pt-wise ordered

$\pi_I : I \times J \to I$

$$\left(\int_j^p \varphi\right) \vdash^I_{p} \psi = \varphi \vdash^{I\times J}_{p} \pi_I^* \psi$$

proof

$$\int_i^{-p} \left(\int_j^p \varphi\right) \multimap \psi = \int_i^{-p} \int_j^{-p} (\varphi \multimap \psi) = \int_{i,j}^{-p} \varphi \multimap \psi. \quad \square$$

Definition 4.11. We say an ~~indexed $\mathcal{V}$-preorder~~ indexed $\mathcal{H}$-preorder satisfyng Theorem 4.7 (existence of p-adjoints to reindexing along projections for all $p \in \mathcal{H}$), Lemma 4.9 (Beck-Chevalley satisfied for all left p-adjoints) and Lemma 4.10 (Frobenius condition with respect to $\otimes$) is **soft-first-order**.

Theorem 4.12. *The functor $L : \mathbf{Prob}^{\mathrm{op}} \to (\oplus^*, \otimes)\text{-}\mathbf{Prd}$ is a soft-first-order hyperdoctrine.*

$$\frac{\left(\int \left(\int f^p\right)^{q/p}\right)^{1/q}}{\left(\int \left(\int f^q\right)^{p/q}\right)^{1/p}}$$

$r - 1$

Sequent calculus

"mixed" L^p-norm $\rightsquigarrow$ $x_1 :_{p_1} X_1, \dots, x_n :_{p_n} X_n \mid \varphi \vdash \psi = \int_{X_n}^{-p_n} \cdots \int_{X_2}^{-p_2} \int_{X_1}^{-p_1} \varphi(x_1, \dots, x_n) \multimap \psi(\dots) \, dx_1$

We ponder sequents of the form

$$x :_P X \mid \Gamma \vdash \Delta \tag{6.1}$$

where

1. $P \equiv (p_1, \dots, p_n)$ is an ordered sequence of harmonic reals $p_i \in [0, \infty]_{\oplus^*}$;
2. $x :_P X \equiv (x_1 :_{p_1} X_1, \cdots, x_n :_{p_n} X_n)$ is a list of typed variable declarations, each tagged by the corresponding $p_i \in P$, called its **softness**,
3. $\Gamma \equiv \{\gamma_1(x), \dots, \gamma_h(x)\}$ and $\Delta \equiv \{\delta_1(x), \dots, \delta_k(x)\}$ are finite sets of formulae using only variables in x.

6.1. **Variable contexts.** The following are not really rules, but rather definitions. The presence of variable contexts is the main difference between this and the propositional calculus. In fact, we recover the propositional fragment of our logic by working in the empty context.

$$\frac{}{() \text{ ctx}} \text{ (EMPTY)} \qquad \frac{x :_P X \text{ ctx} \qquad Y \text{ type} \qquad q \in [0,\infty]_{\oplus^*}}{x, y :_{Pq} X, Y \text{ ctx}} \text{ (INT)} \tag{6.7}$$

where $PQ = (p_1, \dots, p_h, q_1, \dots, q_k)$ is concatenation.

6.5. **Weakening.** Weakening is sound in L:

$$\frac{x :_P X \mid \Gamma \vdash \varphi}{x :_P X, y :_q Y \mid \Gamma \vdash \varphi} \text{ (WEAK)} \tag{6.13}$$

6.3. **Structural.**

$$\frac{x :_P X \mid \Gamma, \Xi \vdash \Delta}{x :_P X \mid \Xi, \Gamma \vdash \Delta}\,(\mathrm{EX_L}) \qquad \frac{x :_P X \mid \Gamma \vdash \Xi, \Delta}{x :_P X \mid \Gamma \vdash \Delta, \Xi}\,(\mathrm{EX_R}) \tag{6.10}$$

For $P \leq Q$,

$$\frac{x :_Q X \mid \Gamma \vdash \Delta}{x :_P X \mid \Gamma \vdash \Delta}\,(\mathrm{RLX}) \tag{6.11}$$

6.4. **Axiom & cut.**

$$\frac{x :_P X \text{ ctx}}{x :_P X \mid \varphi \vdash \varphi}\,(\mathrm{AX}) \qquad \frac{x :_P X \mid \Gamma \vdash \varphi \quad \otimes \quad x :_Q X \mid \Gamma, \varphi \vdash \Delta}{x :_{P \oplus^* Q} X \mid \Gamma \vdash \Delta}\,(\mathrm{CUT}) \tag{6.12}$$

where $P \oplus^* Q = (p_1 \oplus^* q_1, \ldots, p_k \oplus^* q_k)$.

6.5. **Substitution.** The following rule is sound in L:

$$\frac{y :_p Y \mid \Gamma \vdash \Delta \qquad f : X \to Y}{x :_p X \mid \Gamma[y/f(x)] \vdash \Delta[y/f(x)]} \tag{SUBST}$$

where $f : X \to Y$ is a term.

From Rule RLX and Rule SUBST applied to diagonals $\Delta : X \to X \times X$ (Remark 4.4), we can prove:

$$\frac{x :_P X, x :_Q X \mid \Gamma \vdash \Delta}{x :_{P \wedge Q} X \mid \Gamma \vdash \Delta} \tag{CONT}$$

6.6. **Quantifiers.** The following rules are reversible:

$$\frac{x :_P X \mid \Gamma, \exists(y :_Q Y).\varphi(y) \vdash \Delta}{x :_P X, y :_Q Y \mid \Gamma, \varphi \vdash \Delta}\,(\text{EXIST}) \qquad \frac{x :_P X, y :_Q Y \mid \Gamma \vdash \psi}{x :_P X \mid \Gamma \vdash \forall(y :_Q Y).\psi(y)}\,(\text{UNIV}) \tag{6.14}$$

where $\exists(y :_Q Y)$ is short for $\exists y_1 :_{q_1} Y_1 \cdots \exists y_n :_{q_n} Y_n$ and these are to be interpreted semantically as $\exists^{q_i}_{\pi_i}$—and likewise for $\forall$.

6.7. **Multiplicatives.**

$$\frac{x :_P X \mid \Gamma, \varphi, \psi, \Gamma' \vdash \Delta}{x :_P X \mid \Gamma, \varphi \otimes \psi, \Gamma' \vdash \Delta}\ (\otimes_{\mathrm{L}}) \qquad \frac{x :_P X \mid \Gamma \vdash \Delta, \varphi, \psi, \Delta'}{x :_P X \mid \Gamma \vdash \Delta, \varphi \otimes^* \psi, \Delta'}\ (\otimes^*{}_{\mathrm{R}}) \tag{6.18}$$

$$\frac{x : X, z : Z \mid \Gamma \vdash \Delta, \varphi \quad \otimes \quad z :_S Z, y :_Q Y \mid \Gamma' \vdash \psi, \Delta'}{z :_{R \oplus^* S} Z, x :_P X, y :_Q Y \mid \Gamma, \Gamma' \vdash \Delta, \varphi \otimes \psi, \Delta'}\ (\otimes_{\mathrm{R}})$$

$$\frac{z :_R Z, x :_P X \mid \Gamma, \varphi \vdash \Delta \quad \otimes \quad z :_S Z, y :_Q Y \mid \psi, \Gamma' \vdash \Delta'}{z :_{R \oplus^* S} Z, x :_P X, y :_Q Y \mid \Gamma, \varphi \otimes^* \psi, \Gamma' \vdash \Delta, \Delta'}\ (\otimes^*{}_{\mathrm{L}})$$

$P \oplus^* Q \leq P \wedge Q \in ([0,\infty], \infty, \oplus^*, \leq)$

- **Global co/tensor**. No variables are shared, and we get reversible rules:

$$\frac{x :_P X \mid \Gamma \vdash \Delta, \varphi \quad \otimes \quad y :_Q Y \mid \Gamma' \vdash \psi, \Delta'}{x :_P X, y :_Q Y \mid \Gamma, \Gamma' \vdash \Delta, \varphi \otimes \psi, \Delta'}\ (\otimes^g_{\mathrm{R}})$$

$$\frac{x :_P X \mid \Gamma, \varphi \vdash \Delta \quad \otimes \quad y :_Q Y \mid \psi, \Gamma' \vdash \Delta'}{x :_P X, y :_Q Y \mid \Gamma, \varphi \otimes^* \psi, \Gamma' \vdash \Delta, \Delta'}\ (\otimes^{*g}_{\mathrm{L}})$$

- **Local co/tensor**. All variables are shared, recovering the structure of Lemma 4.5:

$$\frac{z :_R Z \mid \Gamma \vdash \Delta, \varphi \quad \otimes \quad z :_S Z \mid \Gamma' \vdash \psi, \Delta'}{z :_{R \oplus^* S} Z \mid \Gamma, \Gamma' \vdash \Delta, \varphi \otimes \psi, \Delta'}\ (\otimes^l_{\mathrm{R}})$$

$$\frac{z :_R Z \mid \Gamma, \varphi \vdash \Delta \quad \otimes \quad z :_S Z \mid \psi, \Gamma' \vdash \Delta'}{z :_{R \oplus^* S} Z \mid \Gamma, \varphi \otimes^* \psi, \Gamma' \vdash \Delta, \Delta'}\ (\otimes^{*l}_{\mathrm{L}})$$

6.8. **Additives.**

$$\frac{x :_Q X \mid \Gamma, \varphi, \Gamma' \vdash \Delta \quad \oplus^{-P} \quad x :_Q X \mid \Gamma, \psi, \Gamma' \vdash \Delta}{x :_Q X \mid \Gamma, \varphi \oplus^P \psi, \Gamma' \vdash \Delta} \tag{$\oplus_{\mathrm{L}}$}$$

$$\frac{x :_Q X \mid \Gamma \vdash \Delta, \varphi, \Delta' \quad \oplus^{-P} \quad x :_Q X \mid \Gamma \vdash \Delta, \psi, \Delta'}{x :_Q X \mid \Gamma \vdash \Delta, \varphi \oplus^{-P} \psi, \Delta'} \tag{$\oplus^*_{\mathrm{L}}$}$$

$$\frac{x :_P X \mid \Gamma, \varphi \oplus^P \psi, \Gamma' \vdash \Delta}{x :_P X \mid \Gamma, \varphi, \Gamma' \vdash \Delta \quad \oplus^{-P} \quad x :_P X \mid \Gamma, \psi, \Gamma' \vdash \Delta} \,(\oplus^P_{\mathrm{R}}) \tag{6.22}$$

$$\frac{x :_P X \mid \Gamma \vdash \Delta, \varphi \oplus^{-P} \psi, \Delta'}{x :_P X \mid \Gamma \vdash \Delta, \varphi, \Delta' \quad \oplus^{-P} \quad x :_P X \mid \Gamma \vdash \Delta, \psi, \Delta'} \,(\oplus^{-P}_{\mathrm{R}}) \tag{6.23}$$

$$\frac{\infty}{x :_P X \mid 0 \vdash \Delta} \,(\mathrm{ZERO}) \qquad \frac{\infty}{x :_P X \mid \Gamma \vdash \infty} \,(\mathrm{INFINITY}) \tag{6.24}$$