Enriched Categories Applied to Qualia Spaces

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Motivating Paper (Tsuchiya, Phillips, Saigo)

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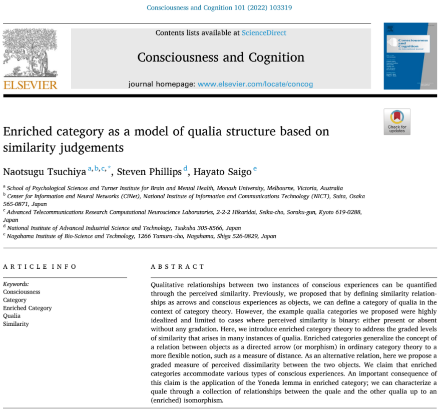
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Enriched category as a model of qualia structure based on similarity judgements

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ABSTRACT

Qualitative relationships between two instances of conscious experiences can be quantified through the perceived similarity. Previously, we proposed that by defining similarity relationships as arrows and conscious experiences as objects, we can define a category of qualia in the context of category theory. However, the example qualia categories we proposed were highly idealized and limited to cases where perceived similarity is binary: either present or absent without any gradation. Here, we introduce enriched category theory to address the graded levels of similarity that arises in many instances of qualia. Enriched categories generalize the concept of a relation between objects as a directed arrow (or morphism) in ordinary category theory to a more flexible notion, such as a measure of distance. As an alternative relation, here we propose a graded measure of perceived dissimilarity between the two objects. We claim that enriched categories accommodate various types of conscious experiences. An important consequence of this claim is the application of the Yoneda lemma in enriched category; we can characterize a quale through a collection of relationships between the quale and the other qualia up to an (enriched) isomorphism.

In 2021, Tsuchiya, Phillips, and Saigo (the authors) introduced enriched category theory to model similarity judgements in qualia spaces.

Qualia is a quality or property as perceived or experienced by a person.

- How you taste lemon
- How you feel pain
- How you see the colour red

Content

Key Ideas of Their Paper

- Introduce enriched categories to model similarity judgements
- Discuss advantages of the enriched categorical approach over standard models
- Advocate for a categorical approach over other structural and relational theories of qualia

Questions

- From the paper itself, "What kinds of relations (including similarity) can form an enriched category?"
- Does their categorical approach work?
- What are other applications of enriched category theory?

Content

1. Motivating Scenario: from Monoidal Categories to Enriched Categories
2. Authors' Result and Reasoning
3. Developments
4. Conclusions + Open Questions

Motivating Scenario

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Suppose we ask participants to rate the similarity between colours.

"Can a collection of qualia with continuous degrees of similarity as morphisms qualify as a category?"

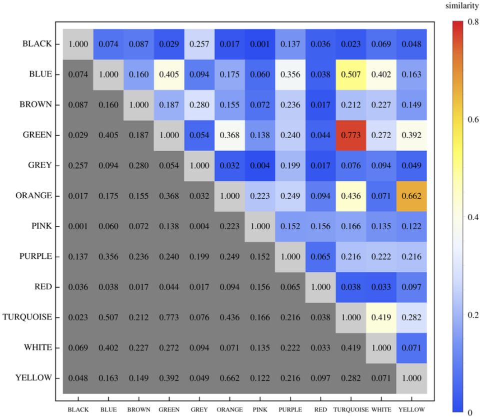

Figure 1: From (Jonauskaite et al. 2019).

Can ordinary categories model similarity?

Given a similarity scale, the authors define a threshold of similarity. In this case, let us say two objects are similar if their similarity value is more than 0.5. We define a morphism $x \to y$ if x is similar to y.

Green is **similar** to Turquoise.
Turquoise is **similar** to Blue.
But Green is **not similar** to Blue.

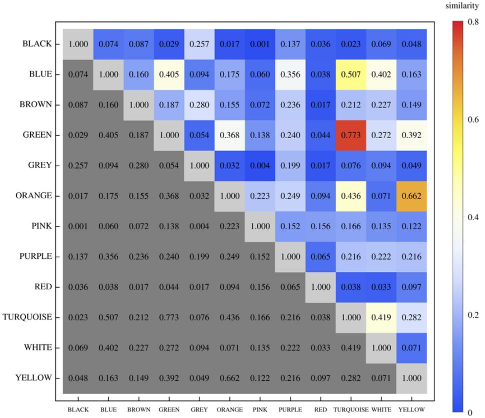

Figure 2: From (Jonauskaite et al. 2019)

The problem?

Similarity as a threshold is too binary to consider.

We need to be able to describe **gradations**.

We need our morphisms to live in some object with more structure than that of a set.

Monoidal Category Theory

Definition (Monoidal Category)

A monoidal category is a tuple $(\mathcal{C}, \otimes, \mathbb{1})$ consisting of:

- a category $\mathcal{C}$,
- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the monoidal product, and
- an object $\mathbb{1} \in \mathcal{C}$ called the monoidal unit.

These satisfy associativity and unity axioms with associator and unitor natural isomorphisms[1].

[1] For more details, see (Johnson and Yau 2021).

Can monoidal categories model dissimilarity?

Dissimilarity category, $\mathcal{D}$

- Objects: $[0, \infty)$,
- Morphisms: $f : x \to y \iff x \geq y$,
- Monoidal Product: $x \otimes y = x + y$ for $x, y \in \mathrm{Ob}(\mathcal{D})$ and $g \otimes f = h : x \to z$ for $f : x \to y$ and $g : y \to z$,
- Monoidal Unit: $0 \in \mathrm{Ob}(\mathcal{D})$ and corresponding 1_0 as its identity morphism.

Given a qualia category $\mathcal{Q}$, if $X, Y, Z \in \mathrm{Ob}(\mathcal{Q})$:

- $d(X, Y), d(Y, Z) \in \mathrm{Ob}(\mathcal{D}) = [0, \infty)$,
- $d(Y, Z) \otimes d(X, Y) \to d(X, Z)$ manifests as $d(X, Z) \leq d(X, Y) + d(Y, Z)$.
- $\mathbb{1} \xrightarrow{i_X} 1_X$ corresponds to $d(X, X) \leq 0$ so $d(X, X) = 0$.

Enriched Categories

To describe gradations, we replace a set of morphisms with a monoidal category $\mathcal{V}$.

Definition (Enriched Category)

A $\mathcal{V}$-enriched category $\mathcal{C}$ consists of:

- a collection $\mathrm{Ob}(\mathcal{C})$ of **objects** in $\mathcal{C}$,
- an object $\hom_{\mathcal{C}}(X, Y) \in \mathcal{V}$ for each $X, Y \in \mathcal{C}$ called the **hom-object**,
- a morphism $\hom_{\mathcal{C}}(Y, Z) \otimes \hom_{\mathcal{C}}(X, Y) \xrightarrow{m_{XYZ}} \hom_{\mathcal{C}}(X, Z)$ in $\mathcal{V}$, called the **composition**, and
- a morphism $\mathbb{1} \xrightarrow{i_X} \hom_{\mathcal{C}}(X, X)$ for each object $X \in \mathcal{C}$, called the **identity** of X.

These satisfy associativity and unity laws[2].

[2] For more details, see (Johnson and Yau 2021).

Authors' Result

The authors show that "a collection of colour qualia with dissimilarity [as morphisms] qualify as an enriched category."

Why use a categorical approach to qualia?

"Finally, we discuss the fundamental motivation of our enriched category framework, that is, the consequence of the Yoneda lemma: to characterise a quale as its relationship to all other qualia."

"To apply our approach, researchers need to make the category explicit, which entails specifying the objects, the relations between objects (morphisms), and how those relations compose."
(N. Tsuchiya, Phillips, and H. Saigo 2022, pg. 10)

"...the essence of the colour experiences resides in relations with other colours..."
(Naotsugu Tsuchiya and Hayato Saigo 2021, pg. 10)

Key Results

1. Can continuous degrees of similarity as morphisms produce an enriched category? Are there other models of similarity?
   Yes.
2. Can we "use" Yoneda's Lemma to classify qualia in terms of their relations?
   Not always — for 2-**ary relations, only when reflexive and transitive**.
3. "What kinds of relations (including similarity) can form an enriched category?"
   Bit more complicated...
4. What are other applications of enriched category theory?
   Enriched quivers are data structures for qualia observations.

Alternative Similarity and Dissimilarity Categories

We can construct a similarity category $\mathcal{S}$ to enrich $\mathcal{Q}$ by considering $\mathcal{D}^{op}$.

Are there other ways to make a dissimilarity category?
With t-norms (Bartoszuk and Gagolewski 2021) (Zwick, Carlstein, and Budescu 1987), which are functions $\otimes : [0,1] \times [0,1] \to [0,1]$ such that for all $a, b, c, d \in [0,1]$:

- Commutativity: $\otimes(a,b) = \otimes(b,a)$,
- Monotonicity: $\otimes(a,b) \leq \otimes(c,d)$ if $a \leq c$ and $b \leq d$,
- Associativity: $\otimes(a, \otimes(b,c)) = \otimes(\otimes(a,b), c)$,
- Unity: $\otimes(a,1) = a$

Examples: minimum, product, Łukasiewicz, Drastic, etc.

Yoneda's Lemma

Let $\mathcal{C}$ be a locally small category with the category of presheaves denoted $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$. For any presheaf $X \in [\mathcal{C}^{\text{op}}, \mathbf{Set}]$, there is a canonical isomorphism

$$\hom_{[\mathcal{C}^{\text{op}}, \mathbf{Set}]}\left(\text{よ}(c), X\right) \cong X(c)$$

between the hom-set of presheaf homomorphisms from the representable[3] presheaf よ(c) to X and the value of X at c, where よ is the Yoneda embedding:

$$\begin{aligned} \mathcal{C} &\xrightarrow{\text{よ}} [\mathcal{C}^{\text{op}}, \mathbf{Set}] \\ c &\mapsto \hom_{\mathcal{C}}(-, c) \\ f &\mapsto \hom_{\mathcal{C}}(-, f) \end{aligned}$$

[3] i.e, naturally isomorphic to a hom-functor $\hom_{\mathcal{C}}(-, X) : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ for some $X \in \mathcal{C}$.

How Yoneda's Lemma Fits

Yoneda's Lemma is useful for the following corollary: if $\mathcal{C}$ is a locally small category, then for $A, B \in \text{Ob}(C)$, $A \cong B \iff \mathcal{C}(-, A) \cong \mathcal{C}(-, B)$ naturally.

"Probes by objects of $\mathcal{C}$ are sufficient to distinguish objects of $\mathcal{C}$: two objects of $\mathcal{C}$ are the same if they have the same probes by other objects of $\mathcal{C}$." (nLab authors 2025)

In a category, an object is defined by its morphisms to all objects - including itself.

Relations

The authors want to use Yoneda's Lemma to show that qualia can be characterised via its relationships to other qualia. How do we describe these relationships?

The typical definition of a relation is a subset of S^2 for a set S. However, we need a definition for n-ary relations — relations that relate n objects together.

Definition (n-ary Relation)

For a set of objects S, an n-ary relation R on S is $R : S^n \to \mathbf{Truth}$, where S^n is seen as a discrete category.

Relations

Define $L(x) \in [0, 10]$ to be the brightness of a colour x.

$R_1(x, y) = T \iff L(x) \leq L(y)$ and $R_2(x, y, z) = T \iff L(x) + L(z) \leq L(y)$.

We can imagine a morphism as an instance of a 2-ary relation.

For a given 2-ary relation R, we get a quiver Q_R — a directed graph whose objects are in S, and there is a directed edge (morphism) from x to y if and only if $R(x, y) = T$.

Lemma

There is a bijective structure-preserving correspondence between 2-ary relations R and thin quivers Q_R.

The Tsuchiya-Saigo Approach

So:

- The authors want to use Yoneda's Lemma to classify an object by its relations.
- These relations with other objects must be morphisms—that is, instances of 2-ary relations.
- There exist relations of higher arity, and relations need not immediately form a composition law.

To see if Yoneda's Lemma can be applied for an arbitrary n-ary relation as claimed, we need to answer when an n-ary relation $R : S^n \to \textbf{Truth}$ 'makes' Q_R into a category $\mathcal{Q}_R$, so we can 'preserve' the information of the relation.

Perspective 1 : Profunctors and Interpreting Morphisms in **Truth**

One way to answer this question is to consider what conditions categories enriched with **Truth** follow.

Truth-Categories

Let us restrict to 2-ary relations $R : S^2 \to \textbf{Truth}$.
A category $\mathcal{C}$ enriched over **Truth** has the same objects as $\mathcal{C}$, and for each $c, c' \in \mathcal{C}$, has the hom-object $\mathcal{C}(c, c') \in \{T, F\}$. One can consider $\mathcal{C}(c, c')$ as being the truth of a relation $R(c, c')$.

To get composition morphisms in this category, for each triple $c, c', c'' \in \mathcal{C}$,

$$\mathcal{C}(c', c'') \otimes \mathcal{C}(c, c') \to \mathcal{C}(c, c'')$$

That is, if $R(c, c')$ and $R(c', c'')$, then $R(c, c'')$, so R is transitive.
Likewise, we need $1 \to \mathcal{C}(c, c)$, so $R(c, c)$ is always true and R is reflexive.
Thus, R is a preorder. Conversely, preorders give rise to **Truth**-categories.

Truth-Categories

Suppose $R : S^n \to \mathbf{Truth}$ is an n-ary relation.

One may consider R as a **Truth**-valued profunctor[4] $R : S^{n-1} \nrightarrow S$, by setting S as a discrete category enriched over **Truth**. Since **Truth** is symmetric-monoidal closed and complete,

$$[S^{n-1} \times S, \mathbf{Truth}] \cong [S^{n-1}, [S, \mathbf{Truth}]] \cong [S, \ldots, [S, \mathbf{Truth}]].$$

Attempting to describe what R is, in this case, is substantially more difficult. It may require the use of enriched multicategories (Leinster 1999). For the case of $n = 2m$, it may be possible to reduce it down by considering the profunctor $S^m \to S^m$ and monads in the bicategory $\mathbf{Prof}_{\mathbf{Truth}}$.

[4] So a functor $R : S^{n-1} \times S \to \mathbf{Truth}$.

Perspective 2 : Quivers to Categories

Another way to answer this question is to consider the quiver Q_R, arising from R, and consider what additional morphisms it requires to become a category.

2-Ary Relations and Quivers

By restricting to 2-ary relations, we see that:

Theorem

Suppose $R : S^2 \to \textbf{Truth}$ is reflexive[5] and transitive[6]. Then, the corresponding thin quiver Q_R may be seen as a thin category $\mathcal{Q}_R$, respecting the structure imposed by R.

Thus, if we have a reflexive and transitive relation R on S, we may view Q_R as a category $\mathcal{Q}_R$. By 'using' Yoneda's Lemma to $\mathcal{Q}_R$, we can obtain all information about the objects of S with respect to R.

Can any arbitrary relation R on S be viewed as a category?

---

[5] $R(x, x) = T$ for all x

[6] $R(x, y) = R(y, z) = T \implies R(x, z) = T$

From Quivers to Categories

We can construct a category $\mathcal{Q}_R$ that ensures reflexivity and transitivity from a quiver Q_R as follows:

- the objects of the category remain the vertices of Q_R,
- the morphisms are paths between objects, where a path is a finite sequence of morphisms connecting one vertex to another.

This defines a free category $\mathcal{Q}_R$ of the quiver Q_R, and we can describe this construction as a functor $F : \textbf{Quiv} \to \textbf{Cat}$, called the free functor.

How unique is this free category $\mathcal{Q}_R$?

Universal Property of Free Categories

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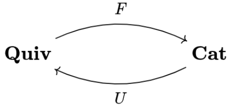
By the Universal Property of Free Categories, we have functors $F : \mathbf{Quiv} \to \mathbf{Cat}$ and $U : \mathbf{Cat} \to \mathbf{Quiv}$ such that $F \dashv U$.

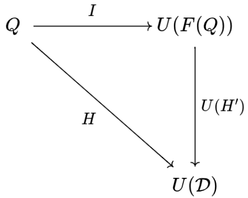


Is there another way of constructing a category, $\mathcal{Q}'_R$, which respects the structure of Q_R? Up to isomorphism, no.

So:

- Given a relation R and its corresponding quiver Q_R, we can construct a unique category $\mathcal{Q}_R$.
- But, $F : \mathbf{Quiv} \to \mathbf{Cat}$ is not injective on objects.

Unfortunately, this is not immediately applicable to our quiver.



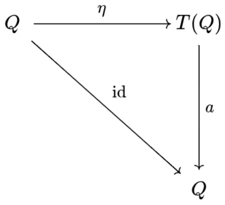
Monad on **Quiv**

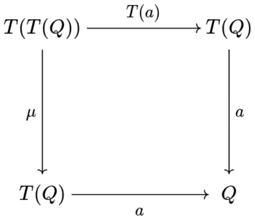
From the adjunction $F \dashv U$, we get the monad $T = UF : \mathbf{Quiv} \to \mathbf{Quiv}$, with $\eta : 1_{\mathbf{Quiv}} \implies T$ and $\mu := U\varepsilon F : T^2 \implies T$. Here, η embeds each arrow as a path of length 1, and μ concatenates a path of paths into a path.

Then, a T-algebra (Q, a) is a quiver Q equipped a map $a : TQ \to Q$ in **Quiv** such that $a \circ \eta = \text{id}$ and $a \circ Ta = a \circ \mu$.

Thus, T-algebras are exactly (small) categories, and T-algebra morphisms are functors.

So, the Eilenberg-Moore category $\mathbf{Quiv}^T$ of T is isomorphic to **Cat**.





When Thin Quivers are Categories

Recall that relations $R : S^2 \to \textbf{Truth}$ and thin quivers Q_R correspond.

The adjunction $F : \textbf{Quiv} \rightleftarrows \textbf{Cat} : U$ provides the monad (T, η, μ). In particular, $\eta_Q : Q \to TQ$ embeds each original arrow as a path of length 1.

Q_R 'can be seen' as a category when there exists $a : T(Q_R) \to Q_R$ such that (Q_R, a) is a T-algebra. For thin quivers, we only require the existence of 'identity' arrows and that every composite path $x \to y \to z$ is already given by an existing arrow $x \to z$. Thus,

- For every vertex x, the length-0 path at x must already be an arrow, so $R(x, x) = T$.
- If $R(x, y) = R(y, z) = T$, then the length-2 path must be given by the unique arrow $x \to z$, so $R(x, z) = T$.

The Two Perspectives

Profunctors and Morphisms
Easier to discover conditions for the relation based on the enrichment category $\mathcal{V}$.

Monads and Algebras
Easier to discern whether a given relation (aka its corresponding quiver) is a category.

What Kinds of Relations Can Form Enriched Categories?

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Definition (Enriched Relations and Enriched Quivers)

Let $\mathcal{V}$ be a category.
A n-ary $\mathcal{V}$-relation R is a functor $R : S^n \to \mathcal{V}$.
A quiver Q is a $\mathcal{V}$-quiver Q that has a collection of objects $\mathrm{Ob}(Q)$ and a $\mathcal{V}$-valued functor $\mathrm{Mor} : \mathrm{Ob}(Q) \times \mathrm{Ob}(Q) \to \mathcal{V}$.

We would like to determine which enriched relations lead to enriched categories.
Once again, we restrict our focus to enriched 2-ary relations.

Lemma

There is a bijective structure-preserving correspondence between 2-ary $\mathcal{V}$-relations R and thin $\mathcal{V}$-quivers Q_R with objects in S.

From $\mathcal{V}$**Quiv** to $\mathcal{V}$**Cat**

Theorem (The Universal Property of Free $\mathcal{V}$-Categories (Wolff 1974))

*Let $\mathcal{V}$ be a cocomplete, symmetric, monoidal closed category, $\mathcal{V}$**Cat** be the category of small $\mathcal{V}$-categories and $\mathcal{V}$**Quiv** the category of small $\mathcal{V}$-quivers.*
There exists a free functor $F : \mathcal{V}\mathbf{Quiv} \to \mathcal{V}\mathbf{Cat}$, such that $F \dashv U$, where $U : \mathcal{V}\mathbf{Cat} \to \mathcal{V}\mathbf{Quiv}$ is the forgetful functor.

Once again, we can consider the monad constructed from this adjunction, (T, η, μ), where $T := UF : \mathcal{V}\mathbf{Quiv} \to \mathcal{V}\mathbf{Quiv}$, $\eta : 1_{\mathcal{V}\mathbf{Quiv}} \implies T$ and $\mu := U\varepsilon F : T^2 \implies T$. Here, η embeds each enriched arrow as a path of length 1, and μ concatenates a path of paths (of enriched arrows) into a path.

When Thin $\mathcal{V}$-Quivers Are $\mathcal{V}$-Categories

A T-algebra is a $\mathcal{V}$-quiver Q equipped a map $a : TQ \to Q$ in $\mathcal{V}$**Quiv** such that $a \circ \eta = \text{id}$ and $a \circ Ta = a \circ \mu$.

For $\mathcal{V}$ a cocomplete, symmetric, monoidal closed category, T-algebras are exactly (small) $\mathcal{V}$-categories, and T-algebra morphisms are $\mathcal{V}$-functors.

Q_R 'can be seen' as a category when $a : T(Q_R) \to Q_R$ such that (Q_R, a) is a T-algebra. For thin $\mathcal{V}$-quivers, we only require the existence of 'identity' arrows and arrows which obey the composition law in $\mathcal{V}$; that is, there exists a morphism $m_{x,y,z} : \text{hom}_{\mathcal{V}}(y, z) \otimes \text{hom}_{\mathcal{V}}(x, y) \to \text{hom}_{\mathcal{V}}(x, z)$ in $\mathcal{V}$.

Examples

Given a Bénabou cosmos $\mathcal{V}$, what condition is placed on R?

$$\mathbb{1} \xrightarrow{i_x} \hom_{\mathcal{C}}(x,x) \in \mathcal{V} \quad \text{and} \quad \hom_{\mathcal{C}}(y,z) \otimes \hom_{\mathcal{C}}(x,y) \xrightarrow{m_{x,y,z}} \hom_{\mathcal{C}}(x,z) \in \mathcal{V}$$

- $\mathcal{V} = (\mathbf{Set}, \otimes, \{\star\})$: Pick an identity element $i_x : \{\star\} \to R(x,x)$ and give a function $m_{x,y,z} : R(y,z) \times R(x,y) \to R(x,z)$. Then, Q_R is a category if and only if the family $\{R(x,y)\}$ can be equipped with identity elements and associative composition.

Examples

- $\mathcal{V} = (\mathbf{Vect}_k, \otimes, k)$: Pick a linear map $k \xrightarrow{k_x} R(x, x)$ and a bilinear composition $\mathrm{m}_{x,y,z} : R(y, z) \otimes_k R(x, y) \to R(x, z)$ satisfying the associativity/unit identities as k-linear maps. Then, Q_R is a category if and only if the family $\{R(x, y)\}$ can become a k-linear category.
- $\mathcal{V} = ([0, \infty], +, 0)$: $0 \xrightarrow{i_x} R(x, x)$ forces $R(x, x) = 0$ and $\mathrm{m}_{x,y,z} : R(y, z) + R(x, y) \to R(x, z)$ forces $R(x, z) \leq R(x, y) + R(y, z)$. Then, Q_R is a category when R acts like a Lawvere metric space.

Enriched Quivers as Data Structures

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1. Take an observer P and Bénabou cosmos $\mathcal{V}$; e.g, $\mathcal{V} = ([0, \infty], +, 0)$.
2. Define a (finite) set S of objects for P to consider; e.g, certain qualia such as colour.
3. Ask P a (finite) number of questions, each relating objects in S to all others; e.g, how cold is x compared to y, how angry is x compared to y.
4. Each question makes a $\mathcal{V}$-relation $R_i : S^2 \to \mathcal{V}$ for $i \in I$, the question index. Each R_i constructs a corresponding $\mathcal{V}$-quiver Q_{R_i}.
5. Once data collection is done, we may construct the product quiver $Q_P = \Pi_{i \in I} Q_{R_i}$.
6. Q_P is a data structure which captures P's perspective without external requirements aside from the choice of $\mathcal{V}$.

Key Results

1. Can continuous degrees of similarity as morphisms produce an enriched category? Are there other models of similarity?
   Yes.
2. Can we "use" Yoneda's Lemma to classify qualia in terms of their relations?
   Not always — for 2-ary relations, only when reflexive and transitive.
3. "What kinds of relations (including similarity) can form an enriched category?"
   For 2-ary $\mathcal{V}$-relations R, when (Q_R, a) is a T-algebra.
4. What are other applications of enriched category theory?
   Enriched quivers are data structures for qualia observations.

Open Questions

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1. Extending these results to n-ary relations.
2. Interpreting alternative $\mathcal{V}$ for data structures.
3. Characterising when a relation R makes Q_R a category, in terms of general profunctors and morphisms.

The Issue with Categorical Consciousness

- "...the essence of the colour experiences resides in relations with other colours..." (Naotsugu Tsuchiya and Hayato Saigo 2021, pg. 10)
- The sacrifice of mathematical rigour
- What if qualia does not constitute a category? How can it?

The Role of Category Theory in Consciousness Science

1. For “characterising” qualia and finding a category of consciousness (Naotsugu Tsuchiya 2025).
2. For explicating structural properties of consciousness theories, such as Integrated Information Theory (Naotsugu Tsuchiya, Taguchi, and Hayato Saigo 2016).
3. For modelling certain philosophical or neuroscientific theories via category theory (Mary 2007).

Any Questions?

1. Can continuous degrees of similarity as morphisms produce an enriched category? Are there other models of similarity?
   Yes.
2. Can we "use" Yoneda's Lemma to classify qualia in terms of their relations?
   Not always — for 2-ary relations, only when reflexive and transitive.
3. "What kinds of relations (including similarity) can form an enriched category?"
   For 2-ary $\mathcal{V}$-relations R, when (Q_R, a) is a T-algebra.
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   Enriched quivers are data structures for qualia observations.

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