Free PLTL Algebras and Coalgebraic Extensions of Hyperdoctrines

2

Dynamic Doxastic Logic

Predicates on interface

$\downarrow$ temporalise

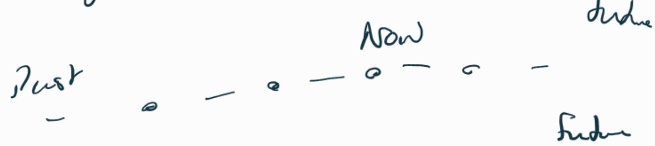
Predicates on systems

Temporal Logic:

interval logic

metric interval logic

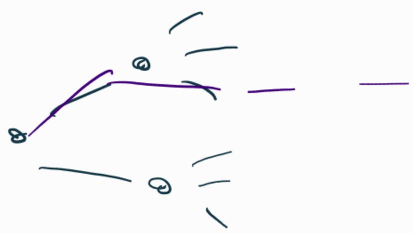
Prior's tense



* propositional linear temporal logic (PLTL)

Computation Tree Logic (CTL)

CTL^*



intuitionistic prop. logic

classical propositional logic

$$\begin{tikzcd}
\mathrm{HeyAlg} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Sets} \arrow[l, bend right, "F"'] \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{BoolAlg} \arrow[l, bend left, "U"]
\end{tikzcd}$$

PLTL: A countable set of propositions, $\mathcal{L}$.

$\wedge, \vee, \rightarrow, \neg$ Classical

X, U temporal

ω-sequences.

$$W = \langle w_0, w_1, w_2, w_3, \dots \rangle$$
$$W_{\geq i} = \langle w_i, w_{i+1}, w_{i+2}, \dots \rangle$$

$$\underset{\text{Now}}{\overset{w_0}{\bullet}} \rightarrow \overset{w_1}{\bullet} \rightarrow \overset{w_2}{\bullet} \rightarrow \overset{w_3}{\bullet} \rightarrow \dots$$

X: $\boxed{\text{neXt}}$ $\quad \overset{w_i}{X\phi} \quad \overset{w_{i+1}}{\phi}$

U: $\boxed{\text{Until}}$ $\quad \phi\, U\, \psi$

w_0	w_1	w_2	w_3
$\phi U \psi$	ϕ	ϕ	ψ
ϕ			

$w \models \alpha$:

$$w \models \top$$

for classical $p \in \mathcal{L}$, $w \models p$ iff $p \in w_0$

$w \models \neg\alpha$ iff $w \not\models \alpha$

$w \models \alpha \wedge \beta$ iff $w \models \alpha$ and $w \models \beta$

$w \models X\alpha$ iff $w_{\geq 1} \models \alpha$

$w \models \alpha \, \mathcal{U} \beta$ iff $\exists\, i \geq 0,\ w_{\geq i} \models \beta$ and for each $0 \leq j < i,\ w_{\geq j} \models \alpha$.

$F\alpha := \top \, \mathcal{U} \, \alpha$ (future)

$G\alpha := \neg F \neg \alpha$ (guarantee)

$\models_{PL}$ $\vdash_{PL}$ T_{PL} a theory of PLTL

Category of PLTL-Algebras PLTL-Alg

Objects: $A = (\underbrace{|A|, \leq_A}_{\text{poset}} ;\ \underbrace{\bot^A, \top^A, \neg^A}, \underbrace{X^A, \mathcal{U}^A}, \wedge^A)$

which satisfy PLTL

Morphisms: $h : A \to B$ preserves all symbols including derived classical + temporal operators.

$$\begin{tikzcd} \mathrm{Pos} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \text{PLTL-Alg} \arrow[l, bend left, "U"] \end{tikzcd}$$

Given a poset $(P, \leq_1)$, $p \leq_1 q$ viewed as axioms stemming from P, and so encode the order of P as a $\boxed{Th_P} := \{p \to^P q : p \leq_1 q\}$

To construct the free PhTL-algebra of P via the Lindenbaum-Tarski Method: (algebraizable logics)

1) We first recursively build a set $F(P)$:

- $P \subseteq F(P)$
- $\phi \wedge^P \psi, \neg^P \phi, \phi U^P \psi, X^P \phi \in F(P)$
- $\bot^P, \top^P \in F(P)$

$T_P \cup \{\phi\} \vdash \psi$

$T_P \cup \{\phi\} \vdash \psi$ iff $T_P \vdash G\phi \to \psi$

2) $T_P := Th_P \cup T_{PL}$

3) $F_{PL}(P) := \dfrac{F(P)}{\sim_P}$, where $\phi \sim_P \psi$ iff $T_P \vdash \phi \leftrightarrow^P \psi$.

We have an ordering $\leq_2$ on $F_{PL}(P)$ via $[\phi] \leq_2 [\psi]$ iff $T_P \vdash \phi \to^P \psi$.

Axioms of PLTL : T_{PL}

Rules:

MP
$$\frac{\alpha \quad \alpha \to \beta}{\beta} \qquad\qquad \frac{\alpha}{G\alpha}$$

Axiom Schema:

C0: any propositional tautology

C1: $\boxed{F\neg\neg\alpha} \overset{=}{\leftrightarrow} \boxed{F\alpha}$

C2: $\boxed{G(\alpha \to \beta)} \overset{\leq}{\to} \boxed{(G\alpha \to G\beta)}$

C3: $G\alpha \to (\alpha \wedge X\alpha \wedge X(G\alpha))$

C4: $X\neg\alpha \leftrightarrow \neg X\alpha$

C5: $X(\alpha \to \beta) \to (X\alpha \to X\beta)$

C6: $G(\alpha \to X\alpha) \to (\alpha \to G\alpha)$

C7: $(\alpha U \beta) \leftrightarrow (\beta \vee (\alpha \wedge X(\alpha U \beta)))$

C8: $\alpha U \beta \to F\beta$

$$\begin{tikzcd}
\underline{\mathrm{Pos}} \arrow[r, bend left, "F_{PL}"] \arrow[r, phantom, "\perp"] & \text{PLTL-Alg} \arrow[l, bend left, "U_{PL}"] \arrow[r, bend right, "U"'] & \text{BoolAlg} \arrow[l, bend right, "F"']
\end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Pos} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{PLTL\text{-}Alg} \arrow[l, bend left, "U"] \end{tikzcd}$$

$$M := UF : \mathrm{Pos} \to \mathrm{Pos}$$

$$(M, \eta^M, \mu^M)$$

unit: $p \longmapsto [p]$

formula-of-formulae $\longrightarrow$ formula

$$\mu^M\left(\underline{[\underline{[\phi \cup \psi]} \to \underline{[\chi]}]}\right) = [(\phi \cup \psi) \to \chi]$$

$$\mathrm{Pos}^M \simeq \mathrm{PLTL\text{-}Alg}$$

__Suppose:__ I is a comonad on $\mathcal{C}$

L is a monad on Pos

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[loop above, "I^{op}"] \arrow[r, "P"] & \mathrm{Pos} \arrow[loop above, "L"] \end{tikzcd}$$

__Fact:__ (Regular) hyperdoctrines are the same as symmetric monoidal double functors / $\boxed{\mathrm{Span}(\mathcal{C})^{op} \to \mathbb{Q}\mathrm{t}(\mathrm{Pos})}$

with a companion commuter property.

<u>Fact:</u> The 2-cat Dbl^{ps} admits the construction of algebras: $i : \mathrm{Dbl}^{ps} \longrightarrow \mathrm{Mnd}(\mathrm{Dbl}^{ps})$ has a right adjoint $\mathbb{A}\mathrm{lg}(-) : \mathrm{Mnd}(\mathrm{Dbl}^{ps}) \to \mathrm{Dbl}^{ps}$.

If I preserves pullbacks, then I^{op} induces a monad on the double cat of spans over $\mathcal{C}$:

$$I^* : \mathbb{S}\mathrm{pan}(\mathcal{C})^{op} \longrightarrow \mathrm{Span}(\mathcal{C})^{op}.$$

<u>Similarly:</u>

$L^* : \mathbb{Q}\mathrm{t}(\mathrm{Pos}) \longrightarrow \mathbb{Q}\mathrm{t}(\mathrm{Pos})$ in Dbl^{ps}.

We get a map

$\mathbb{A}\mathrm{lg}(I^*) \longrightarrow \mathbb{A}\mathrm{lg}(L^*)$ whenever we have a map of monads from I^{op} to L.

<u>Fact:</u> $\underline{\mathbb{S}\mathrm{pan}(\mathrm{Coalg}(I))^{op}} \simeq \mathbb{A}\mathrm{lg}(I^*)$.

A map of monads ρ from I^{op} to L induces a pseudo-double functor from $\mathbb{S}\mathrm{pan}(\mathrm{Coalg}(I))^{op}$ to

$\mathrm{Alg}(h^*)$, which looks like an existential hyperdoctrine.

$$\begin{tikzcd} \mathbf{Set}^{op} \arrow[loop above, "I^{op}"] \arrow[r, "P"] & \mathbf{Pos} \arrow[loop above, "M"] \end{tikzcd}$$

I as the stream comonad on $\mathbf{Set}^{op}$

$p(X) := X$

cofree comonad of X is the stream comonad I

where $I(A) = A^{\mathbb{N}}$,

$\varepsilon_A(\alpha) = \underline{\alpha(0)}$

$\delta_A(\alpha) = \left(\alpha^{(k)}\right)_{k \in \mathbb{N}}$.

$\alpha^{(k)}(n) = \alpha(n+k)$

$P \overset{\text{Sub}}{=}$ is part of a map of monads.

$\lambda : MP \to PI^{op}$ s.t.:

$$\begin{tikzcd}
P \arrow[d, "\eta^M P"'] \arrow[dr, "P_{\eta}^{I^{op}}"] & \\
MP \arrow[r, "\lambda"'] \arrow[ur, phantom, "2"] & PI^{op}
\end{tikzcd}
\qquad
\begin{tikzcd}
MMP \arrow[r, "M\lambda"] \arrow[d, "\mu^M P"'] & MPI^{op} \arrow[r, "\lambda I^{op}"] \arrow[d, phantom, "2"] & PI^{op}I^{op} \arrow[d, "P_{\mu}^{I^{op}}"] \\
MP \arrow[rr, "\lambda"'] & & PI^{op}
\end{tikzcd}$$

Fix a set A.

syntax

$$\varphi \longmapsto \lambda_A(\varphi)$$

semantics

$$\lambda_A : \underbrace{MPA}_{} \longrightarrow \underbrace{PI^{op}A}_{}$$

free PLTL algebra of $P(A)$

poset of predicates on the set $A^{\mathbb{N}}$, thus describing properties of streams/possible behaviours regarding sequences of states.

$\alpha \in A^{\mathbb{N}}$

$\lambda_A(\phi)$ determines whether $\alpha \models_A \phi$.

The behavior of a PLTL formula is via its predicate on streams:

$$[\![\varphi]\!]_A := \lambda_A(\varphi) : A^{\mathbb{N}} \to \Omega$$

truth poset $\{f \leq T\}$

Viewing I as an interface that exposes an entire trajectory of states and P as providing predicates on I, λ allows us to interpret each pLTL formula as the set of streams satisfying it.

Predicate on X $\longrightarrow$ Predicate on IX $= X^{\omega}$.

$$P(X) = BX$$