Topos UK – Oxford Seminar

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jww David Jaz Myers

2-classifiers for 2-algebras

or, the abstract study of compositionality

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Motivation

This work is propedeutic to the wider DCST (or DOTS) program.

Indeed, the two main tenets of DCST are: (1) systems organize in algebras of symmetric monoidal double categories (or double operads) and (2) behaviour is specified by functors into 'behavioural theories', which are Set-like theories of systems 'à là Willems'.

Such functors are often 'corepresentable', but without a formal theory of 'copresheaves of systems theories' this is just a suggestive term. So we set to study the concept of 'copresheaf', i.e. classifying map for a discrete opfibration, for 2-algebraic structures.

The ultimate goal is to characterize the compositionality of a corepresentable behaviour in terms of the properties of the theory of systems it is defined over and the corepresenting object that generates it.

2-classifiers in enhanced 2-categories

Enhanced 2-categories, introduced by Lack and Shulman in Reference [lack-2012-enhanced], are 2-categories equipped with a distinguished wide sub-2-category whose 1-cells are called *tight*.

We work will with with such 2-categories for much of the same reasons Lack and Shulman do, that is, to single out the well-behaved *strict* morphisms of 2-(co)algebras amongst (co)lax ones.

Definition 2.2.1. [mc-000S] [edit]
An **enhanced 2-category**, or $\mathscr{F}$**-category**, is a 2-category $\mathcal{K} \equiv: \mathcal{K}_\lambda$ whose 1-cells are called **loose** and a wide and locally full subcategory $J_{\mathcal{K}} : \mathcal{K}_\tau \hookrightarrow \mathcal{K}$.

In other words, being tight is a mere property of 1-cells of an enhanced 2-category, and 2-cells between tight 1-cells are the same as the 2-cells between the same 1-cells considered as loose. In particular, $\mathcal{K}_\tau \hookrightarrow \mathcal{K}_\lambda$ is identity-on-objects, faithful and locally fully faithful. This is equivalent to the definition as a category enriched in the category $\mathscr{F}$ of full embeddings of categories, as already noted by Reference [lack-2012-enhanced].

From the definition as enriched categories, we also get that *enhanced 2-functors* are 2-functors that preserve tightness, while *enhanced 2-natural transformations*, or *tight natural transformations*, are 2-natural transformations whose components are all tight.

Definition 2.2.3. Enhanced 2-monad [mc-0010] [edit]

An **enhanced 2-monad** is an $\mathscr{F}$-monad, thus a 2-monad (T, i, m) such that T preserves tightness and where i and m have tight components.

Definition-example 2.2.4. Enhanced 2-category of T-algebras and lax morphisms [mc-000L] [edit]

The **enhanced 2-category of T-algebras and lax T-morphism** $\mathcal{A}\mathsf{lg}_l(T)$ for an enhanced 2-monad T on the enhanced 2-category $\mathcal{K}$ is the enhanced 2-category so comprised:

1. its objects are strict T-algebras whose structure map is tight in $\mathcal{K}$,
2. its loose maps are lax T-morphisms,

$$\begin{array}{ccc} TA & \overset{Tf}{\rightsquigarrow} & TB \\ {\scriptstyle\alpha}\big\downarrow & \Swarrow & \big\downarrow{\scriptstyle\beta} \\ A & \underset{f}{\rightsquigarrow} & B \end{array}$$

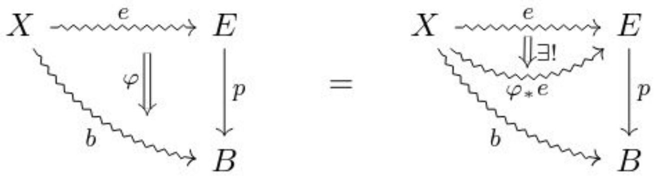
3. its tight maps are strict T-morphisms whose underlying map is tight in $\mathcal{K}$,

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ A & \xrightarrow[f]{} & B \end{array}$$

4. its 2-morphisms are T-2-morphisms.

Definition 2.3.1. (Tight) discrete opfibration [mc-000X] [edit]

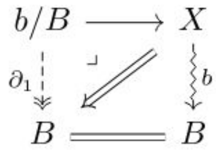
A **(tight) discrete opfibration** in a(n enhanced) 2-category $\mathcal{K}$ is a (tight) map $p : E \to B$ that admits unique (*opcartesian*) lifts:



The chief example of discrete opfibration is given by projections out of a comma:

Definition 2.3.2. Representable discrete opfibration [mc-001T] [edit]

A discrete opfibration is **representable** if it is equivalent to the projection out of a comma object dashed below:



We say $b/B \xrightarrow{\partial_1} B$ is **represented by the object** $b : X \to B$. When $X = 1$, we say it is **globally representable**. When $b = \mathrm{id}_B$, we get the **domain opfibration** associated to B.

For an enhanced 2-category, we are guaranteed that a representable discrete opfibration is tight when the comma is tight, or when b is a tight map and the comma is left-tight.

The following is an instantiation of the notion of *good 2-classifier* as given in Definition 2.15 of Reference [mesiti-2024-classifiers] for the pullback-stable property P of being a tight discrete opfibration.

Definition 2.3.3. Enhanced 2-classifier [mc-0012] [edit]

An **enhanced 2-classifier** in an enhanced 2-category $\mathcal{K}$ with tight terminal object 1 and left-tight commas is a tight map

$$\tau : 1 \to \Omega$$

such that the functor $\tau/-$ induced by taking comma objects is fully faithful and essentially surjective on tight discrete opfibrations:

$$\begin{array}{ccc} \mathcal{K}(B,\Omega) & \xrightarrow[\text{ff}]{\tau/-} & \mathsf{dF}ib(B) \\ & \overset{\sim}{\searrow} & \uparrow \\ & & \mathsf{dF}ib(B)_\tau \end{array}$$

This definition deliberately mimicks that of subobject classifier in a topos, where τ is thought of as picking out a *verum* inside Ω. And in fact the above can be rephrased more informally as saying that, for every discrete opfibration $p : E \to B$, there is a unique classifying morphism $\mathrm{d}p : B \to \Omega$ making the square below a comma:

$$\begin{array}{ccc} E & \longrightarrow & 1 \\ {\scriptstyle p}\big\downarrow & \llcorner \;\; \swarrow\!\!\!\swarrow & \big\downarrow{\scriptstyle \tau} \\ B & \underset{\mathrm{d}p}{\longrightarrow} & \Omega \end{array}$$

Crucially, $\mathrm{d}p$ does not need to be tight.

When p is representable by b, then we also say $\mathrm{d}p$ is representable and denote it by y^b.

Example 2.3.4. [mc-002R] [edit]

The archetypal example of 2-classifier is $1 \xrightarrow{1} \mathsf{Set}$ in $\mathcal{C}\mathsf{at}$ (which, for us, is the 2-category of locally small categories). This recovers the well-known discrete opfibration of elements construction:

$$\begin{array}{ccc} \int F & \longrightarrow & 1 \\ {\scriptstyle \partial_1}\big\downarrow & \lrcorner \;\swarrow & \big\downarrow{\scriptstyle 1} \\ B & \xrightarrow[F]{} & \mathsf{Set} \end{array}$$

Construction-example 2.3.5. Mesiti's result [mc-002P] [edit]

In Reference [mesiti-2024-classifiers], Mesiti shows that $[\mathsf{C}, \mathcal{C}\mathsf{at}]$ admits a 2-classifier:

$$
\begin{array}{ccc}
\mathsf{C} & \xrightarrow{\quad\Omega\quad} & \mathcal{C}\mathsf{at} \\[1em]
c & & [c/\mathsf{C}, \mathsf{Set}] \\
{\scriptstyle f}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle (f^*)^*} \\
c' & & [c'/\mathsf{C}, \mathsf{Set}]
\end{array}
$$

with $\tau : 1 \to \Omega$ picking $1 \in [c/\mathsf{C}, \mathsf{Set}]$ for each $c \in \mathsf{C}$.

Example 2.3.6. 2-classifier on $\mathcal{G}\mathsf{raph}(\mathcal{C}\mathsf{at})$ [mc-002Q] [edit]

Following Mesiti's recipe for $\mathcal{G}\mathsf{raph}(\mathcal{C}\mathsf{at}) = [\{e \rightrightarrows v\}, \mathcal{C}\mathsf{at}]$, one finds that:

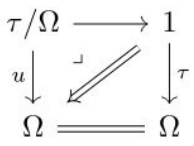
$$
\Omega(e) = \left\{ \begin{array}{ccc} & \mathrm{id}_e & \\ \swarrow & & \searrow \\ s & & t \end{array} \right\} \to \mathsf{Set}
$$

$$
\Omega(v) = \{\mathrm{id}_v\} \to \mathsf{Set}
$$

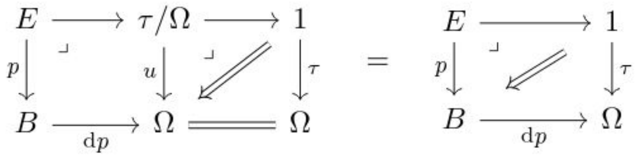
with $\Omega(s)$ and $\Omega(t)$ given by the evident restrictions. Thus Ω is the (large) graph of spans of sets, and $\tau : 1 \to \Omega$ picks out the trivial span $1 = 1 = 1$.

In the presence of a 2-classifier, there always is a special fibration:

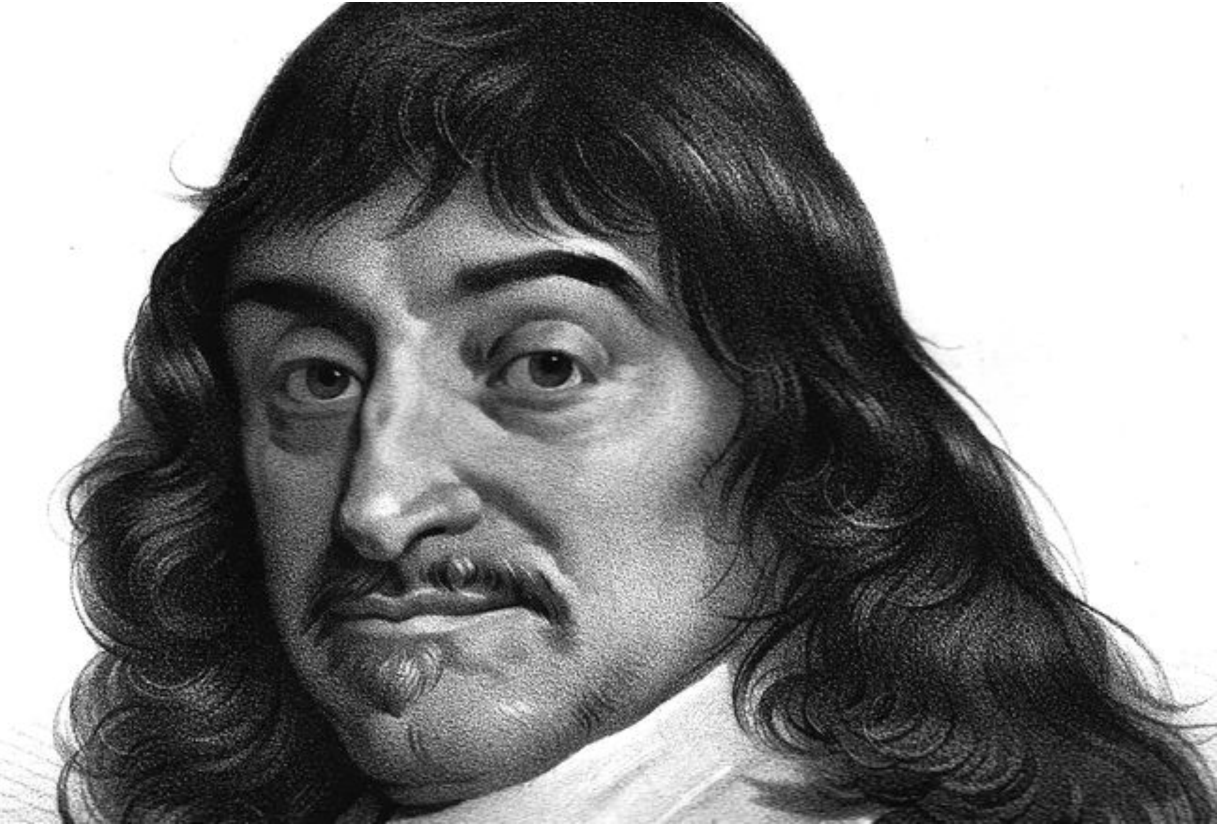
Construction 2.3.4. Generic discrete opfibration [mc-000A] [edit]
The discrete opfibration classified by the identity or, equivalently, represented by τ, is called the **generic discrete opfibration**:



As the name suggest, one can equivalently use u to classify discrete opfibrations, now by taking strict pullbacks rather than commas. Specifically, it is the case that $p = \mathrm{d}p^*u$, by a straightforward application of the pasting lemma for commas (Theorem 2.1):



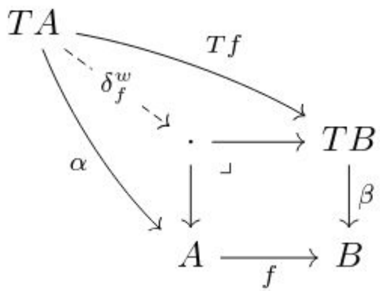
This is the original treatment given by Weber in Reference [weber-2007-yoneda]. As remarked by Mesiti in Reference [mesiti-2024-classifiers], the main advantage of using Definition [djm-00D6] instead is a rather straightforward definition of the functor turning classifying morphisms into discrete opfibrations, which in the case of a generic discrete opfibration needs instead to be define `by hand' using the lifting property of u *qua* opfibration, rather than following automatically from the universal property of commas. This streamlines the proof of Theorem 4.10, though in other places we do still use Weber's definition.



Cartesianity

Definition 3.1. [mc-002C] [edit]

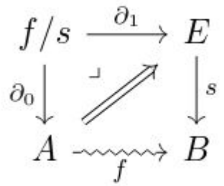
Let $f : A \to B$ be a strict T-morphism. We call **T-cartesianity defect of** f the canonical comparison map induced by the pullback below:



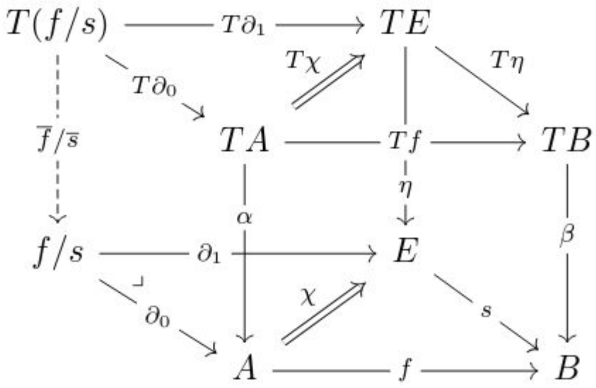
We say f is **T-cartesian** when its defect is invertible.

Proposition 3.3. **Reference [lack-2005-lax] (Proposition 4.6)** [mc-000M] [edit]

The forgetful 2-functor $\mathcal{A}\mathsf{lg}_c(T) \to \mathcal{K}$ creates comma objects of the form f/s, where f is colax and s is strict:



This basically means the structures on f and s equip f/s with a T-algebra structure, thus the above diagram (which exists in $\mathcal{A}\mathsf{lg}_c(T)$) corresponds to the following diagram in $\mathcal{K}$, where the dashed map is the T-algebra in question:



Definition 3.4. [mc-0004] [edit]

Let (A, α) be a T-algebra and $a : X \to A$ a colax T-morphism. We say α is **cartesian at** a when the representable discrete opfibration at a (Definition 2.3.2) is a T-cartesian morphism:

$$\begin{array}{ccc} T(a/A) & \xrightarrow{T\partial_1} & TA \\ {\scriptstyle \overline{a}/\alpha}\downarrow & \lrcorner & \downarrow{\scriptstyle \alpha} \\ a/A & \xrightarrow{\partial_1} & A \end{array}$$

Example 3.5. [mc-001E] [edit]

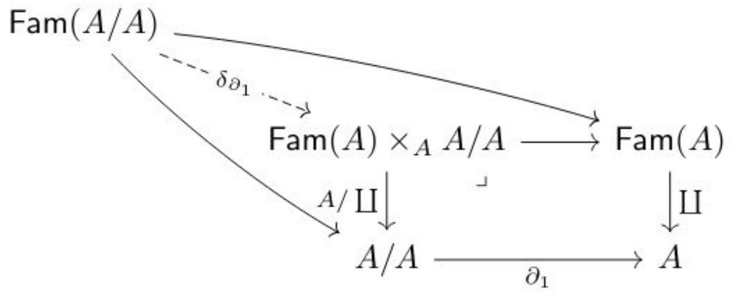
Let $T = \mathsf{Fam}$ be the free coproduct completion 2-monad on Cat. Then a category with coproducts A, i.e. a Fam-algebra, is cartesian at id_A iff it is *strictly* extensive.

By *strictly extensive* we mean that there is an *isomorphism* of categories $A/(a+b) \cong A/a \times A/b$, rather than the usual equivalence (Reference [carboni-1993-introduction]). We end up with this weaker notion of extensivity as a consequence of working with strict 2-pullbacks, rather than `strict bipullbacks' (i.e. pullbacks whose universal property holds only up to equivalence).

To prove the claim we made, observe the comma algebra $A/\coprod : \mathsf{Fam}(A/A) \to A/A$ is given by

$$(I \in \mathsf{Finset}, (\varphi_i : x_i \to y_i)_{i \in I}) \mapsto \coprod_{i \in I} \varphi_i : \coprod_{i \in I} x_i \to \coprod_{i \in I} y_i$$

Consider



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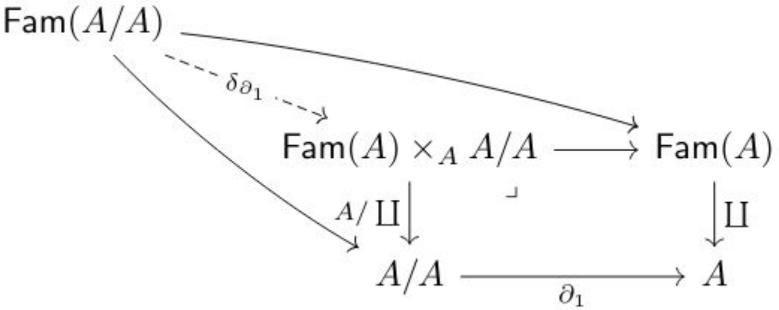


The Fam-cartesianity defect of $\partial_1 : A/A \to A$ sends $(I, (\varphi_i : x_i \to y_i)_i)$ to $((I, (y_i)_i), \coprod_i \varphi_i : \coprod_i x_i \to \coprod_i y_i)$. To say this is an isomorphism means saying that for all $\psi : X \to \coprod_i y_i$ there exists a unique family $(\psi_i : x_i \to y_i)_i$ such that $\coprod_i \psi_i = \psi$. Clearly, this is precisely saying that A is strictly extensive, i.e. that the canonical map $\prod_i A/y_i \to A/(\coprod_i y_i)$ is an isomorphism of categories.

Remark 3.7. [mc-002I] [edit]

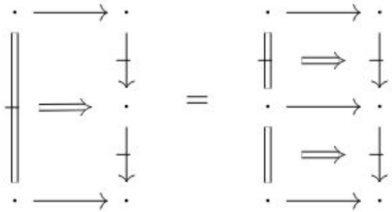
In the special case $X = 1$, we get that

$$A(a, t(b_1, \ldots, b_n)) \cong t(A(a, b_1), \ldots, A(a, b_n))$$

For T the free symmetric monoidal category 2-monad on $\mathcal{C}\mathsf{at}$ is easy to see this corresponds to the characteristic representability of products (we will come back to this example shortly, in Example 3.15). In general, the above definition is quite literally expressing the fact that the corepresentables at a, which normally are just lax T-morphisms, be actually *strong*.

Example 3.8. Spanish double categories [mc-002O] [edit]

A **spanish** double category is one for which every `triangle' as below left factors uniquely as below right:



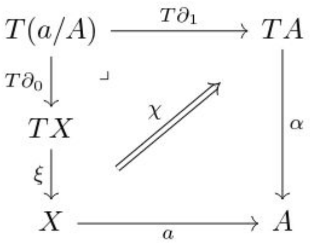
Consider the free double category 2-monad fc on $\mathcal{G}\mathsf{raph}(\mathcal{C}\mathsf{at})$. A strict object $a : 1 \to A$ in a double category A is simply an object of A, and A is spanish exactly when it is fc-cartesian at all its objects. Indeed, an object of $\mathsf{fc}(a/A)$ is a composite of triangles such as above right, while $\alpha \times_A \partial_1$ is comprised of single triangles as above left. The cartesian defect of a composes a series of triangles, and asking this map invertible means having unique factorizations as required.

Proposition 3.9. [mc-000D] [edit]

Let $a : X \to A$ be a colax morphism of T-algebras. The following are equivalent:

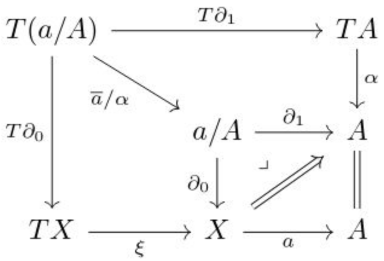
1. A is cartesian at a,
2. the following is a comma square:

   Diagram. [mc-0025] [edit]



Proof. [mc-000E] [edit]

By Proposition 3.3, the square on the left commutes:



The claim is then a direct application of the pasting lemma for commas (Theorem 2.1) to the two squares on the right, constituting Diagram [mc-0025].

Generalized 'Reverse Fox' Theorem

Observe that Diagram [mc-0025] above doesn't mention the colax structure of a. In fact, when T is `cartesian at the representable discrete opfibrations', such colax structure is uniquely determined. Here's what we mean:

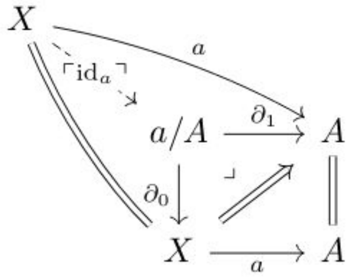
Definition 3.10. [mc-0008] [edit]
A 2-natural transformation $\phi : F \Rightarrow G : \mathcal{K} \to \mathcal{H}$ is **cartesian at a morphism** $f \in \mathcal{K}$ when the 2-naturality square of ϕ at f is a strict 2-pullback in $\mathcal{H}$.

Definition 3.11. [mc-001Z] [edit]
A 2-monad T is **cartesian at a class of maps** $\mathcal{M} \subseteq \mathcal{K}$ when its unit and multiplication, as 2-natural transformations, are cartesian at the maps in question.

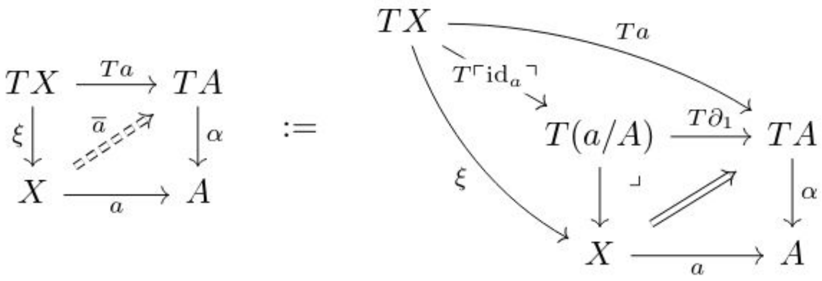
Theorem 3.12. Costructure for free [mc-001W] [edit]
Suppose T has unit and multiplication i and m cartesian at representable discrete opfibrations. Let (X, ξ) and (A, α) be T-algebras, and let $a : X \to A$ be an object of A. Consider the discrete opfibration represented by a (Definition 2.3.2) and suppose Diagram [mc-0025] is a comma. Then a is equipped with a unique T-colax structure.

Proof. [mc-001V] [edit]

We construct the colaxator in two steps. First, consider the universal map $\ulcorner \mathrm{id}_a \urcorner : X \to a/A$ induced by factoring the identity square of a through the comma square defining the discrete opfibration represented by a:



Then, whisker Diagram [mc-0025] by $T\ulcorner \mathrm{id}_a \urcorner$ to get the desired 2-cell $\overline{a}$:



Therefore, for these well-behaved monads, we get a less demanding notion of cartesianity at an object. Indeed, for a T-algebra (A, α), we could distinguish between **unstructured objects**, which are just objects $a : X \to A$ of its carrier A, and **T-co/structured objects**, which are co/lax T-morphisms $a : X \to A$. Then Definition 3.4 defined *T-cartesianity at a T-costructured object*, whereas Theorem 3.12 defines *T-cartesianity at an (unstructured) object*. We record this definition:

Definition 3.13. [mc-0022] [edit]
Let T be a 2-monad cartesian at the representable discrete opfibrations. Given a T-algebra (A, α) and a (plain) morphism $a : X \to A$, we say α is **cartesian at a** when Diagram [mc-0025] is a comma.

And we have:

Corollary 3.14. [mc-0024] [edit]
(A, α) is T-cartesian at the (unstructured) object a if and only if it is cartesian at the (uniquely) T-costructured object a.

Lifting 2-classifiers

Definition 4.1. Plumbus [mc-001K] [edit]

A **plumbus** is an enhanced 2-category admitting the following 2-limits:

1. all left-tight pullbacks of tight discrete opfibrations,
2. all left-tight commas,
3. a tight terminal object.

Proposition 4.3. [mc-0011] [edit]

When $\mathcal{K}$ is a plumbus and (T, i, m) an enhanced 2-monad, the 2-category $\mathcal{A}\mathsf{lg}_l(T)$ is a plumbus too.

Proof. [mc-002F] [edit]

We take $\mathcal{A}\mathsf{lg}_l(T)$ with its canonical enhanced structure of Definition-Example 2.2.4 given by the choice of T-strict tight morphisms. It admits tight-over-loose commas because $\mathcal{K}$ does and by Proposition 3.3 we know these are created in $\mathcal{A}\mathsf{lg}_l(T)$. Finally, pullbacks of tight discrete opfibrations exists because, first of all, they do in $\mathcal{K}$ and then by (the dual of) Theorem 5.10 from Reference [capucci-2024-contextads]. Indeed, the latter result says that strict normal tight opfibrations admit pullbacks along arbitrary lax T-morphisms in $\mathcal{A}\mathsf{lg}_l(T)$. Since discrete opfibrations are automatically normal, we get our desired claim.

From now on, fix a plumbus $\mathcal{K}$ with enhanced 2-classifier $\tau : 1 \to \Omega$ (Definition [djm-00D6]). In the following, we will be concerned in lifting this latter piece of structure to the plumbus $\mathcal{A}\mathsf{lg}_l(T)$. This requires the 2-monad to play along. We now single out two sets of conditions that need to be verified.

Definition 4.4. Opfibrantly cartesian enhanced 2-monad [mc-000K] [edit]
An enhanced 2-monad (T, i, m) is **opfibrantly cartesian** when

1. T preserves pullbacks of tight discrete opfibrations,
2. T preserves tight discrete opfibrations,
3. i and m are cartesian at all tight discrete opfibrations.

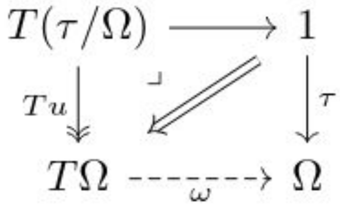
Remark 4.5. [mc-002K] [edit]
Opfibrantly cartesian 2-monads is an enhanced and slightly weaker notion of `cartesian' 2-monad, i.e. a 2-monad required to respect 2-limits and whose structure maps are cartesian at the tight maps. Indeed, every 2-monad which preserves commas preserves discrete opfibrations.

Lemma 4.6. [mc-0017] [edit]

An enhanced 2-monad (T, i, m) is opfibrantly cartesian if and only if:

1. ' T preserves pullbacks of the generic discrete opfibration,
2. ' Tu is a (tight) discrete opfibration,
3. ' i and m are cartesian at the generic discrete opfibration.

Diagram. [mc-0029] [edit]



Lemma 4.7. [mc-0015] [edit]

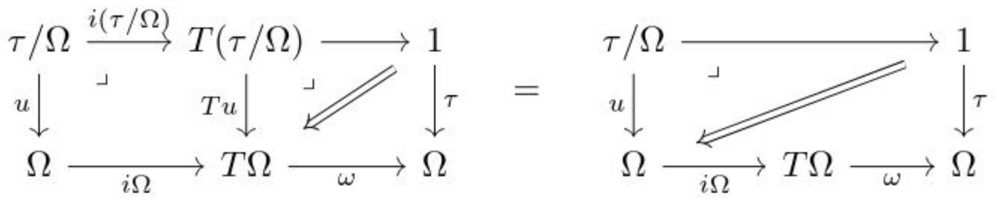
The map $\omega : T\Omega \to \Omega$ defined in the proof of Lemma 4.6 is a ~~strict~~ T-algebra.

* in fact, only pseudo!

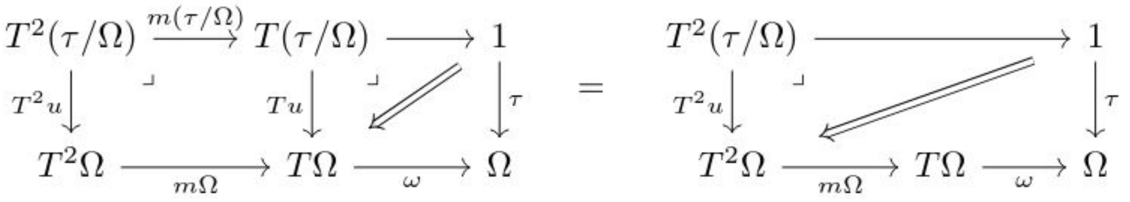
Proof. [mc-0016] [edit]

In the proof below, the definition of 2-classifier only implies the two classifying maps are isomorphic

Strict unitality is proven by observing that



and thus $\omega(i\Omega) = \mathrm{d}u = \mathrm{id}_\Omega$. In the above, the left hand side pullback square is given by cartesianity of i at the tight discrete opfibration Tu. We obtain strict associativity in the same way, appealing to cartesianity of m instead. Indeed we have



and therefore, applying repeatedly the identity proven in Proof [mc-0018], $\omega(m\Omega) = \mathrm{d}(T^2u) = \omega(T\mathrm{d}(Tu)) = \omega(T\omega)$.

Lemma 4.8. [mc-0019] [edit]

The 2-classifier $\tau : 1 \to \Omega$ is a colax T-morphism and (Ω, ω) is cartesian at τ.

Proof. [mc-0023] [edit]

The first part follows from Theorem 3.12. Indeed T satisfies the necessary assumptions and Diagram [mc-0025] instantiates to the comma square classifying Tu, i.e. Diagram [mc-0029]. The second part is precisely Corollary 3.14.

What does this mean? It means that `points of a term are terms of the points', and thus that ω is directly defined by its classification property.

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What does this mean? It means that `points of a term are terms of the points', and thus that ω is directly defined by its classification property.

Proposition 4.9. [mc-002A] [edit]

An enhanced 2-monad (T, i, m) is opfibrantly cartesian if and only if:

1. ' T preserves pullbacks of the generic discrete opfibration,
2. '' Ω is equipped with a T-algebra ω which is cartesian at τ.

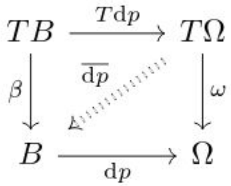
However, in order to lift τ to $\mathcal{A}\mathsf{lg}_l(T)$, we must assume τ is in fact *strict*, otherwise we cannot even consider it as lax T-morphism, and as seen Proposition 3.3 we cannot take commas with strong T-morphisms in general. This explains the hypotheses of the following, which is the main theorem of the paper:

Theorem 4.10. [mc-0003] [edit]
Let T be an opfibrantly cartesian 2-monad (Definition 4.4) on a plumbus $\mathcal{K}$ (Definition 4.1) such that $\tau : 1 \to \Omega$ is a strict T-morphism (with respect to the algebra structure of Lemma 4.7). Then its enhanced 2-category of T-algebras and lax morphisms (Definition-Example 2.2.4) admits an enhanced 2-classifier.

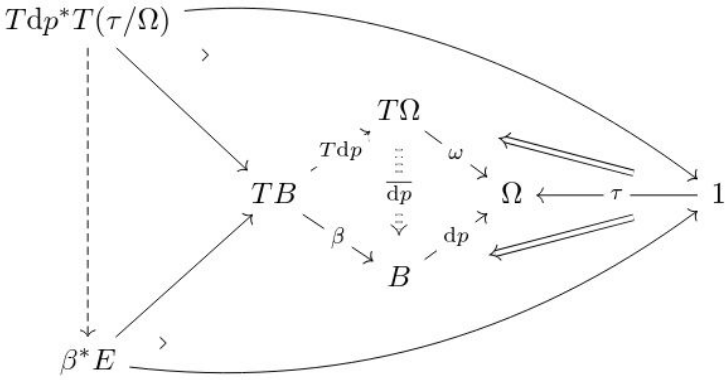
Proof. [mc-000J] [edit]
Let $p : E \to B$ be a tight discrete opfibration in $\mathcal{A}\mathsf{lg}_l(T)$, and denote by η and β, respectively, the algebra structures on E and B. Consider the classifying map $\mathrm{d}p : B \to \Omega$ obtained in $\mathcal{K}$ from $U(p)$. By Proposition 3.3, if p can be classified, it must be by a lax T-morphism carried by $\mathrm{d}p$.

Thus we only need to show $\mathrm{d}p$ admits a lax T-structure (see (1.1)--(1.3) in Reference [blackwell-1989-twodimensional]), which first of all amounts to a 2-cell:

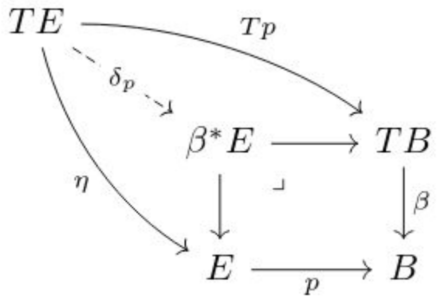
Diagram. [mc-000N] [edit]



To see this, consider the above square as a 2-cell between classifying maps. Such 2-cells, by the structure of a 2-classifier, are in one-to-one correspondence with maps as dashed below, where the (non-dotted) 2-cells are classifying comma squares (squashed beyond recognition):



By the proof of Lemma 4.6, we know that $T\mathrm{d}p^*Tu \cong Tp$. Thus we get $\overline{\mathrm{d}p}$ by exhibiting a map $TE \to \beta^* p$ over TB, and indeed there is a unique such map, namely the T-cartesianity defect of p (Definition 3.1):



Corollary 4.12. [mc-001R] [edit]

Let $p : E \to B$ be a tight discrete opfibration in $\mathcal{A}\mathsf{lg}_l(T)$. Its classifying map $\mathrm{d}p : B \to \Omega$ is strong if and only if p is T-cartesian (Definition 3.1).

Proof. [mc-001S] [edit]

It follows by inspection of the proof of the main theorem: the laxator of $\mathrm{d}p$ classifies the strict T-cartesianity defect of p.

In particular, a representable tight discrete opfibration (Definition 2.3.2) $b/B \xrightarrow{\partial_1} B$ is T-cartesian precisely when B is T-cartesian at its representing object $b : X \to B$, by definition (Definition 3.4). Thus:

Corollary 4.13. [mc-001U] [edit]

The representable copresheaf $y^b : B \to \Omega$ associated to a representable discrete opfibration $b/B \xrightarrow{\partial_1} B$ is always a lax T-morphism, and it is strong precisely when B is T-cartesian at b.

Thanks!