Composing Flavoured Petri Nets

David Jaz Myers
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Topos Institute

in the framework of

Double Operadic Theory of Systems

Sophie Libkind + DJM, arXiv: 2505.18329

Building on the work of

- John Baez + Jade Master, arXiv: 1808.05415
- Jade Master, arXiv: 1904.09091
- Fabrizio Genovese + David Spivak, arXiv: 2002.02762
- Joachim Kock, arXiv: 2005.05108

What is a Petri Net?

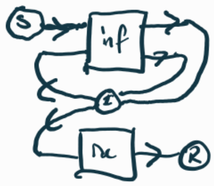
- A graph-like structure for presenting dynamical systems
  - "Species" or 'things'
  - "transitions" or 'basic processes'

  at each transition $t : s_1, \ldots, s_n \longrightarrow s'_1, \ldots, s'_m$ transforms (potentially w/ repeats)

E.g. the SIR-model from Epidemiology

3 species
- S susceptible
- I infected
- R recovered

$$\begin{cases} \text{infection} : S, I \longrightarrow I, I \\ \text{recovery} : I \longrightarrow R \end{cases}$$



Mass-action Eqn

$$\begin{cases} \frac{dS}{dt} = - r_{inf} SI \\ \frac{dI}{dt} = +2r_{inf} SI - r_{inf} SI - r_{rec} I \\ \frac{dR}{dt} = r_{rec} I \end{cases}$$

Why do category theorists love Petri nets?

- B/c Petri nets <u>present</u> symmetric monoidal categories

Petri net : SMC :: Graph : Category

What actually is a Petri Net?

∘ "Whole grain" Petri Nets by Kock

$$\begin{tikzcd} & I \arrow[dl] \arrow[dr, "\text{finite fibers}"] & & O \arrow[dl, "\text{finite fibers}"'] \arrow[dr] & \\ S & & T & & S \end{tikzcd}$$

$I_t \cong$ ... t

ie

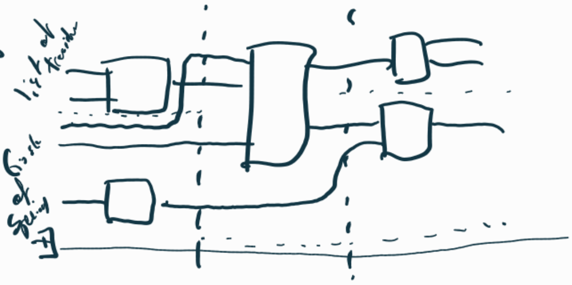
$$\underbrace{\mathrm{Fin}^{\cong}/S}_{SMC(S)} \xleftarrow{I} T \xrightarrow{O} \underbrace{\mathrm{Fin}^{\cong}/S}_{SMC(S)}$$

$$\begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ SMC(S) & & SMC(S) \end{tikzcd}$$

$$\begin{tikzcd} I_t \arrow[d] & t \arrow[l, mapsto] \arrow[r, mapsto] & O_t \arrow[d] \\ S & & S \end{tikzcd}$$

Processes in a Petri Net

Given a petri net $P = SMC(S) \xleftarrow{I} T \xrightarrow{O} SMC(S)$ a <u>process</u> is a sequence of possible transitions



Def: Proc(P) is the free SMC generated by P.

Double

$$\begin{tikzcd} & T \arrow[dl, "I"'] \arrow[dr, "O"] & \\ SMC(S) & & SMC(S) \end{tikzcd} \quad T \in Cat$$

⇝

$$\begin{tikzcd} & SMC(T) \arrow[dl, "I^{\text{lift}\otimes}"'] \arrow[dr, "O^{\text{lift}\otimes}"] & \\ SMC(S) & & SMC(S) \end{tikzcd} \quad SMC(T) \in SM(Cat)$$

$\in$ Graph(SM(Cat))

free (pseudo) category ⇝

Symmetric monoidal (graded) cat of process

$$\begin{tikzcd} & ? \arrow[dl] \arrow[dr] & \\ SMC(S) & & SMC(S) \end{tikzcd}$$

$\underline{Cat(SM(Cat))}$

$SM(\underbrace{Cat(Cat)}_{Dbl})$ $\cong$

SMC : Cat → Cat

Lack's Dubuc's approach to free monoids

$$\begin{tikzcd} & SMC(T) \arrow[dl, "I^{\text{lift}\otimes}"'] \arrow[dr, "O^{\text{lift}\otimes}"] & \\ SMC(S) & & SMC(S) \end{tikzcd} \quad SMC(T) \in SM(Cat)$$

$\in$ Graph(SM(Cat))(SMC(S))

$\circlearrowleft$

SMC(S)

SMC(S + T)

$\underline{SMC(S) \oplus SMC(T)}$

⇝

$$\begin{tikzcd} & SMC(S) \oplus SMC(T) \arrow[dl] \arrow[dr] & \\ SMC(S) & & SMC(S) \end{tikzcd}$$

$* \xrightarrow{y} y$

Ptd(...)

$$* \xrightarrow{y} y \rightrightarrows^{yy}_{yy} y^2 \to y^3 \ \cdots$$

$\underline{y \otimes y}$

$Proc(P)_1 = \mathrm{Colim}_n\, y^n$

Takeaways:

$$\begin{tikzcd} \mathrm{Petri} \arrow[r, "\mathrm{Proc}"] \arrow[d] & SM(\mathrm{Dbl}) \arrow[loop below, "(-)^{-1}"] \\ \mathrm{Graph}(\mathrm{SMCat}) & \end{tikzcd}$$

Point: $\underline{\underline{\mathrm{Proc}(P)^{\otimes} \xrightarrow{F} \mathbb{C}}}$ symmetric monoidal pseudo double functors (strict to partner)

Species $S \longmapsto F(S) \in \mathbb{C}$

 $\longmapsto \bigotimes_i F(s_i) \xrightarrow{F(t)} \bigotimes_j F(s_j')$

Def: A Flavoured petri net is SMDbl functor $\mathbb{F} : \mathrm{Proc}(P) \longrightarrow \mathbb{C}$ (Flavouring) (Double category of colors)

→ Every species S gets a color $F(S)$

→ Every Transition gets a loose morphism $\bigotimes_i F(s_i) \longrightarrow \bigotimes_j F(s_j')$

A morphism of Flavoured Petri nets is a triangle

$$\begin{tikzcd} F^P(S) \arrow[d, "f_S"] & \bigotimes_i F^P(s_i) \arrow[r, "F^P(t)"] \arrow[d] \arrow[dr, phantom, "\Downarrow f_t"] & \bigotimes_j F^P(s_j') \arrow[d] \\ F^Q(f(S)) & \bigotimes_i F^Q(f(s_i)) \arrow[r, "F^Q(f(t))"'] & \bigotimes_j F^Q(f(s_j')) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Proc}(P) \arrow[d, "\mathrm{Proc}(f)"'] \arrow[dr, bend left, "F^P"] \arrow[dr, phantom, "\Downarrow f"] & \\ \mathrm{Proc}(Q) \arrow[r, bend right, "F^Q"'] & \mathbb{C} \end{tikzcd}$$

$\mathbb{C} = \mathbb{S}\mathrm{q}(\mathrm{Set})$ ↪ Colored Petri Nets

$F(S) \in \mathrm{Set}$ "set of colors"

$\prod_i F(s_i) \xrightarrow{F(t)} \prod_j F(s_j')$

$T(n,m)$: $F(0) = \{0\}$, $F(1) = \{1_j\}$, $F(2) = !$

$$\begin{tikzcd} \mathrm{Proc}(T(n,m)) \arrow[d, "\mathrm{Proc}(t)"'] \arrow[dr, "\mathbb{1}"] \arrow[dr, phantom, "\Downarrow c"] & \\ \mathrm{Proc}(P) \arrow[r, "F"'] & \mathbb{S}\mathrm{q}(\mathrm{Set}) \end{tikzcd}$$

$t \in P$ of that arity, $s_i := \mathrm{Proc}(t)(0_i)$

$$\begin{tikzcd} \prod_i \{0_i\} \arrow[r, "!"] \arrow[d, "\prod c_i"'] & \prod_j \{1_j\} \arrow[d, "\prod c_j"] \\ \prod_i F(s_i) \arrow[r, "F(t)"'] & \prod_j F(s_j') \end{tikzcd}$$

$\mathbb{C} = \mathbb{K}\mathrm{l}(\mathrm{Maybe})$ $\cong$ $\mathbb{S}\mathrm{pan}(\mathrm{Set}, (\mathrm{mono}, \mathrm{all}))$ ← Guarded Petri Nets

$F(s)$

$$\begin{tikzcd} \prod \{0_i\} \arrow[d, "\prod c_i"'] & \mathrm{Excites} \arrow[l, hook'] \arrow[r] \arrow[d] & \prod \{1_j\} \arrow[d] \\ \prod_i F(s_i) & G(t) \arrow[l, hook'] \arrow[r, "F(t)"] & \prod_j F(s_j') \end{tikzcd}$$

$$\begin{tikzcd} \prod [s_i, s_i] \arrow[d, "\prod_i c_i"'] & \prod \{b_i\} \arrow[l, hook', "b_i"'] \arrow[r, "a_j"] \arrow[d] & \prod [s_j, s_j] \arrow[d] \\ \prod_i F(s_i) & G(t) \arrow[l, hook'] \arrow[r] & \prod_j F(s_j') \end{tikzcd}$$

$\mathbb{C} = \mathbb{S}\mathrm{pan}(\mathrm{Manifold}, (\mathrm{opening}, \mathrm{all}))$ | foot replaced allow ODE → manifold ← based off Rachunk & Clark at Kenny Courser's thesis

We get a DOTS doctrine

$SM(\mathrm{Dbl}) \longrightarrow SM(\mathrm{lMod}_r)$

$\mathrm{Proc} : \mathbb{K}\mathrm{leisli}(\mathrm{Proc}) \longrightarrow \mathbb{Q}\mathrm{t}(SM(\mathrm{Dbl}))$

both drawings not copies

$\mathbb{C} \longmapsto \mathbb{Q}\mathrm{t}(SM(\mathrm{Dbl}))^{co}(\mathbb{C}, \mathrm{Proc}(-))$

$P \longrightarrow \mathrm{Proc}(Q) \xrightarrow{F} \mathbb{C}$