Composing Flavoured Petri Nets II

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in the framework of

Double Operadic Theory of Systems

Sophie Libkind + DJM, arXiv: 2505.18329

Building on the work of

- John Baez + Jade Master, arXiv: 1808.05415
- Jade Master, arXiv: 1904.09091
- Fabrizio Genovese + David Spivak, arXiv: 2002.02762
- Joachim Kock, arXiv: 2005.05108

Recall:

- A Petri Net is a span

$$\begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ \mathrm{Fin}^{\cong}/S & & \mathrm{Fin}^{\cong}/S \end{tikzcd}$$

in Cat, w/ S, T sets

transitions

species / places

free symmetric monoidal category on S

equivalently: a span of free SMCs

$$\begin{tikzcd} & \mathrm{SMC}(T) \arrow[dl, "\text{strict}"'] \arrow[dr, "\text{strict}"] & \\ \mathrm{SMC}(S) & & \mathrm{SMC}(S) \end{tikzcd}$$

in SymMonCat

- From a petri net P, we get a symmon double cat Proc(P) of "processes in P"

  Proc(P) is the free SymMonCat-internal category on P.

- Flavoured Petri nets (e.g. Guarded Petri Nets, Coloured Petri Nets, Stochastic Petri nets, Dynamically Coloured SDCPNs)
  - Consider a symmon double cat of "colors" $\mathbb{C}$
  - A $\mathbb{C}$-flavoured Petri net is $\mathrm{Proc}(P) \xrightarrow{F} \mathbb{C} \in \mathrm{SymMonDbl}$ $\qquad \mathrm{Petri}_{\mathbb{C}} := \mathrm{Proc} \downarrow^{\mathrm{lax}} \mathbb{C}$
  - Objects of $\mathbb{C}$ are Spaces of colors. to an place $p \in P$, F_p which is a color-space for p.
  - Loose morphisms in $\mathbb{C}$ are guards + transition between color-spaces to each $t \in P$, $t : p_1 \cdots p_n \to p'_1 \cdots p'_m$ $\qquad F_t : \bigotimes_i F_{p_i} \to \bigotimes_j F_{p'_j}$

A tight morphism in $\mathbb{C}$ is a homomorphism of color-spaces

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A square is a compatibility of guards/transitions w/ homomorphisms.

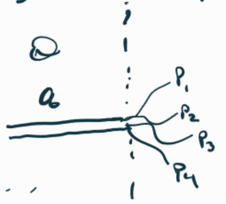
Suppose we have a $\mathbb{C}$-flavored petri net $F : \mathrm{Proc}(P) \to \mathbb{C}$

$$\begin{tikzcd} \mathrm{Proc}(P) \arrow[r, "F"] & \mathbb{C} \\ P \arrow[u, phantom, "\wr"] \arrow[ur] & \end{tikzcd}$$

Given a Kleisli morphism $Q \xrightarrow{i} \mathrm{Proc}(P)$

extends to $\mathrm{Proc}(Q) \xrightarrow{i^{\text{lift}}} \mathrm{Proc}(P) \rightsquigarrow F \circ i^{\text{lift}} : \mathrm{Prop}(Q) \to \mathbb{C}$

What is $i : Q \to \mathrm{Prop}(P)$



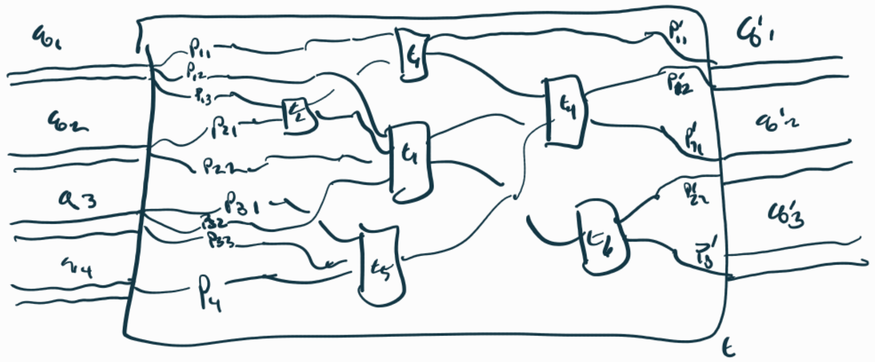
$i(q) = \bigotimes_{j=1}^{n} p_j$

$q \xmapsto{F \circ i} \bigotimes_{j=1}^{n} F p_j$

$i(t) : \overbrace{\bigotimes_{j_1} p_{j_1}}^{i(q_1)} \otimes \cdots \otimes \overbrace{\bigotimes_{j_n} p_{j_n}}^{i(q_n)} \longmapsto \overbrace{\bigotimes_{j_1'} p_{j_1'}}^{i(q_1')} \otimes \cdots \otimes \bigotimes_{j_m'} p_{j_m'}$

$t : q_1 \cdots q_n \to q_1' \cdots q_m'$

$t \longmapsto$ the composite of all the $F t_i$



$F_1 : \mathrm{Proc}(P_1) \to \mathbb{C}$

$F_2 : \mathrm{Proc}(P_2) \to \mathbb{C}$

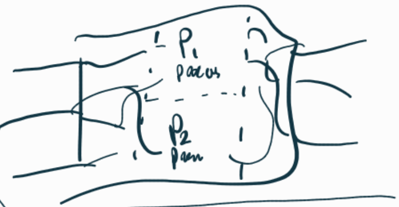
$F_1 \times F_2 : \overbrace{\mathrm{Proc}(P_1) \times \mathrm{Proc}(P_2)}^{} \to \mathbb{C} \times \mathbb{C} \xrightarrow{\otimes} \mathbb{C}$

$\qquad\qquad\qquad\ \ \wr\!\wr$

$F_1 \otimes F_2 : \mathrm{Proc}(P_1 + P_2) \longrightarrow \mathbb{C}$

$Q \xrightarrow{i} \mathrm{Proc}(P_1 + P_2)$

$q \longmapsto \bigotimes_{j} p_{1j} \otimes \bigotimes_{j'} p_{2j'}$



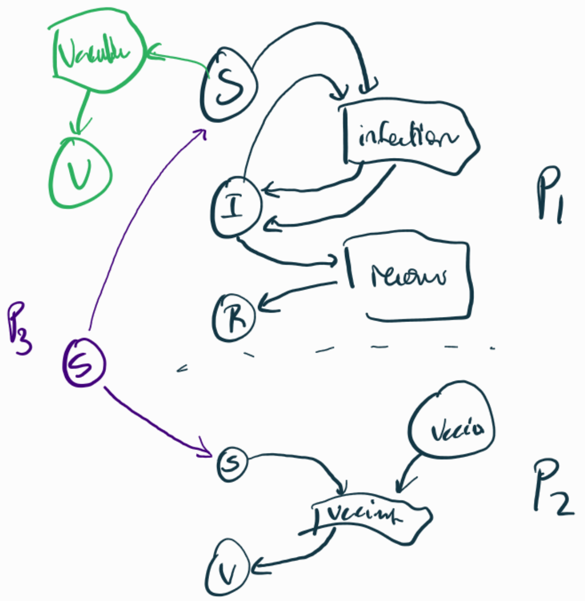
$$\begin{tikzcd} & \mathbb{C} & \\ & \mathbb{C} \arrow[u, phantom, "t"] & \\ & (x,y) \arrow[u, dashed, "F_1 \times \otimes F_2 y"'] & \\ & \mathrm{Proc}(P_1) \times \mathrm{Proc}(P_2) & \\ \mathrm{Proc}(P_1) \arrow[ur, "{x, y_0}"] \arrow[uuuur, bend left, "F_1"] & & \mathrm{Proc}(P_2) \arrow[ul, "{y_0, x}"'] \arrow[uuuul, bend right, "F_2"'] \end{tikzcd}$$

Doctrine of flavored petri nets

$\mathbb{C} \longmapsto \mathrm{Cat}(\underline{\mathrm{SymMonDbl}})(\check{-}, \mathbb{C})$ Cartesian

$$\begin{tikzcd} \text{Sys Theory} = \mathrm{SymMonDbl} \arrow[r] \arrow[d] & \underline{\text{SymMonoidalMaps}} \arrow[d] \\ \text{Interaction Theory} = \{\bullet\} \arrow[r, "\mathrm{Cat}(\underline{\mathrm{FreeSymMonDbl}})"'] & \mathrm{SymMonDbl} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Proc}(Q) \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\Downarrow S"] & \mathrm{Proc}(P) \arrow[r, "F", "\text{System}"'] \arrow[d] \arrow[dr, phantom, "\Downarrow \text{homomorphism}"] & \mathbb{C} \\ \mathrm{Proc}(Q') \arrow[r, "i'"'] & \mathrm{Proc}(P') \arrow[ur, "F'"'] & \end{tikzcd}$$



$S = \{S, I, R\}$

$T = \{\text{infection}, \text{recovery}\}$

$$P = \begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ \mathrm{Fin}^{\simeq}/S & & \mathrm{Fin}^{\simeq}/S \end{tikzcd}$$

$$\begin{tikzcd} & P_1 \arrow[dr] & \\ P_3 \arrow[ur] \arrow[dr] & & P_1 +_{P_3} P_2 \\ & P_2 \arrow[ur] & \end{tikzcd}$$

Since Proc is a free construction, it should preserve pushouts. So long as the flavors of P_1 and P_2 agree P_1, we set a flavoring for the pushout

$$\begin{tikzcd} & \mathrm{Proc}(P_1) \arrow[d] \arrow[drr, bend left, "F_1"] & & \\ \mathrm{Proc}(P_3) \arrow[ur] \arrow[dr] & \mathrm{Proc}(P_1 +_{P_3} P_2) \arrow[rr, dashed, "\exists (F_1, F_2)"] & & \mathbb{C} \\ & \mathrm{Proc}(P_2) \arrow[u] \arrow[urr, bend right, "F_2"'] & & \end{tikzcd}$$

The main point is that $\mathrm{Proc} \downarrow \mathbb{C}$ has pushouts.