

DESCENT IN PROBABILITY THEORY: THE FIRST STEPS DOWNWARD

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Joint work with

M. Di Meglio

JS Lemay

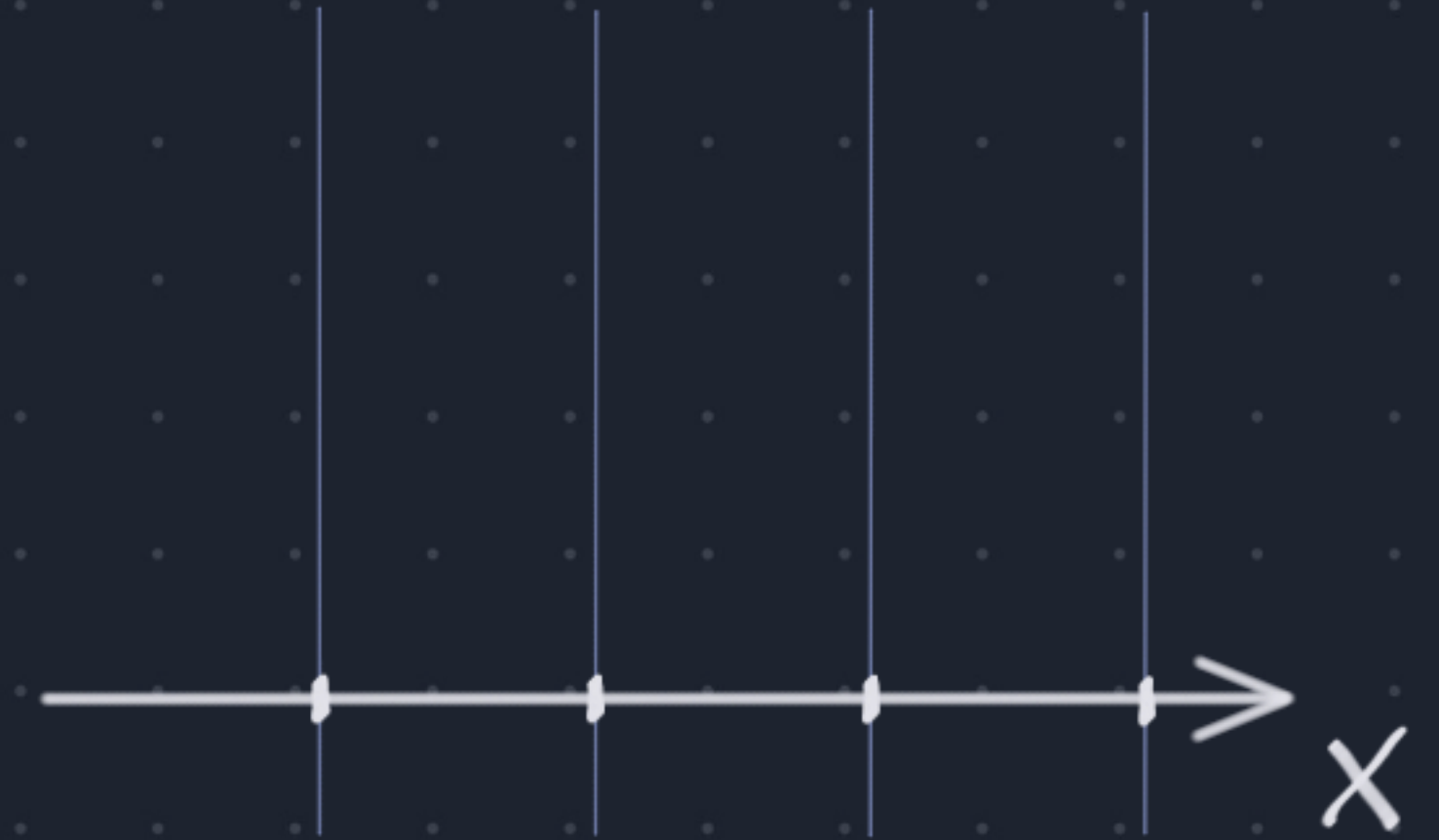
C. Heunen

D. Stein

1. THE IDEA OF DESCENT
2. RANDOMNESS AS IGNORANCE
3. SHEAVES IN PROBABILITY
4. STATISTICAL EXPERIMENTS

1. THE IDEA OF DESCENT • Descent to the quotient

Definition. Let $(X, \sim)$ be a set with an equivalence relation.



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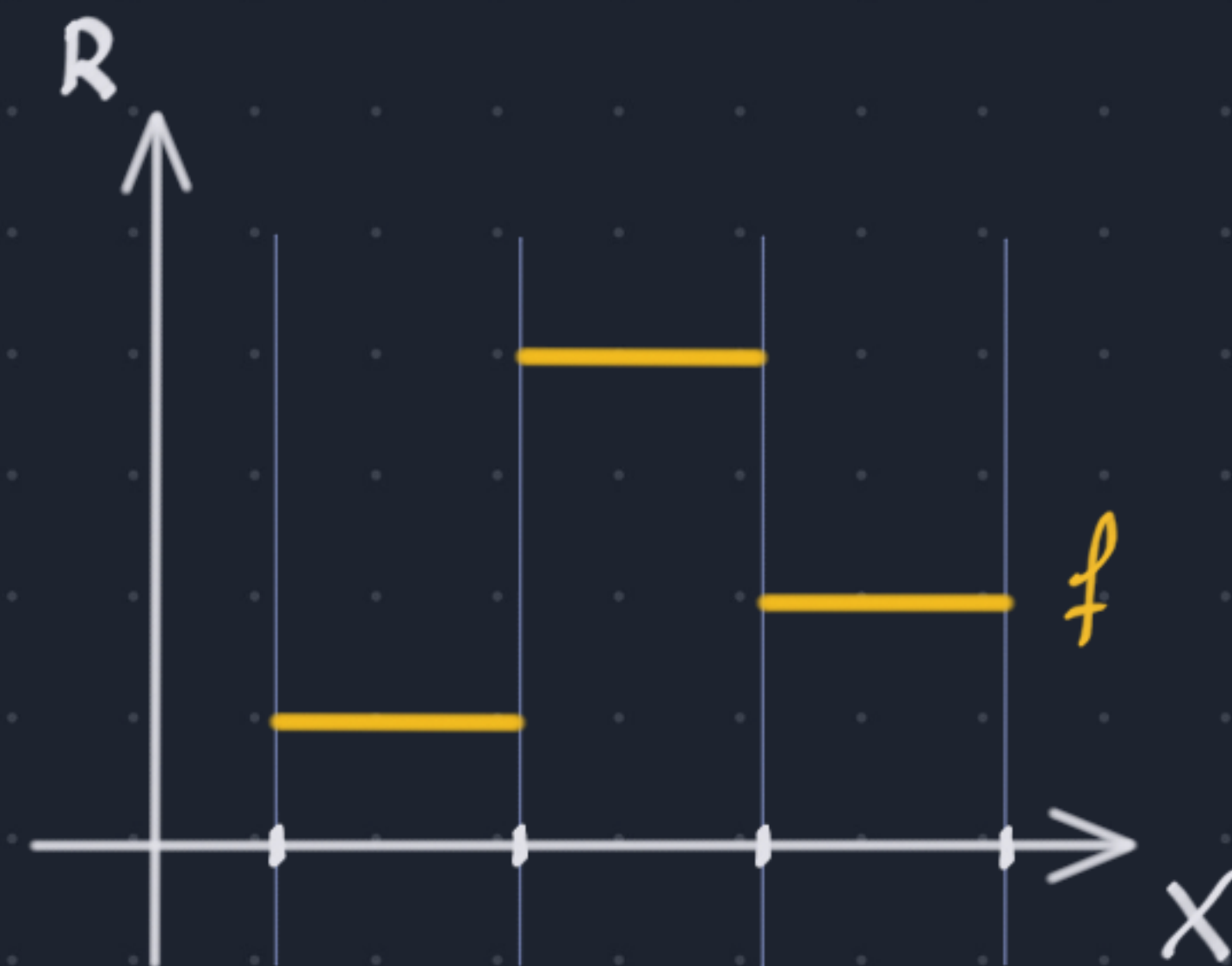
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We say that $X \xrightarrow{f} R$ descends to the quotient

if there exists $X/\sim \xrightarrow{\tilde{f}} R$ such that

this commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & R \\ \pi \downarrow & \nearrow \tilde{f} & \\ X/\sim & & \end{array}$$



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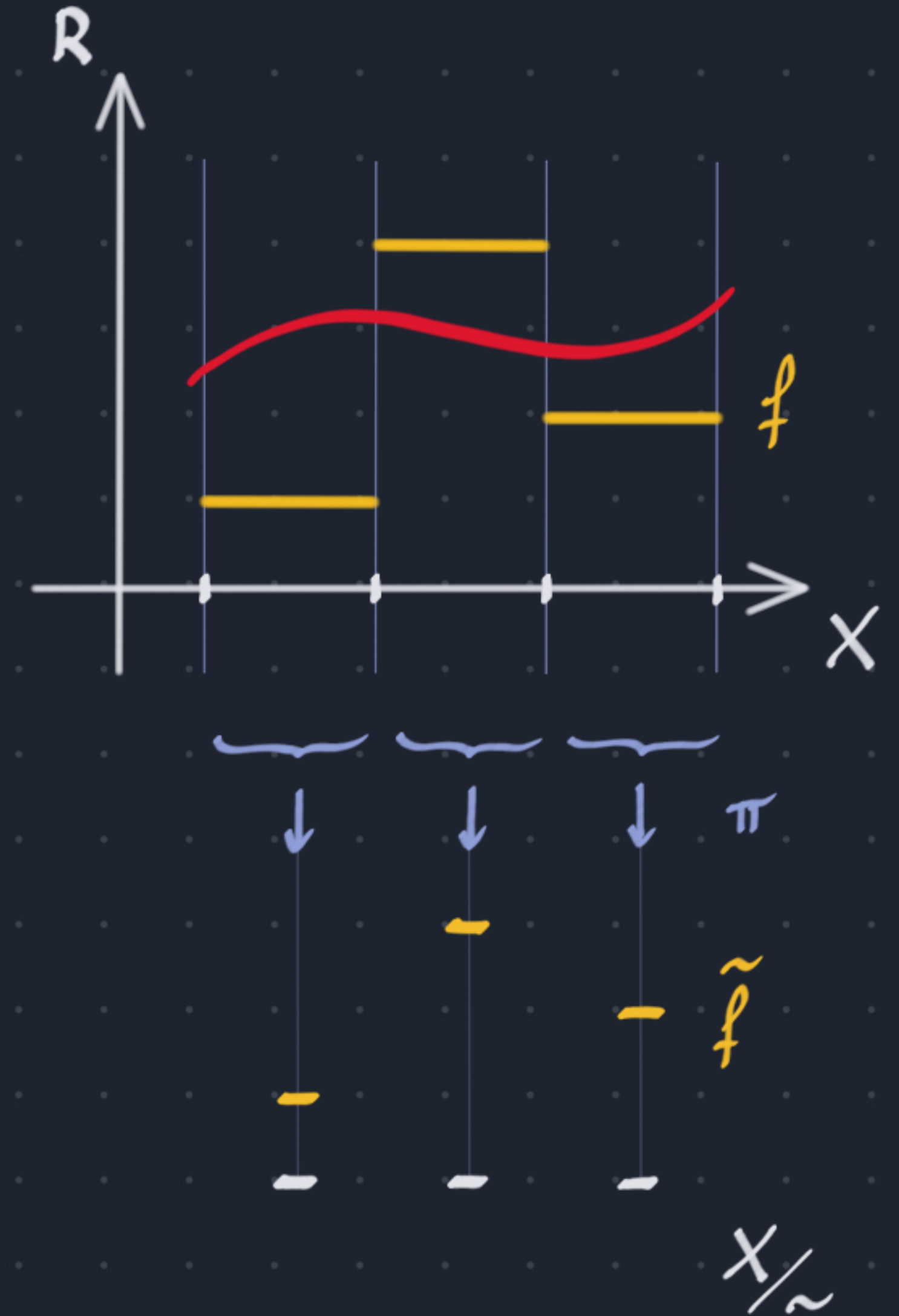
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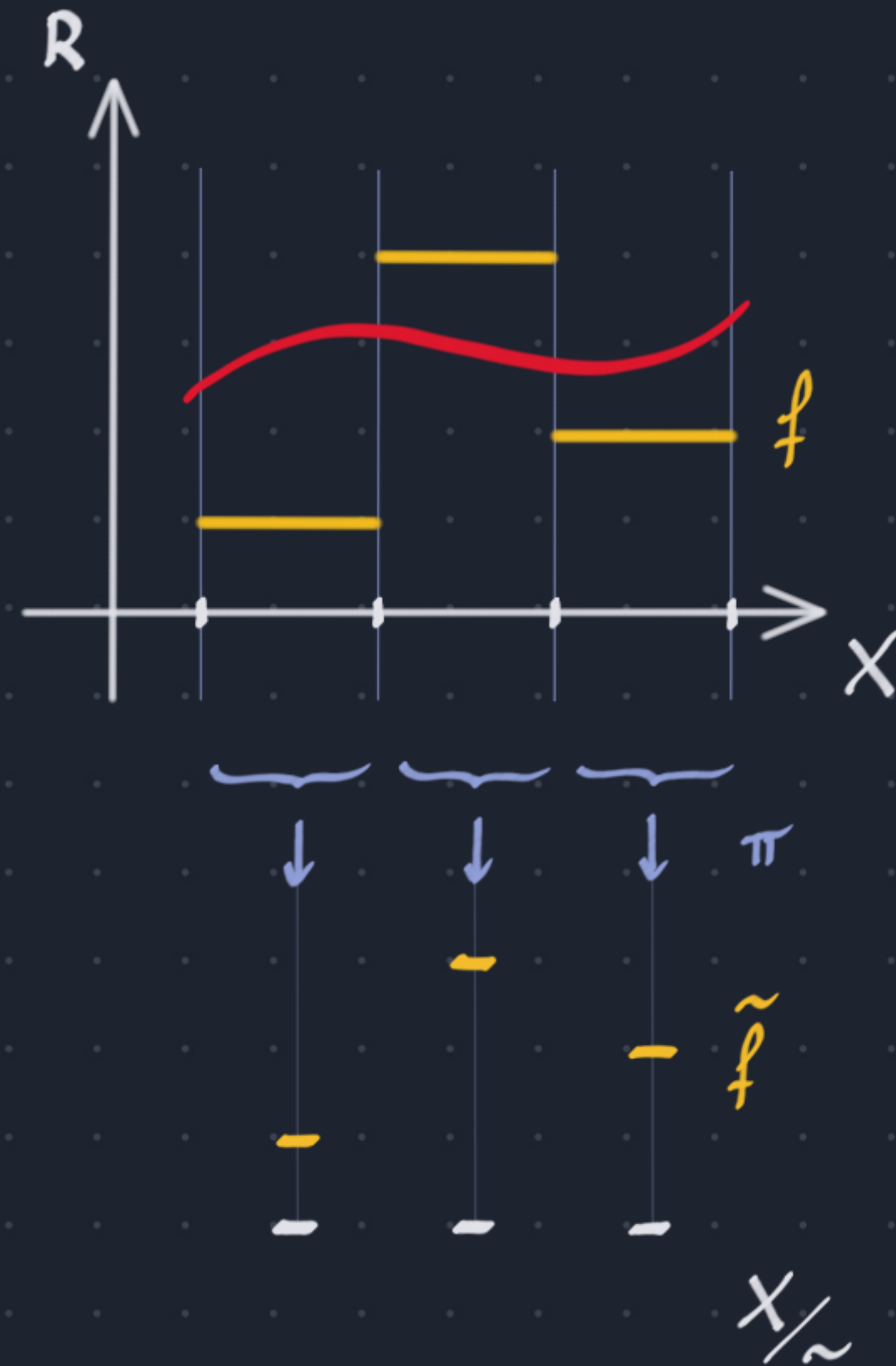
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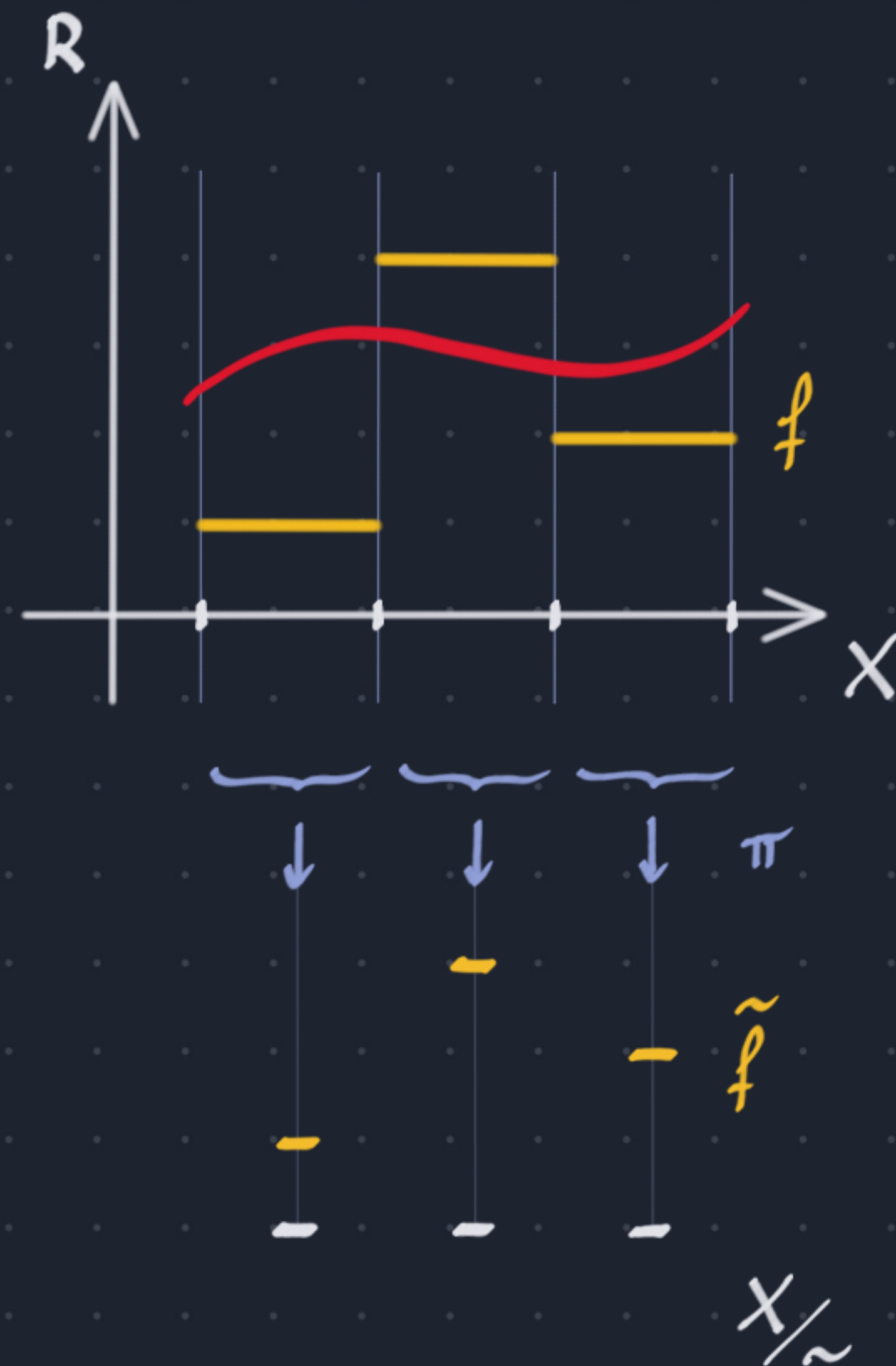
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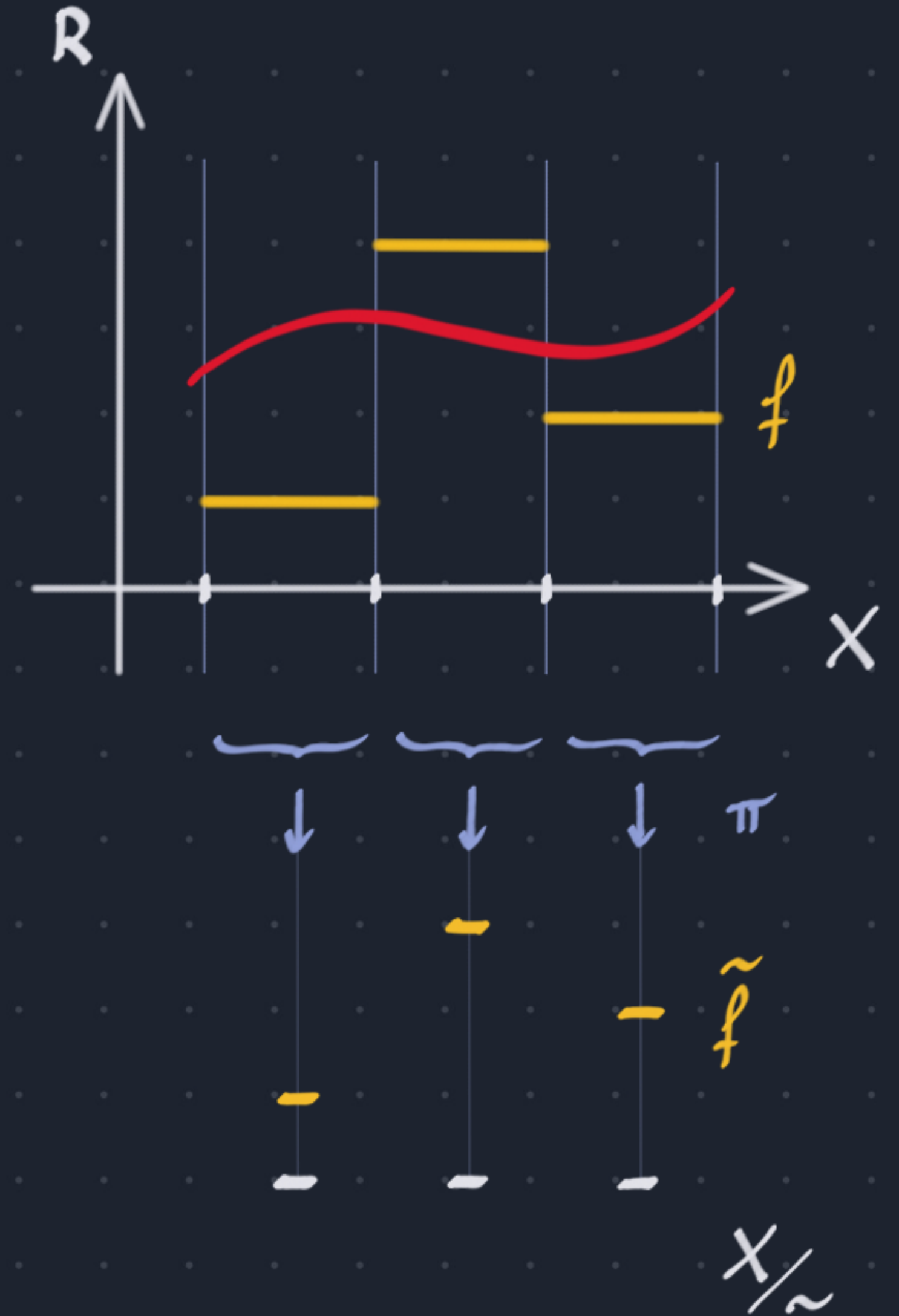
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 \quad \text{DD}(\pi, R) \longleftrightarrow \mathcal{L}(X, R)$$

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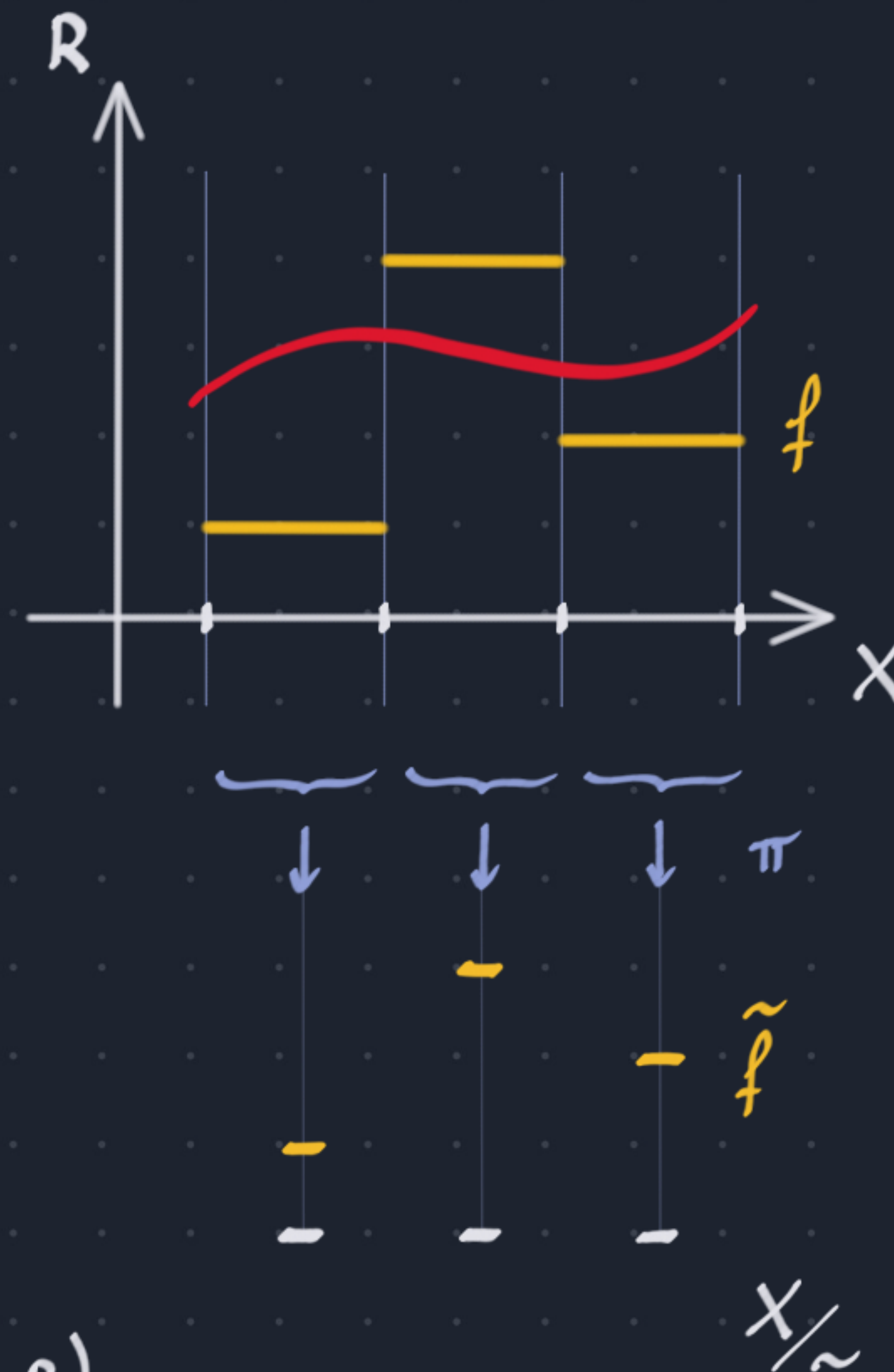


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Corollary. π induces a bijection: $\text{Descent data } (\pi, R) \cong \text{Functions } (X/\sim, R)$



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Example 3. Linear maps $V \otimes W \rightarrow \mathbb{R}$ are "the same as"
linear maps $\mathbb{R}(V \times W) \xrightarrow{f} \mathbb{R}$ s.t.

$$f(v+v', w) = f(v, w) + f(v', w)$$
$$f(v, w+w') = f(v, w) + f(v, w')$$
$$f(av, w) = af(v, w) = f(v, aw) .$$

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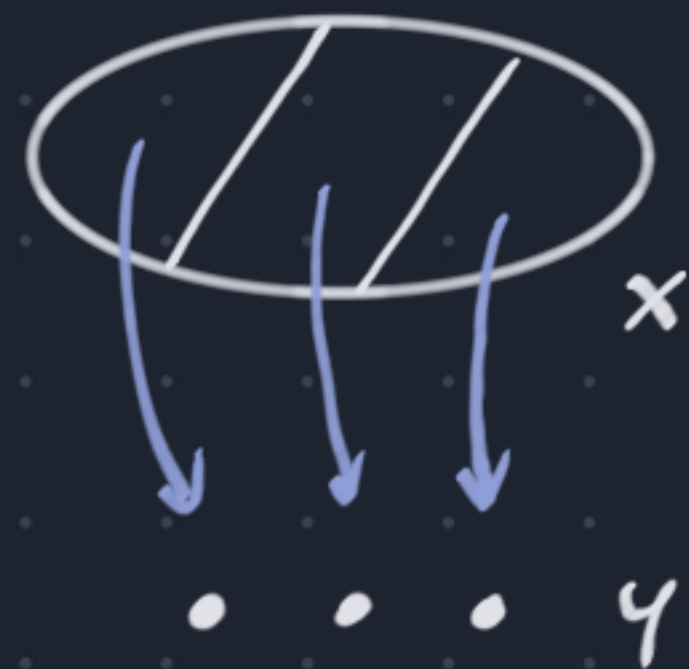
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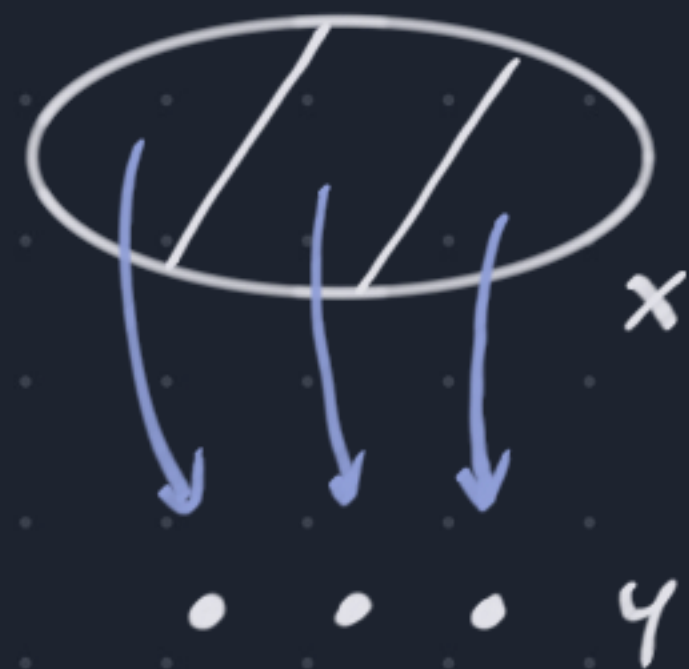
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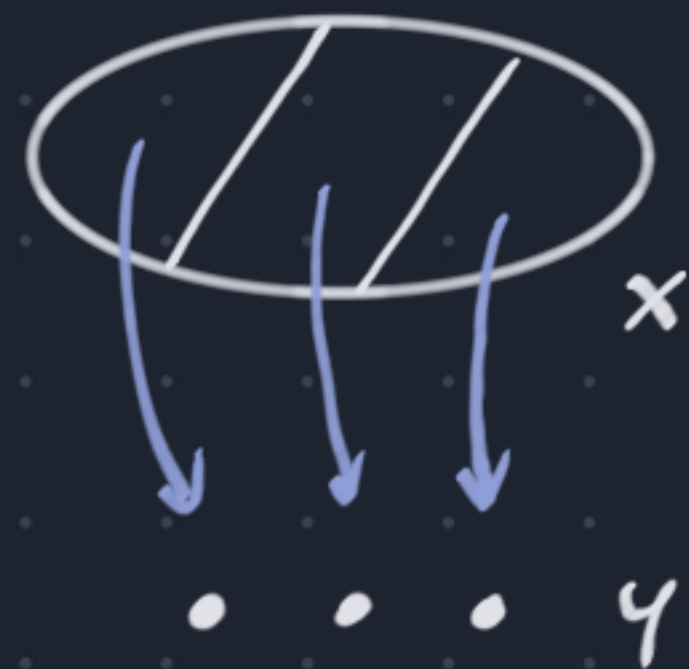
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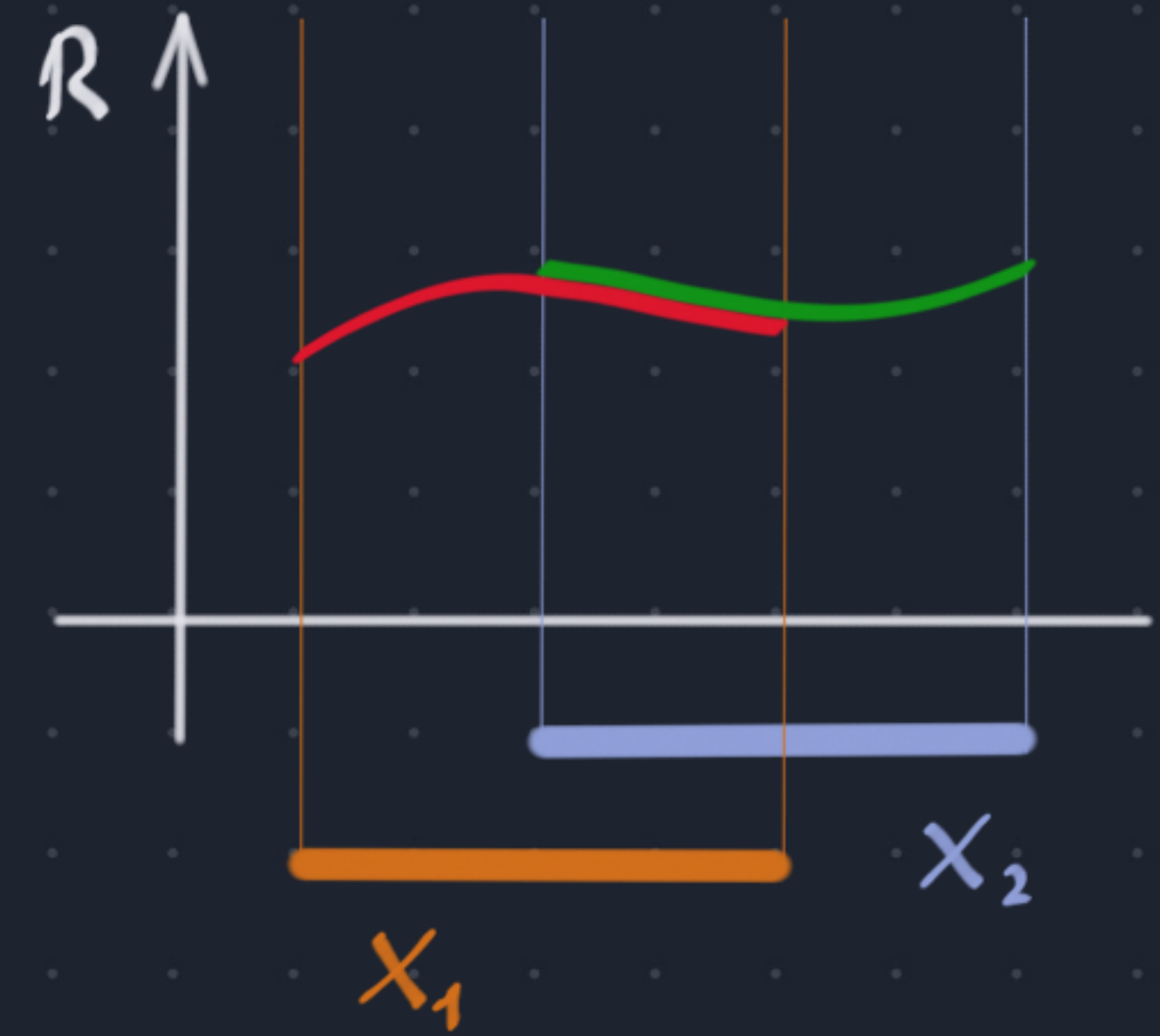
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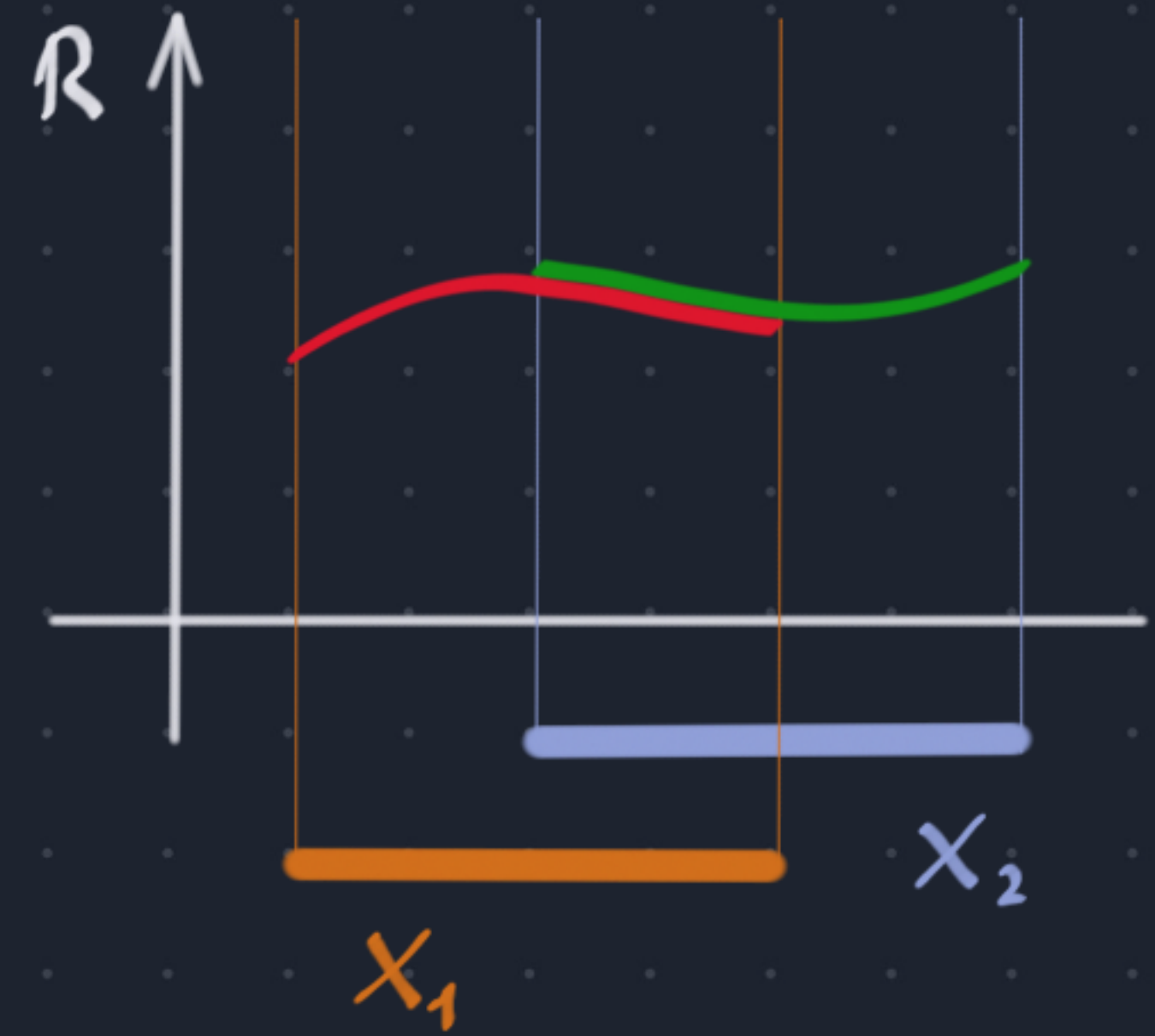
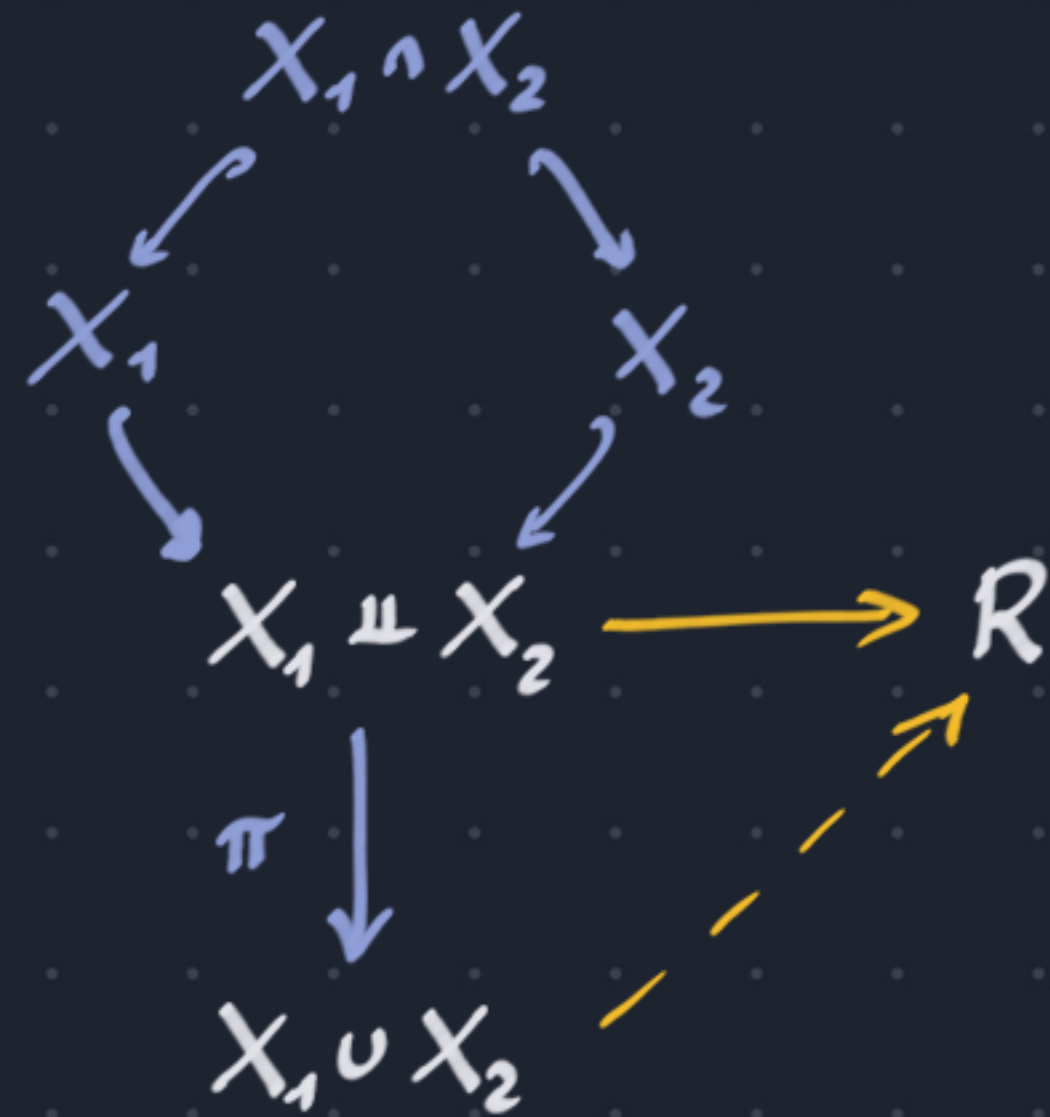
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which agree on the intersection.



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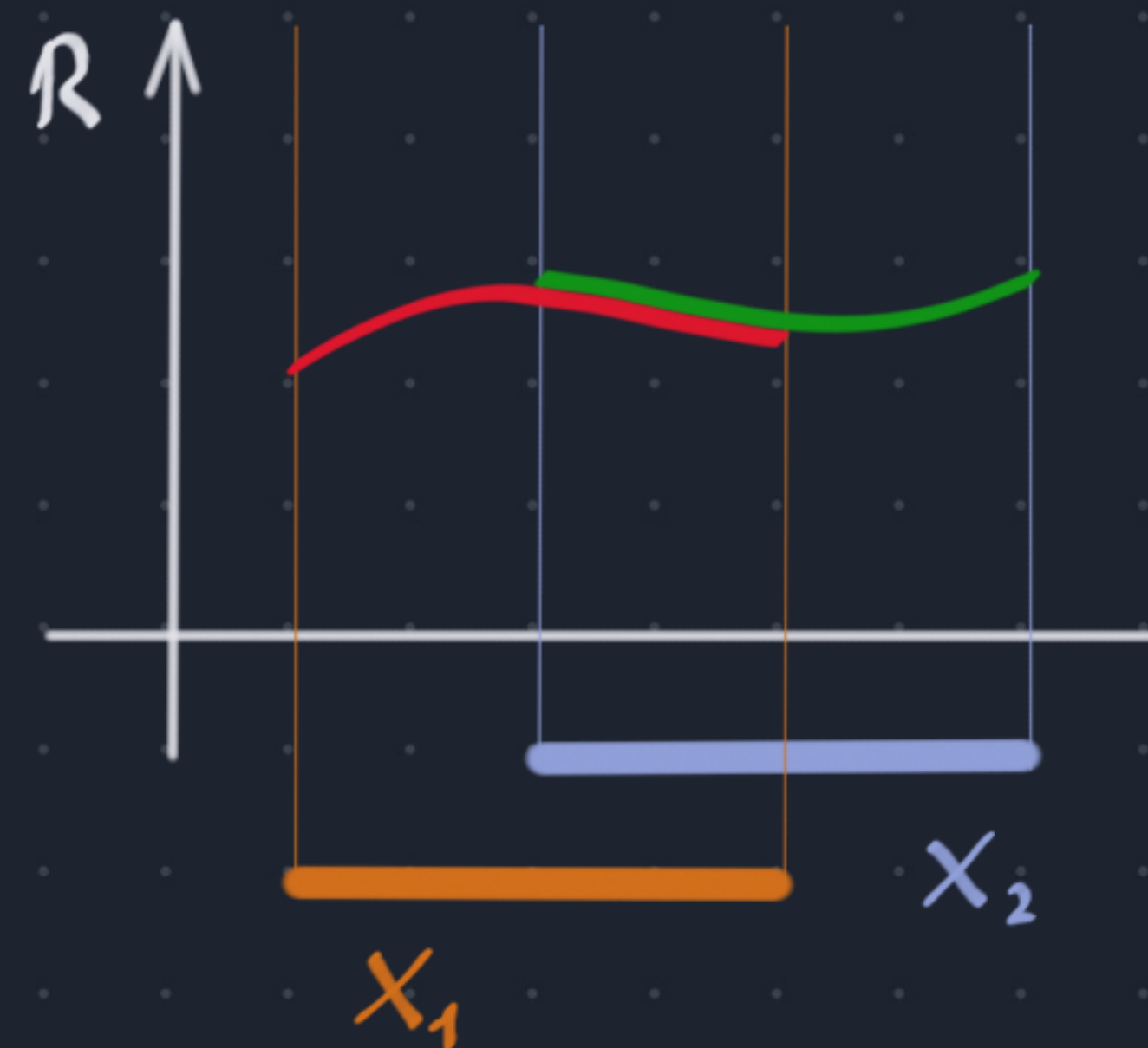
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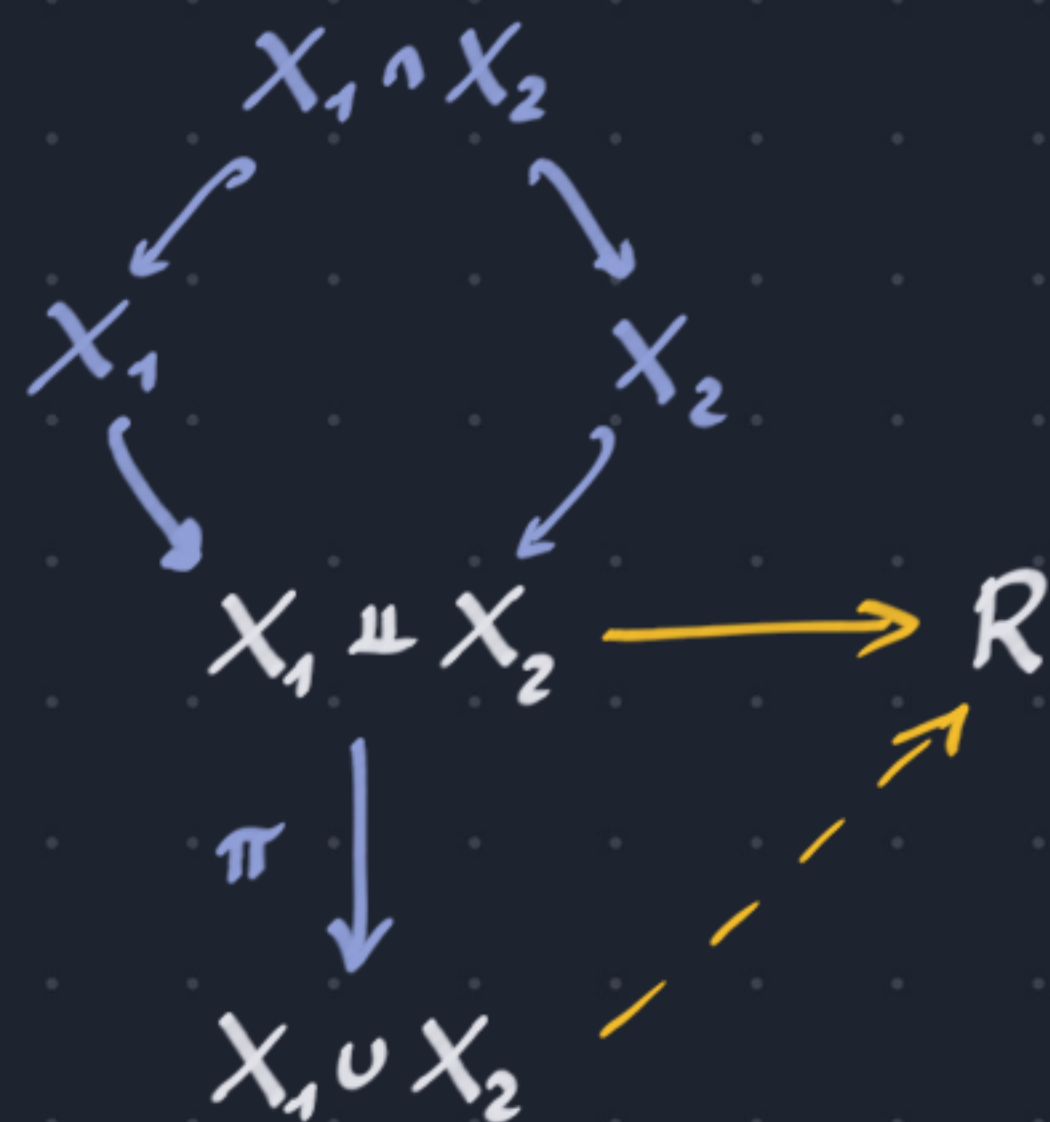


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Or, for categories w/o the right colimits, we can generalize the idea of descent to sets of arrows



For the purposes of this talk, single arrows will suffice.

1. THE IDEA OF DESCENT • Categorical definition

Definition. Let $X \xrightarrow{\pi} Y$ be any morphism. We say that $X \xrightarrow{f} R$ descends along π if there exists $Y \xrightarrow{\tilde{f}} R$ such that this commutes:

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$$\Rightarrow f \circ g = f \circ h.$$

Definition. We say that R satisfies descent along π if π induces a bijection

$$\text{Descent Data}(\pi, R) \cong \mathcal{C}(Y, R).$$

1. THE IDEA OF DESCENT • Categorical definition

Remarks.

- In general, " π epi" is not enough.
(E.g. take $\pi = \text{id}$, X discrete, Y codiscrete.)

We need a choice of "nice epi"

→ Coverage / Grothendieck topology

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- If pullbacks exist, descent data can be equivalently written as follows:

$$f: X \rightarrow R \mid f \circ \pi_1 = f \circ \pi_2 \quad (\Rightarrow \text{fork!})$$

So, iff π is coeq. $\Leftrightarrow$ Every R satisfies descent along π .

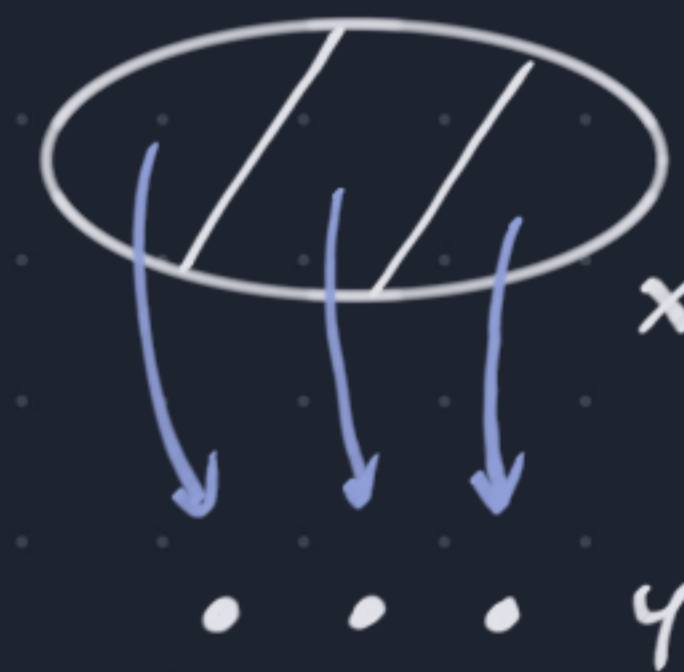
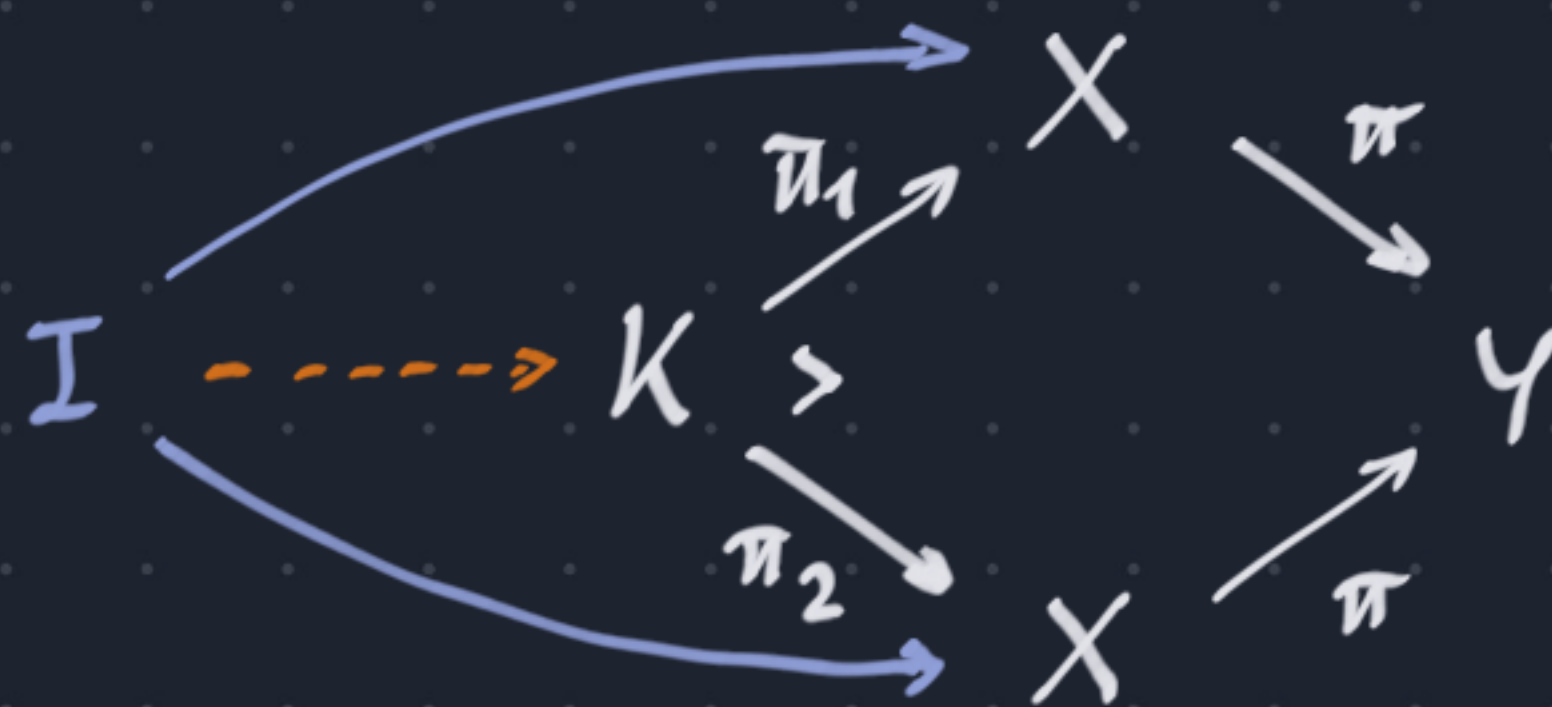
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$$\begin{array}{ccc} \text{Ker}(\pi) & \xrightleftharpoons[\pi_2]{\pi_1} & X \xrightarrow{f} R \\ & \pi \downarrow & \nearrow \tilde{f} \\ & Y & \end{array}$$

1. THE IDEA OF DESCENT • Categorical definition

Definition. The kernel pair of $X \xrightarrow{\pi} Y$ is the pullback of π along itself.



It is a congruence (internal equivalence relation).

$$\text{id} \Rightarrow \pi_2 \circ \pi_1^+ \quad \pi_2 \circ \pi_1^+ = \pi_1 \circ \pi_2^+ \quad \pi_2 \circ \pi_1^+ \circ \pi_2 \circ \pi_1^+ \Rightarrow \pi_2 \circ \pi_1^+ \quad \text{in cat. of spans.}$$

It generalizes the "equivalence relation induced by a map".

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Idea. We can replace "arrows into R " with more general $\Phi: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$. [E.g. functions $\text{FinSet} \rightarrow \mathbb{R}$]

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Definition. A descent datum for Φ along π is an element $\varphi \in \Phi(X)$

s.t. $\forall g, h: I \rightarrow X, g^* \varphi = h^* \varphi$.

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Definition. We call Φ a sheaf iff it satisfies descent along every chosen cover π .

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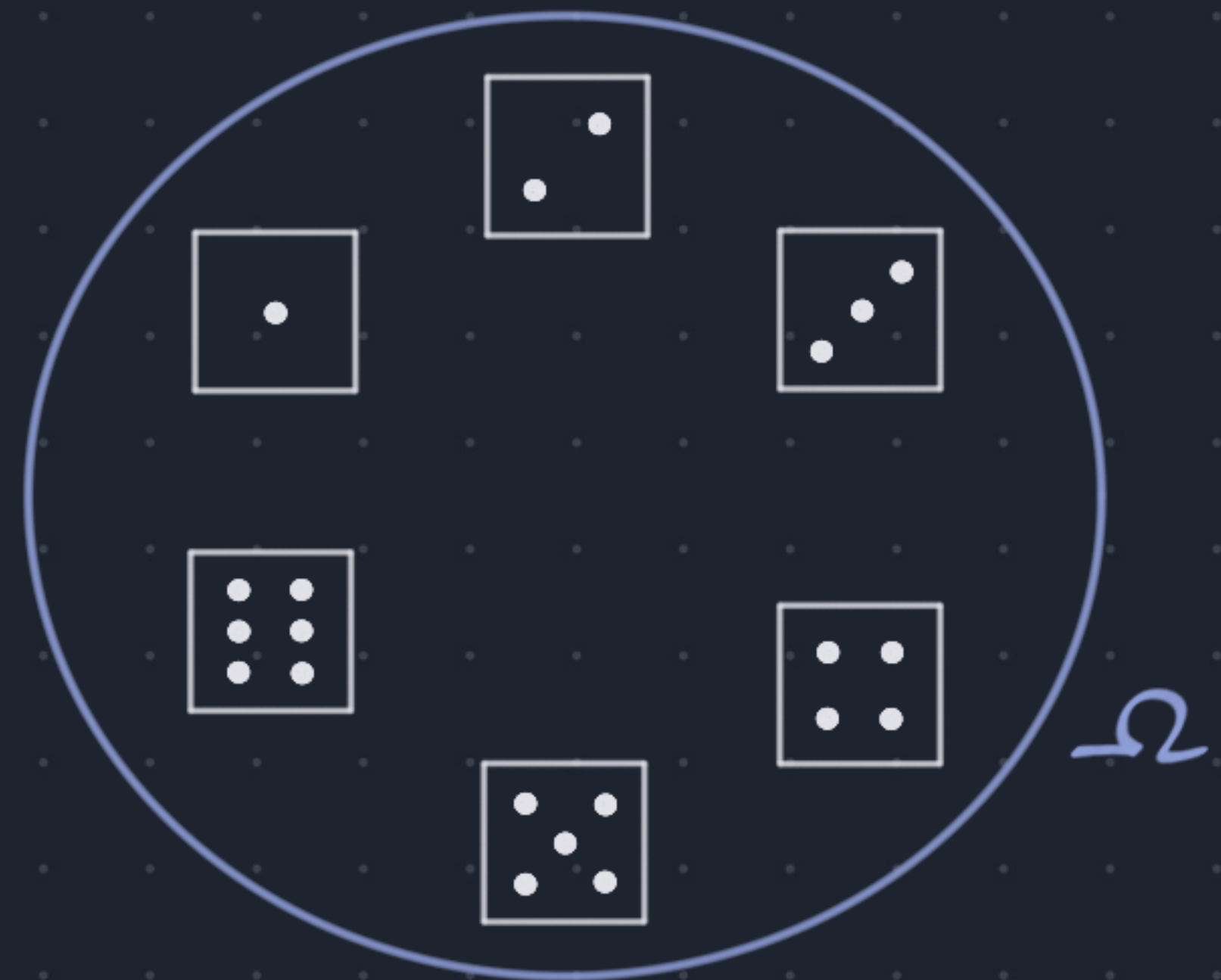
How do probabilists model probability?

Idea. Consider a set Ω of "possible states of the world". [Not the subject classifier!]

(Called sample space or outcome space.)

"What happens" is a point $\omega \in \Omega$.

We don't know which one.



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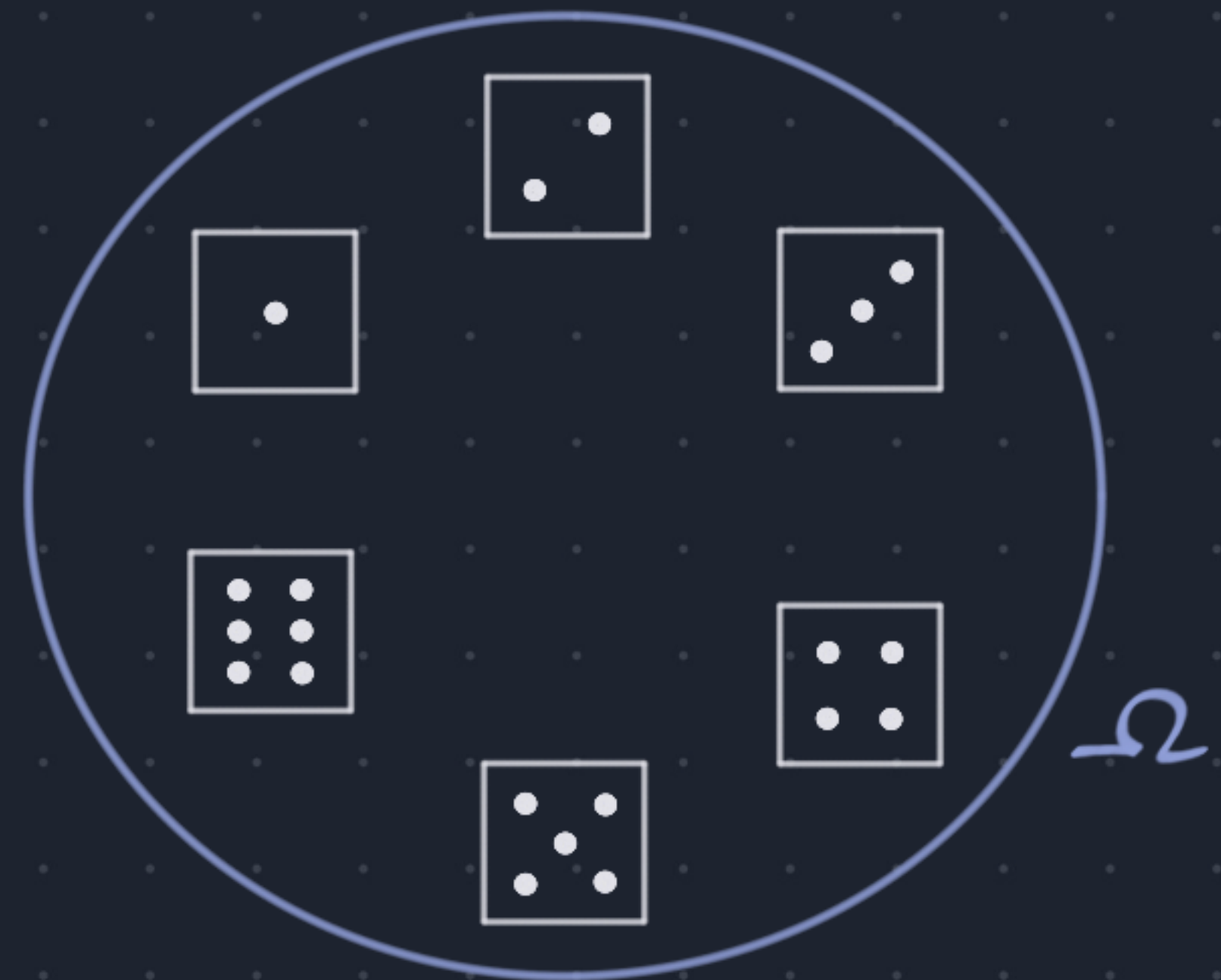
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Ω is equipped with:

- A σ -algebra
- A probability measure.



2. RANDOMNESS AS IGNORANCE · Probability spaces

Definition. A σ -algebra on a set X is a set $\mathcal{A}$ of subsets of X

closed under complements and countable unions.



Intuition: "Distinctions we could make (if we knew everything)"

A measurable space is a set equipped with a σ -algebra.

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Definition. A probability measure on $(X, \mathcal{A})$ is an assignment

$$p: \mathcal{A} \rightarrow \mathbb{R} \text{ such that } \cdot p\left(\bigsqcup_n A_n\right) = \sum_n p(A_n) \cdot$$

$$\cdot p(X) = 1.$$

Intuition: "how large/likely" a certain region is. (More on that soon.)

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Definition. $(X, \mathcal{A}) \xrightarrow{\pi} (Y, \mathcal{B})$ is measurable if $\forall B \in \mathcal{B}, \pi^{-1}(B) \in \mathcal{A}$.

Intuition: "does not break or refine the partition".



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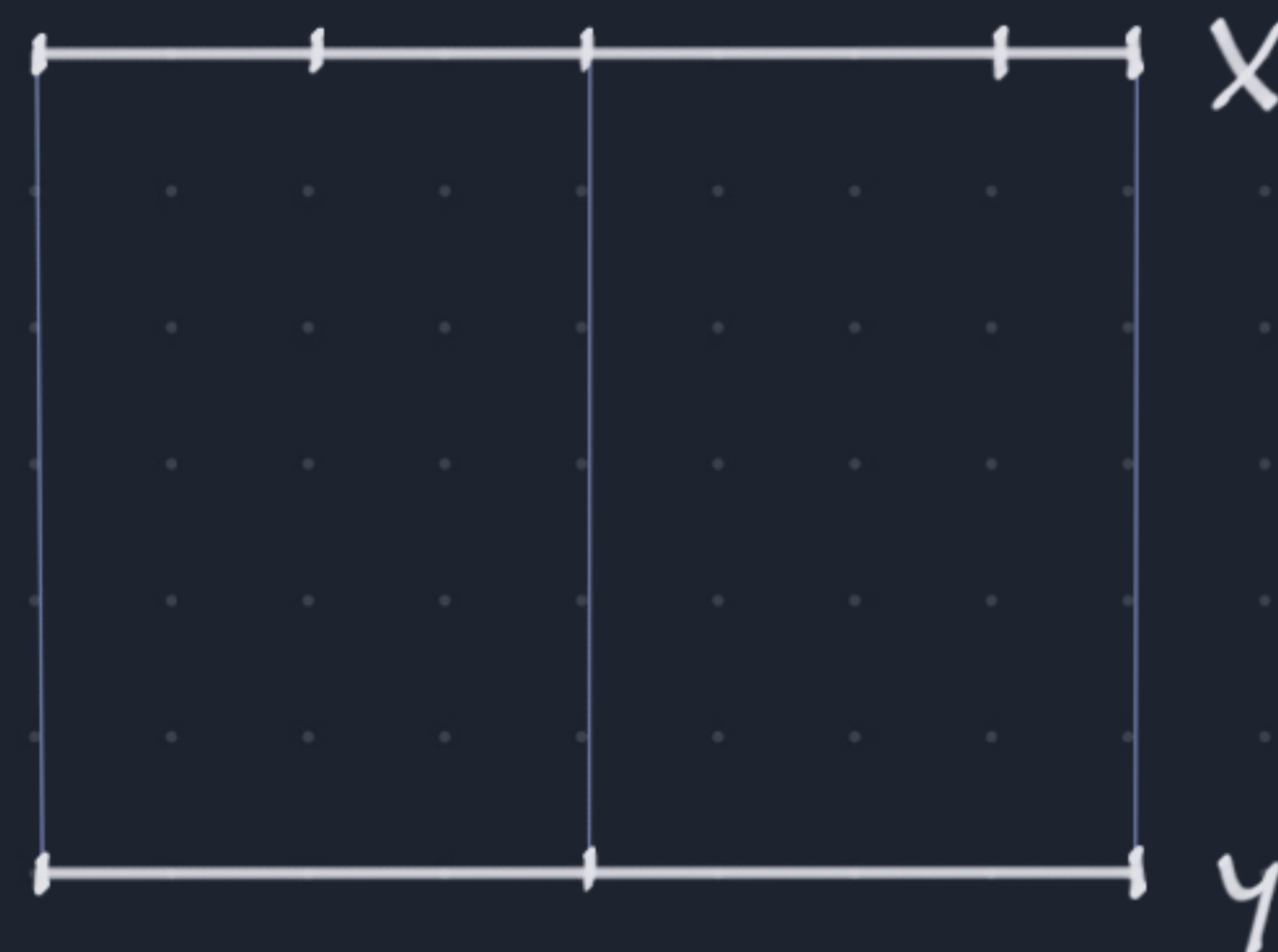
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Given such π , the set $\pi^{-1}(\mathcal{B}) = \{ \pi^{-1}(B) \mid B \in \mathcal{B} \}$

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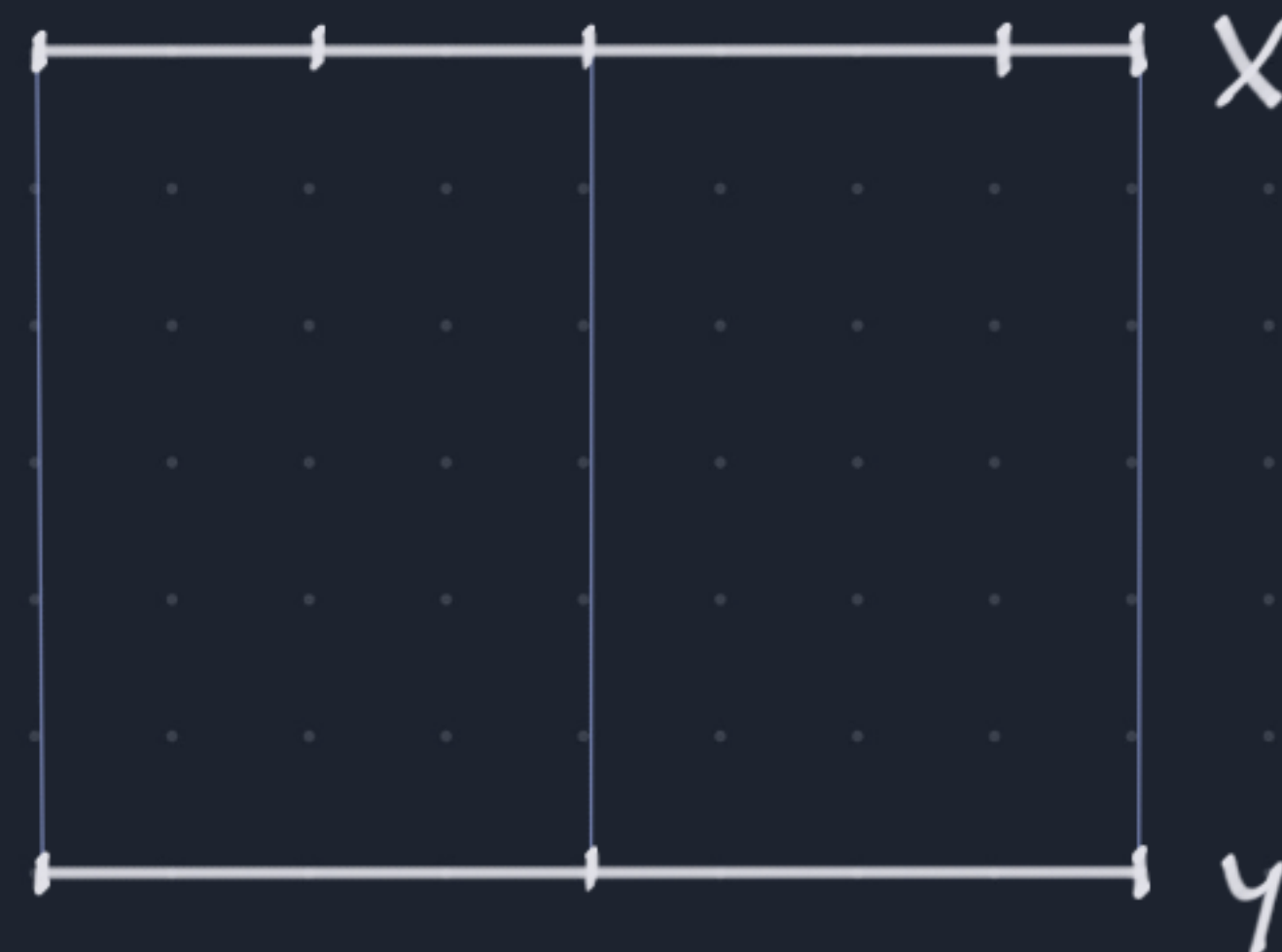
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In probability theory, this models "what we know".

(And π is an "observation" or "experiment" we can make.)



2. RANDOMNESS AS IGNORANCE • Fuzzy predicates

Definition. A random variable on Ω is a measurable function $\Omega \xrightarrow{f} \mathbb{R}$.

Intuition: f is "compatible with the state of our knowledge".

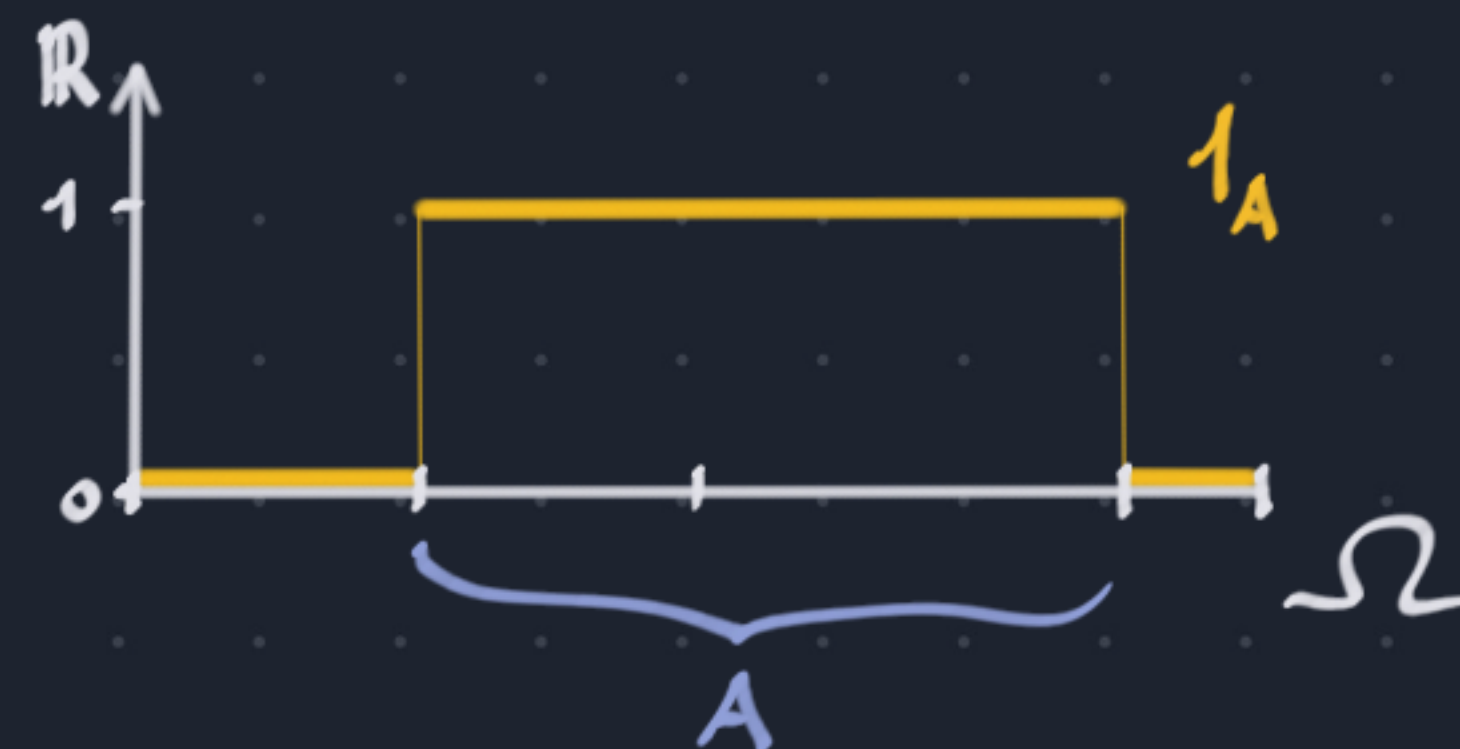


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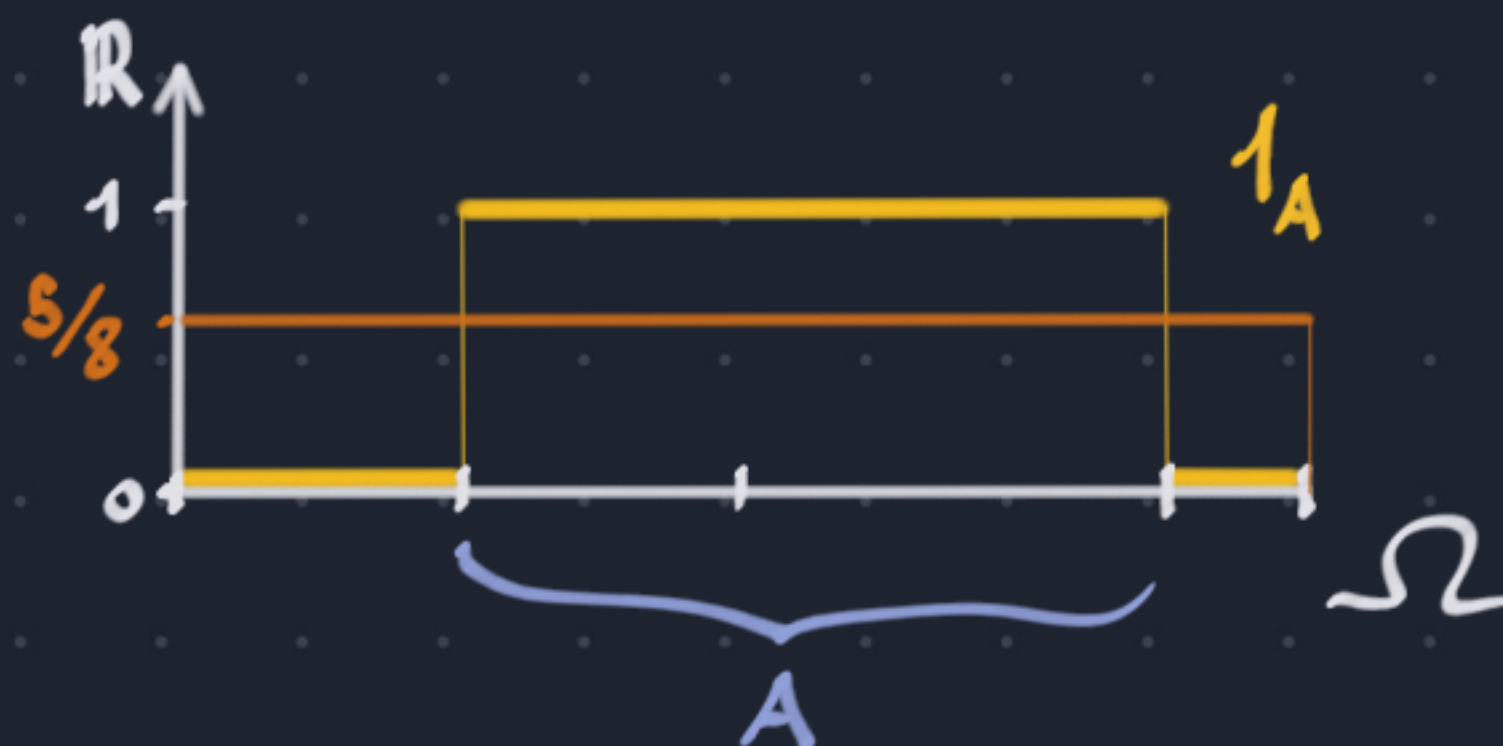


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We have: $\int_{\Omega} 1_A d\mathbb{P} = \int_A 1 d\mathbb{P} = \mathbb{P}(A) \quad (= 5/8 \uparrow)$

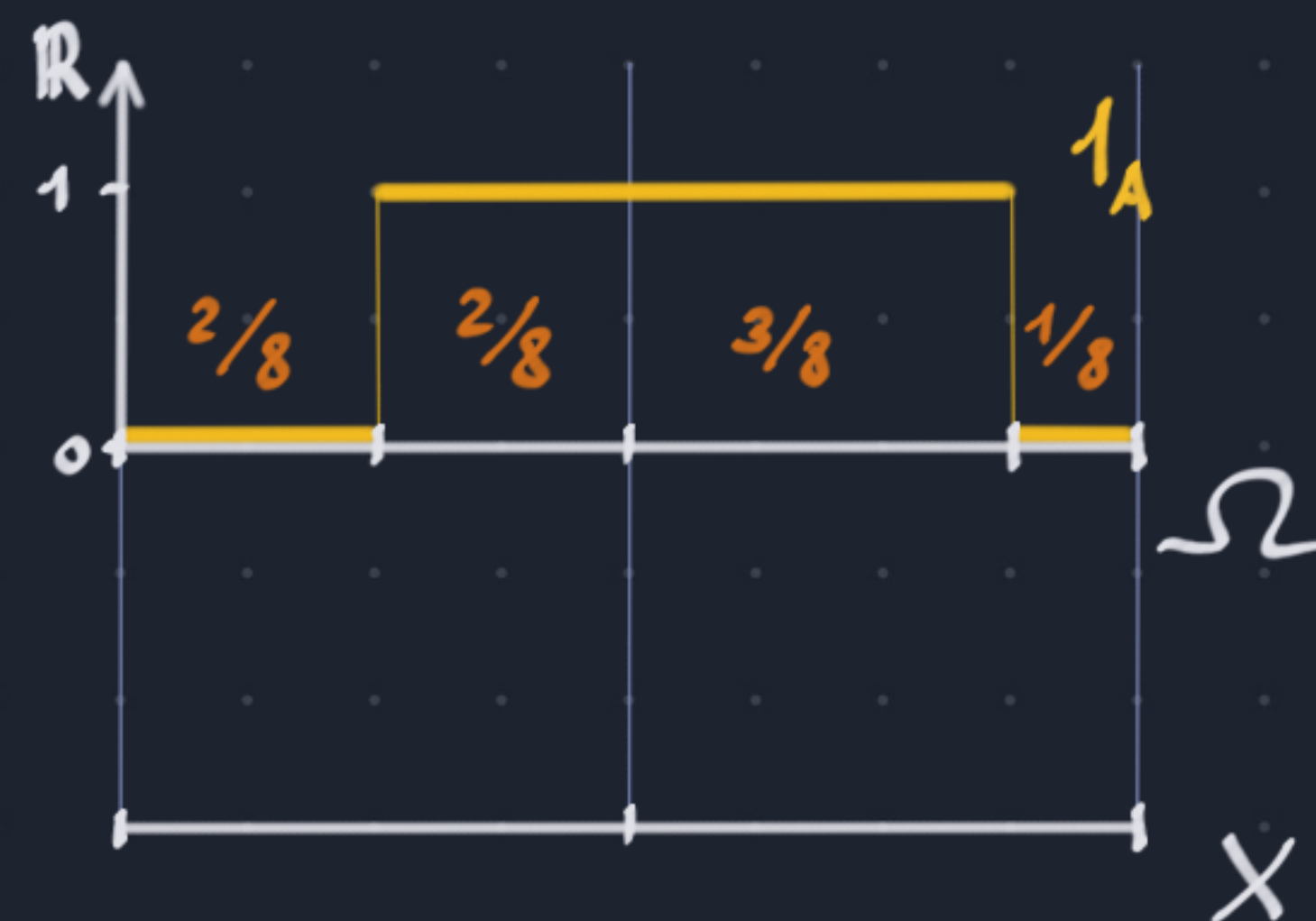
Intuition: we are "blurring/spreading/averaging what we don't know".

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Consider now an observation $\Omega \xrightarrow{\pi} (X, \mathcal{B})$

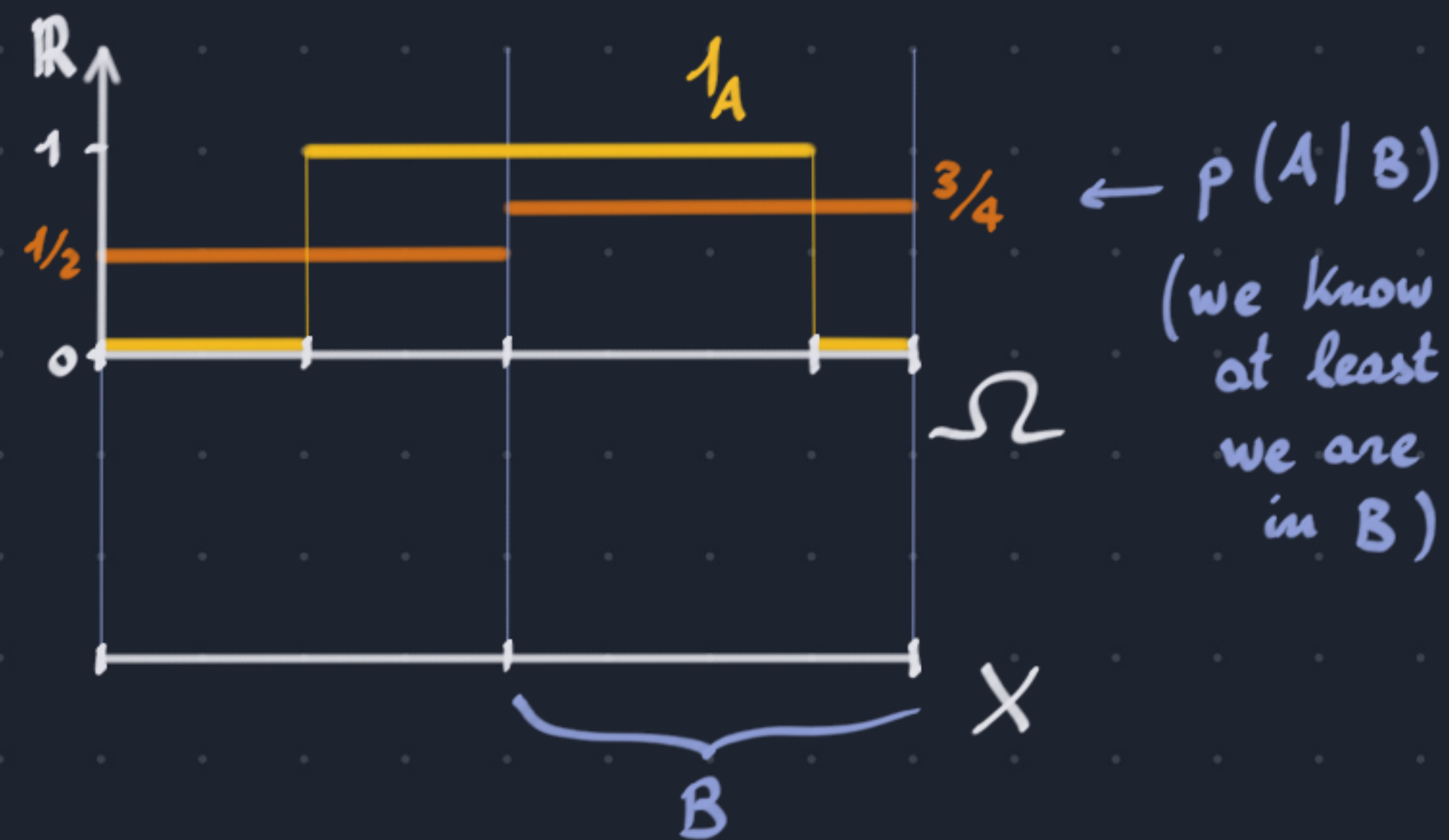


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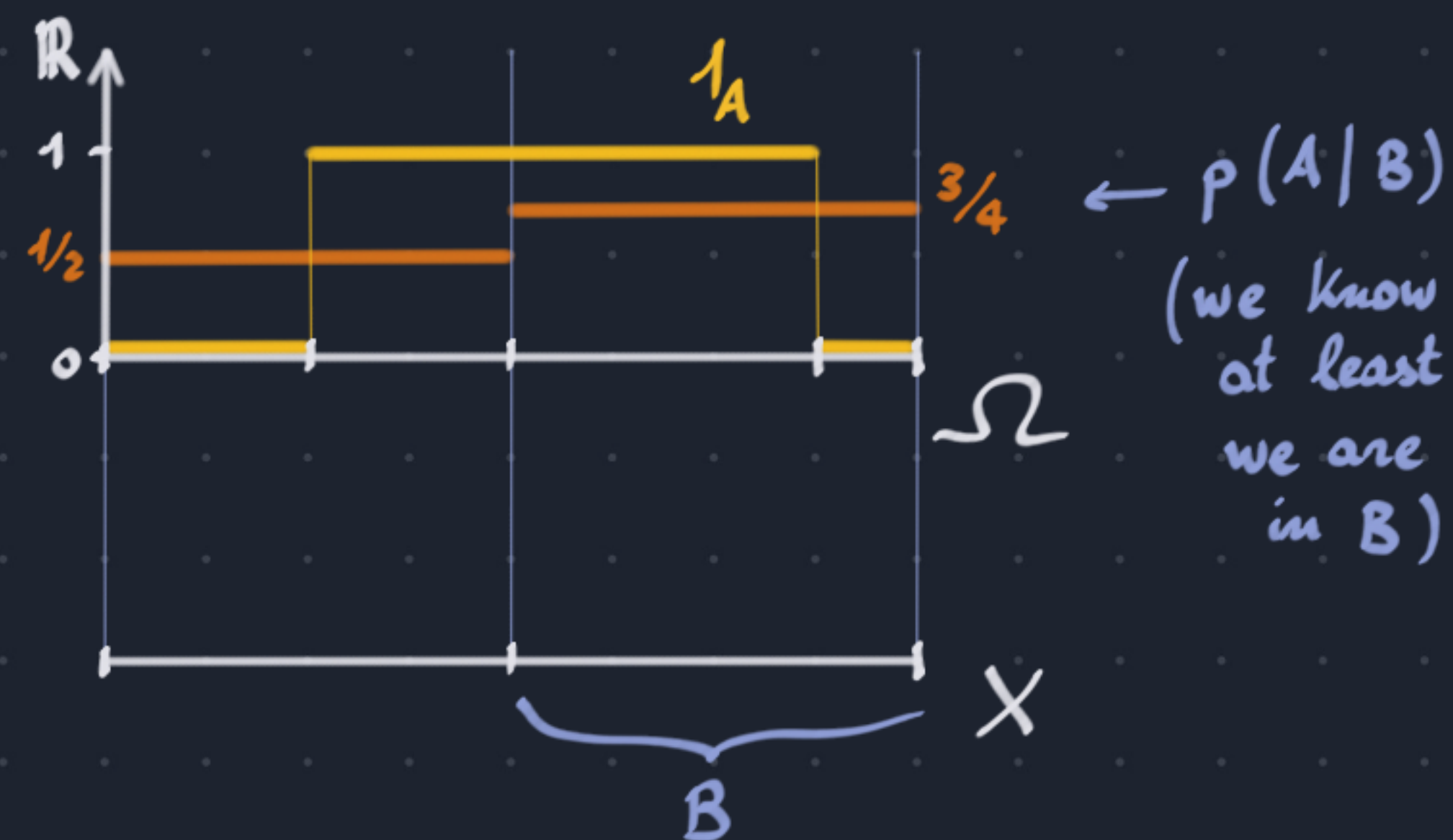
Consider now an observation $\Omega \xrightarrow{\pi} (X, \mathcal{B})$



For every $B \in \pi^{-1}(\mathcal{B})$: we want to average only within B .

2. RANDOMNESS AS IGNORANCE · Fuzzy predicates

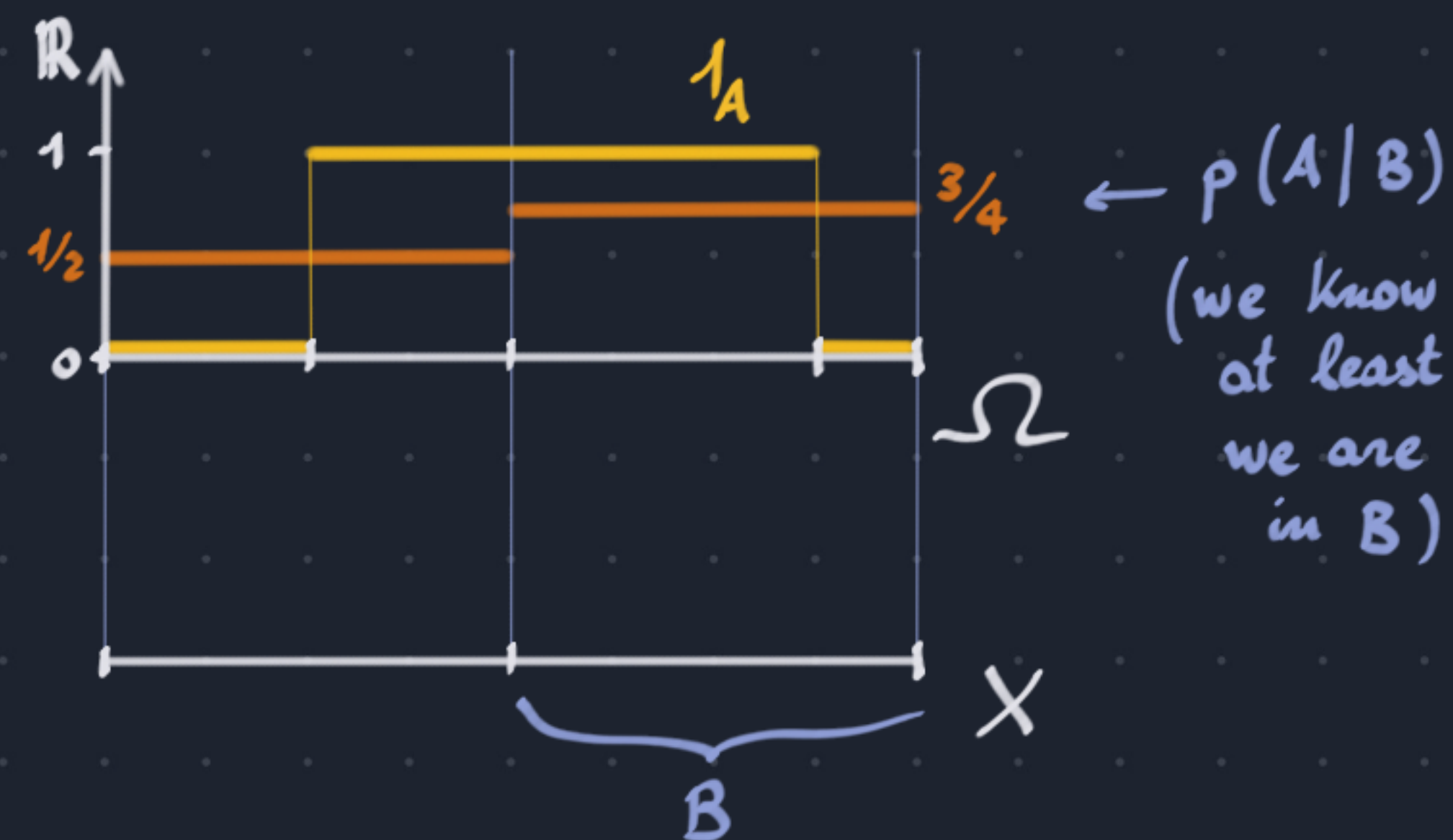
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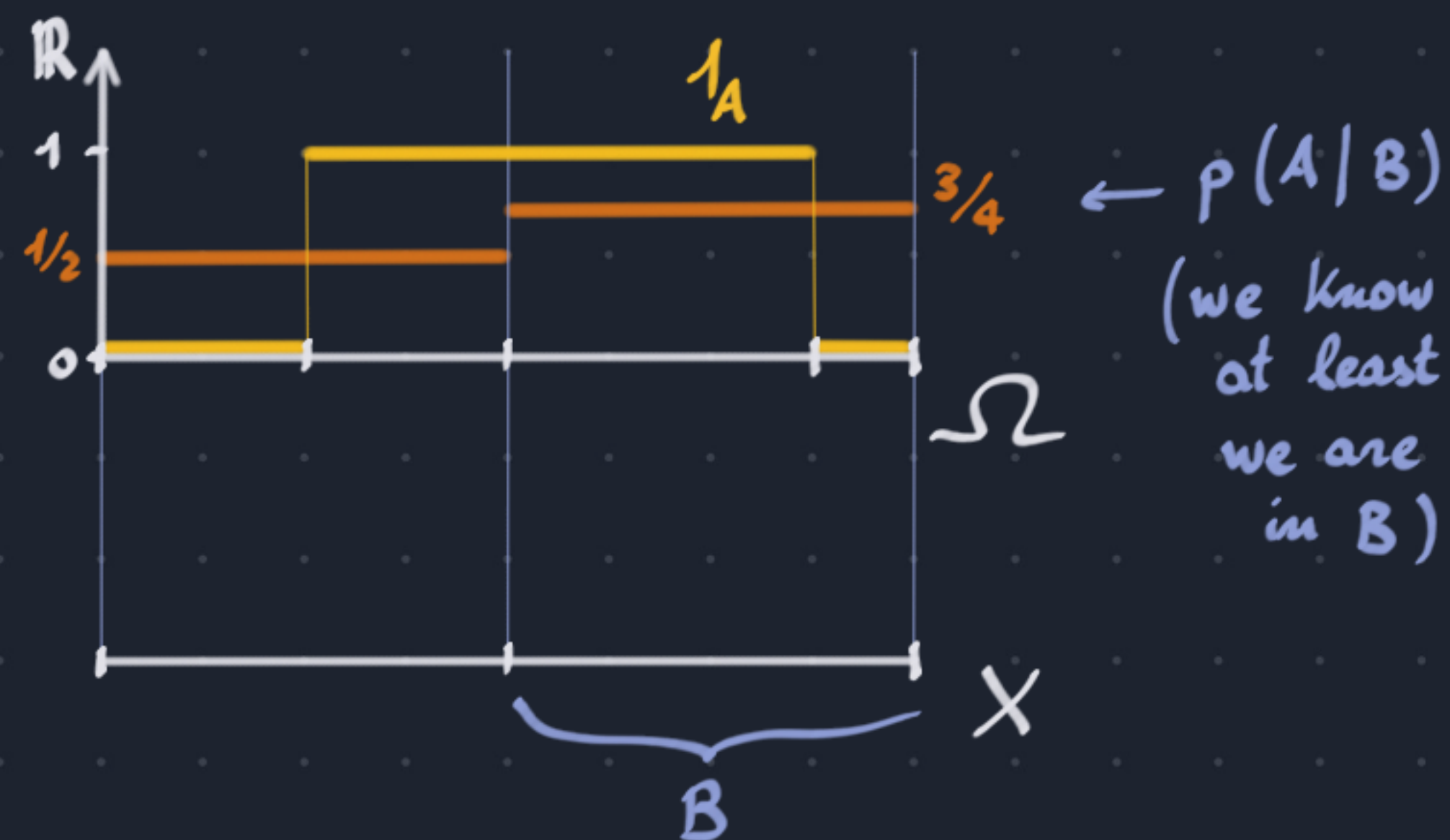


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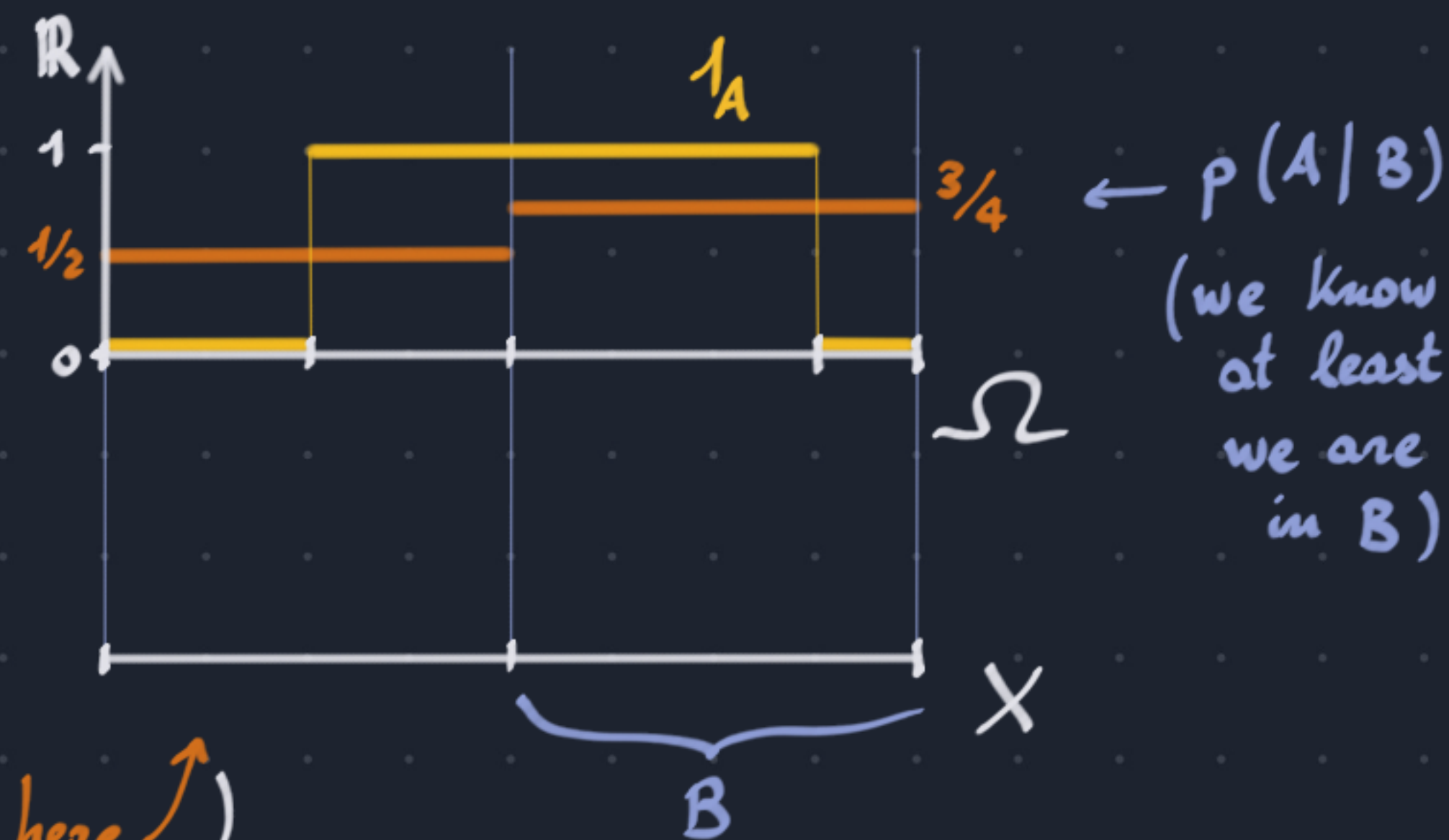
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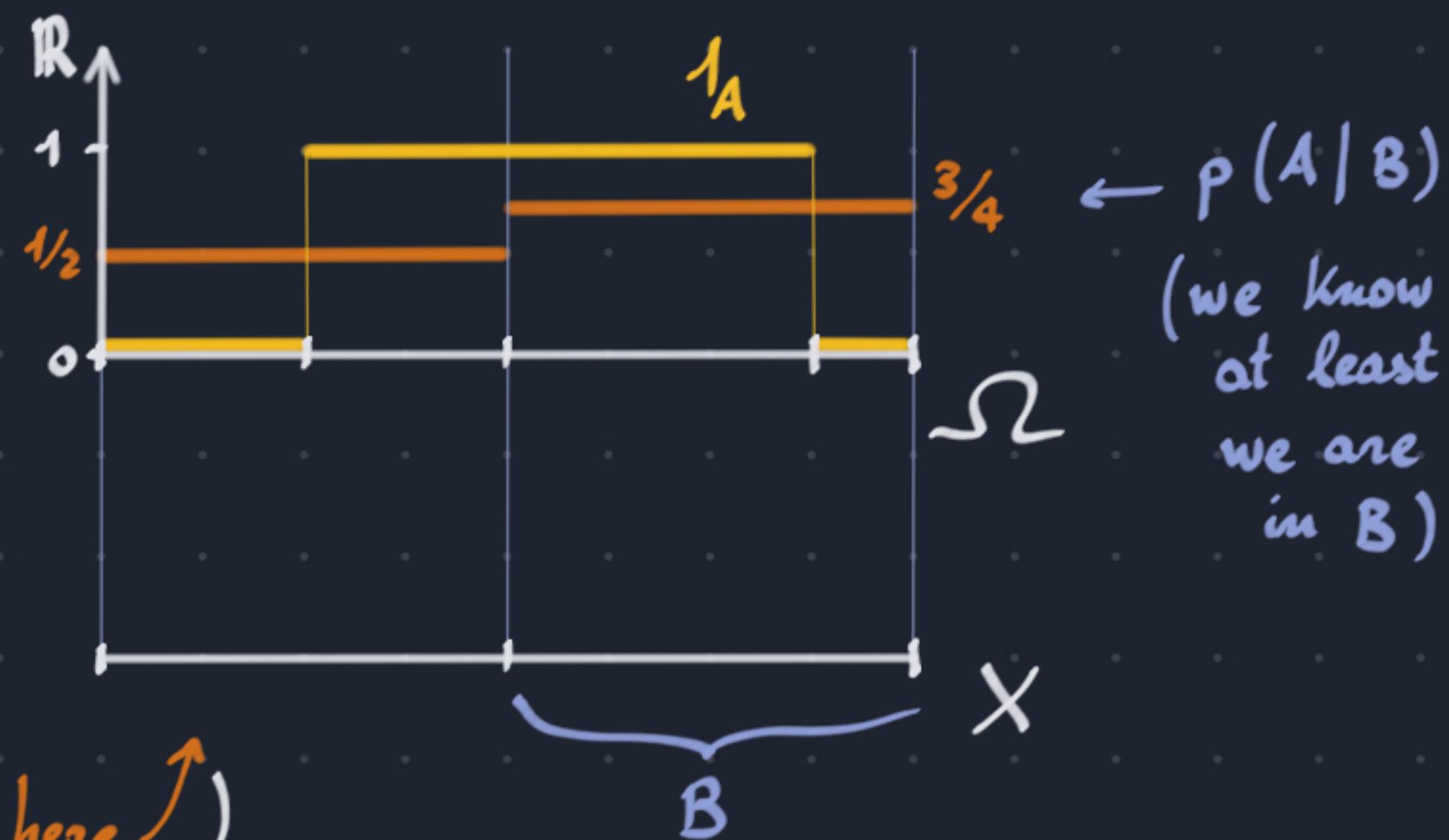
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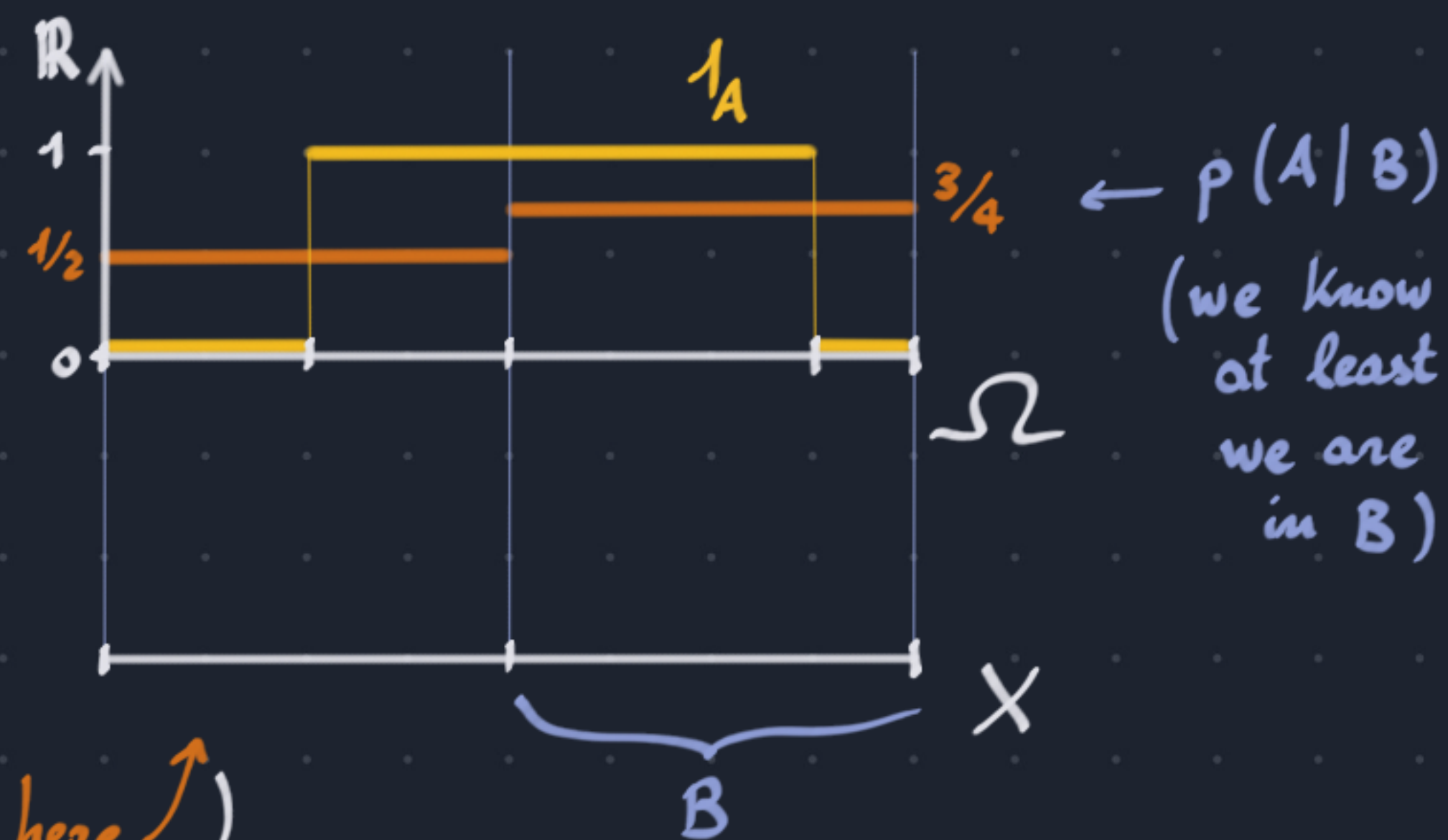
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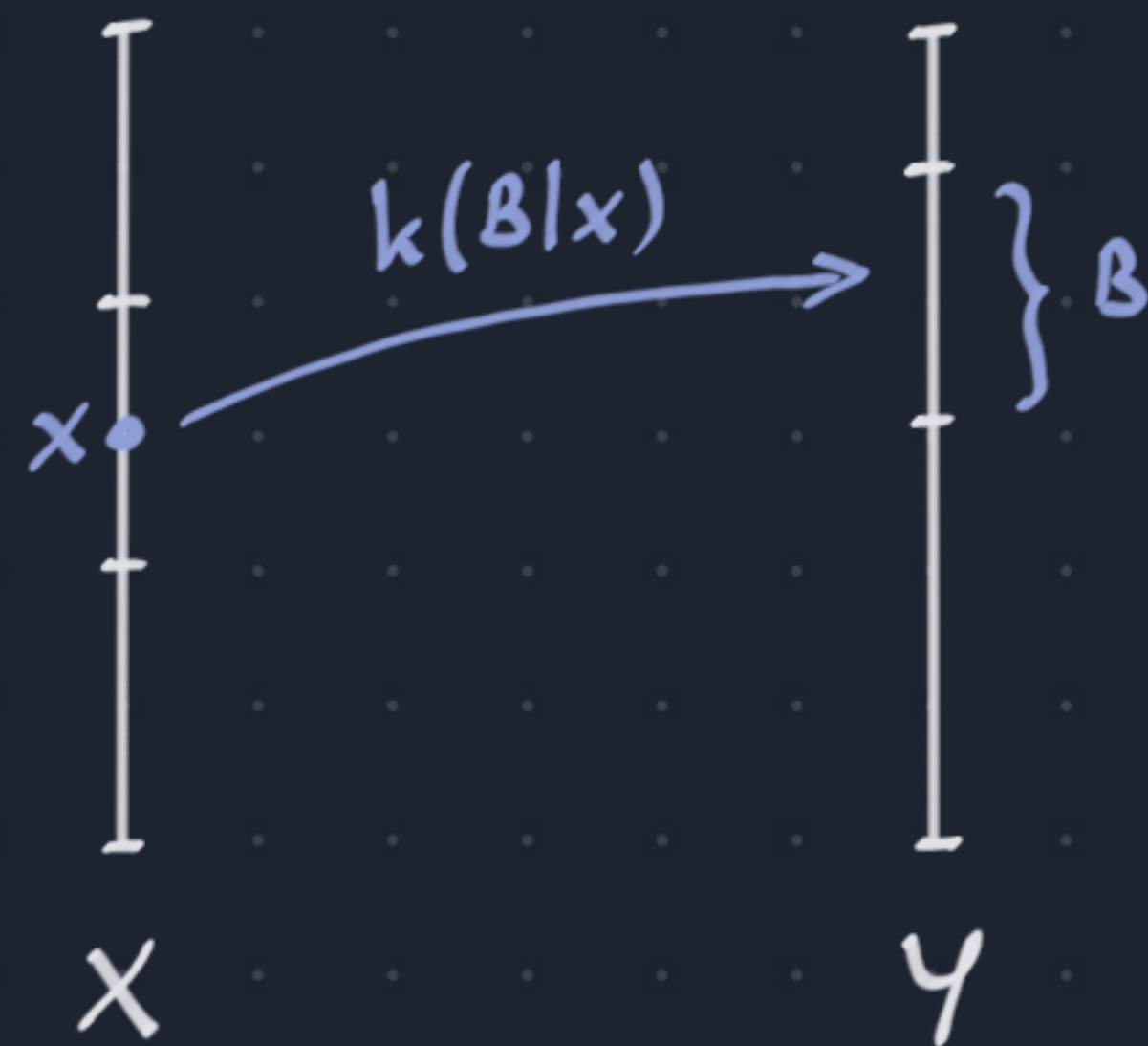
• $P[A|\pi]$ is a function $X \rightarrow \mathbb{R}$. (In orange here ↗)

• It is not unique. (Only up to measure zero.)

Question. Is $P[A|\pi]$ a probability measure (in A)?

2. RANDOMNESS AS IGNORANCE · Markov kernels

Definition. A Markov kernel $(X, \mathcal{A}) \xrightarrow{k} (Y, \mathcal{B})$ is an assignment $X \times \mathcal{B} \longrightarrow \mathbb{R}$

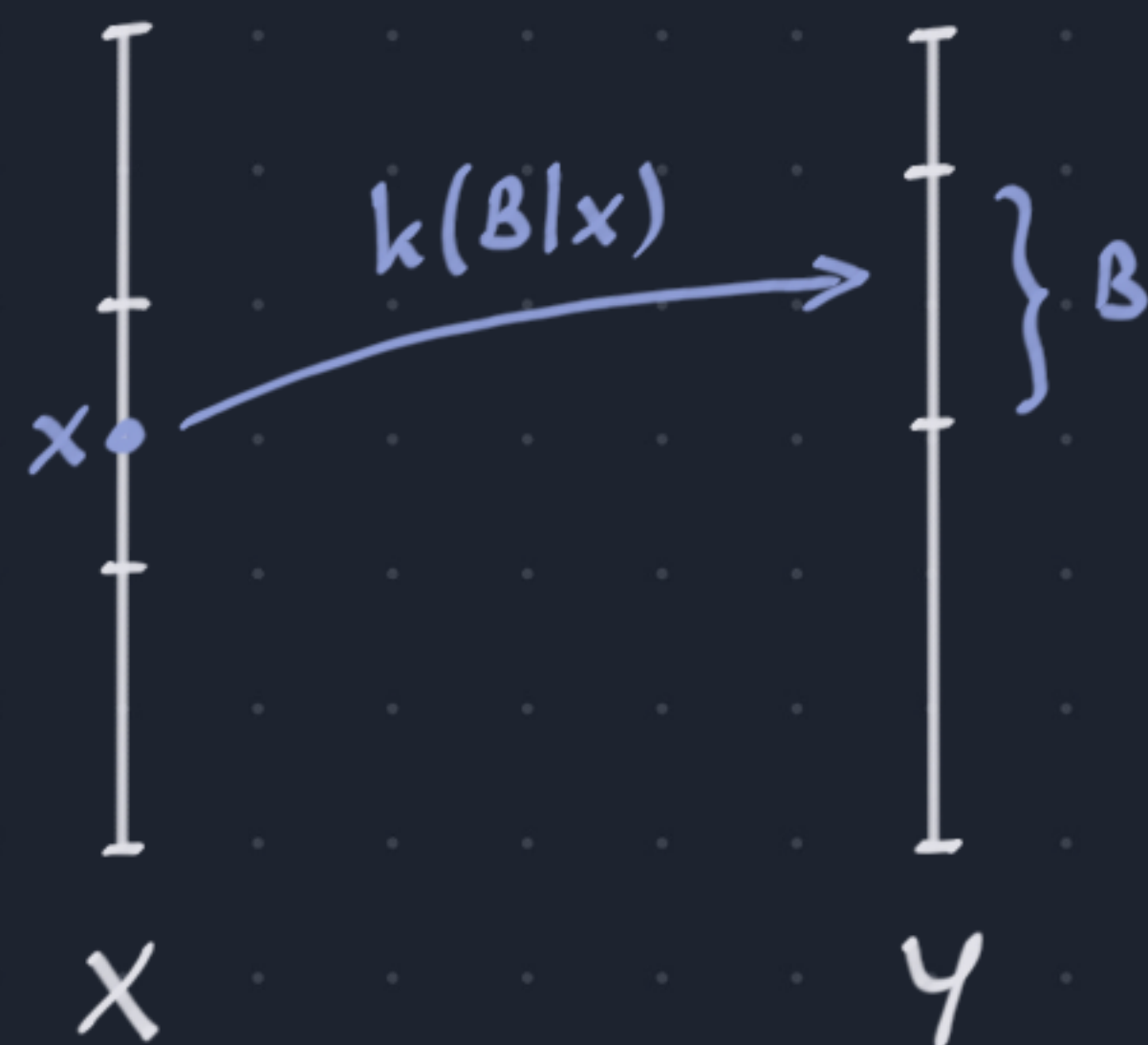


$$(x, \mathcal{B}) \longmapsto k(\mathcal{B}|x)$$

a prob. measure in $\mathcal{B}$
measurable in X

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$$(x, B) \longmapsto k(B|x)$$

$\swarrow$ a prob. measure in B
 $\nwarrow$ measurable in X

Theorem (Disintegration).

For $(X, \mathcal{A}, \rho)$ standard Borel, $\forall$ meas. $\Omega \xrightarrow{\pi} X$ there exists a kernel $\pi^+: X \rightarrow \Omega$ such that $\forall A \in \mathcal{A}$, $k(A|x)$ is a conditional probability.

1. THE IDEA OF DESCENT
2. RANDOMNESS AS IGNORANCE
3. SHEAVES IN PROBABILITY
4. STATISTICAL EXPERIMENTS

3. SHEAVES IN PROBABILITY · Measurability as descent data

Credits. These ideas first appeared in the work of Alex Simpson.

Here we present refinements due to Dario Stein (and later, myself).

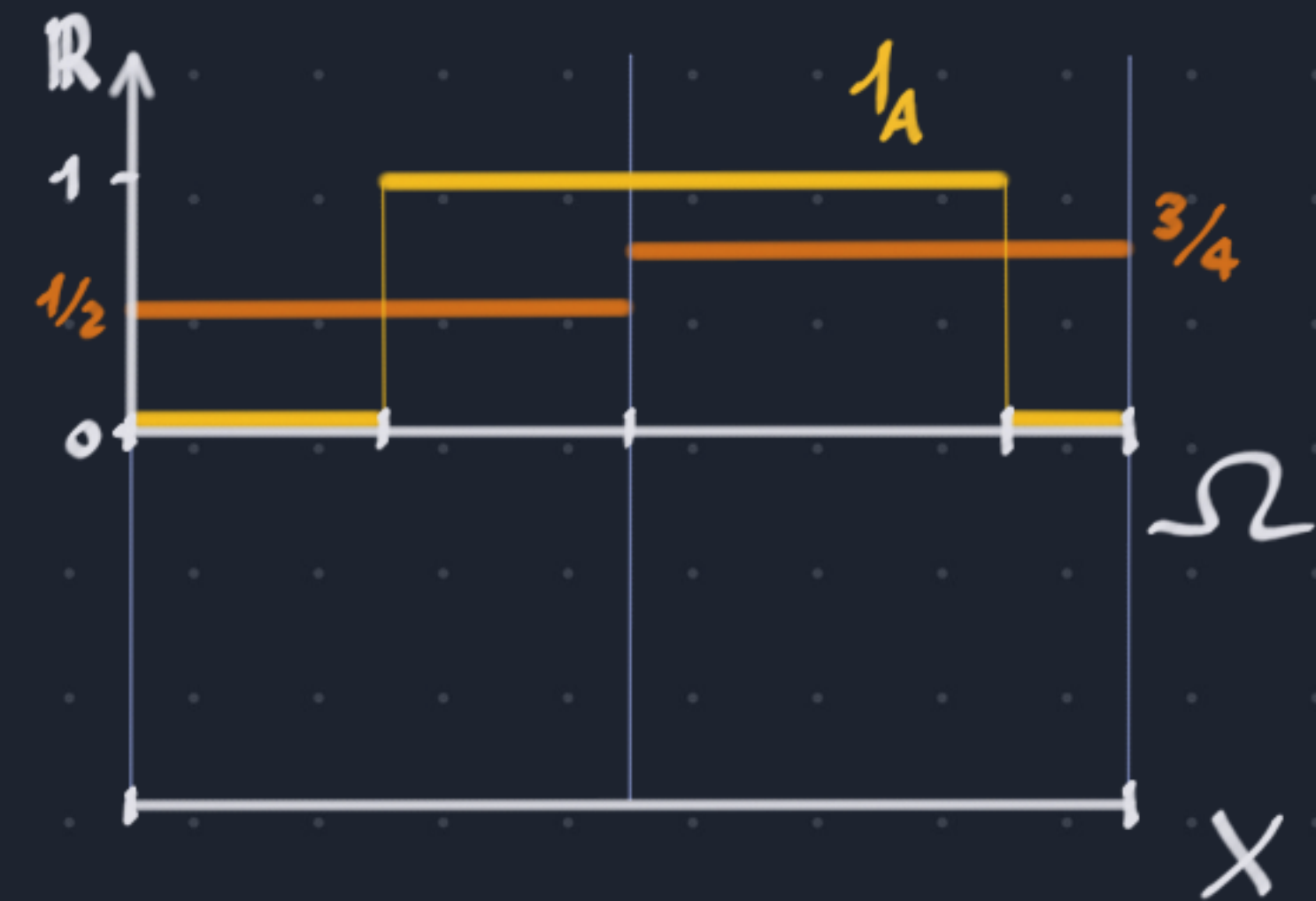
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Idea. Given $\Omega \xrightarrow{\pi} X$, RVs which are measurable for $\pi^{-1}(\mathcal{B})$ are a lot like descent data!
("Respect the distinctions of π ".)

Atomic coverage: every map is a cover



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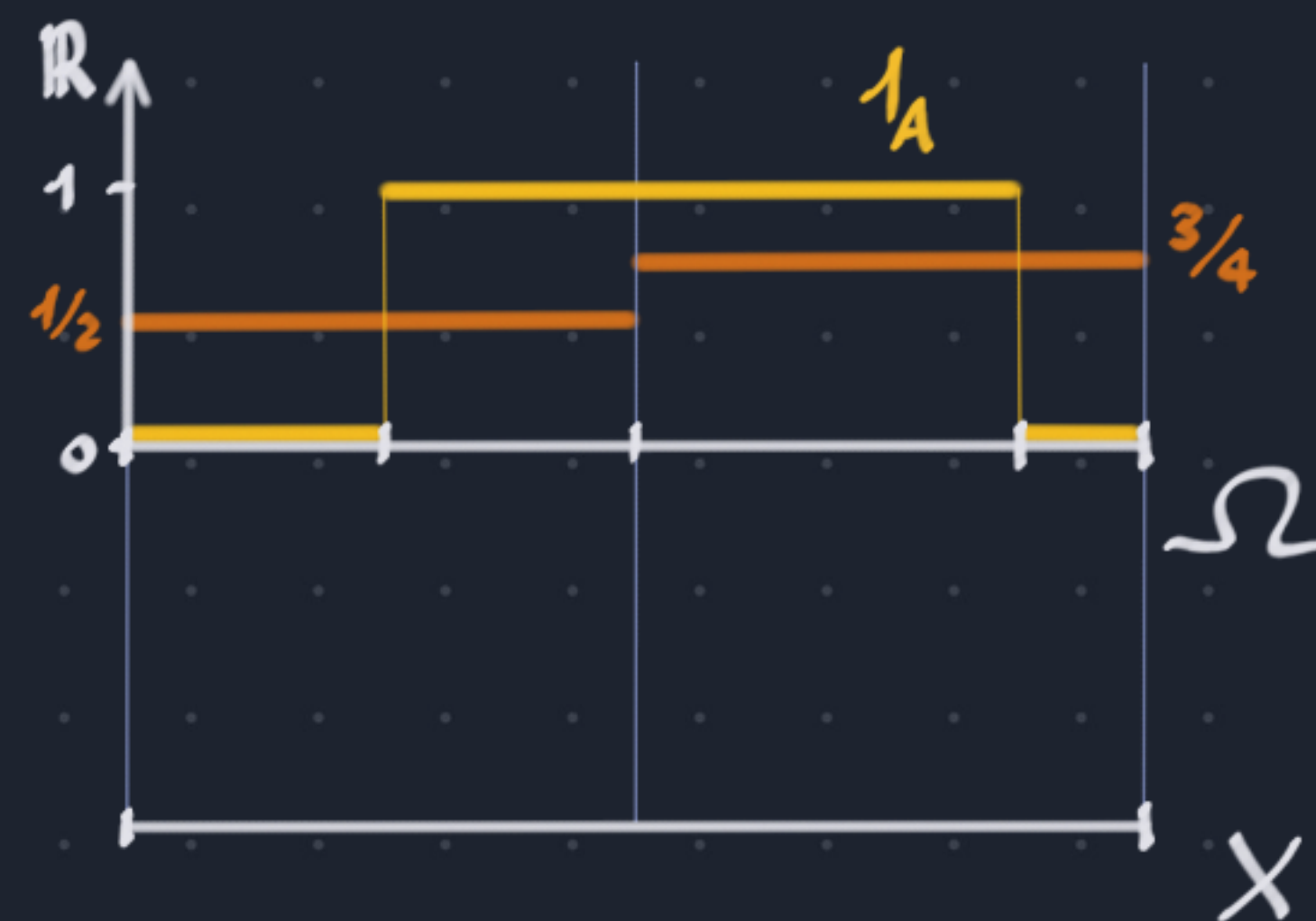
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Atomic coverage: every map is a cover

But, on what categories?



3. SHEAVES IN PROBABILITY · Categories of probability spaces

Definition. The category **Prob** has:

- As objects, std. Borel prob. spaces
- As morphisms, measure-preserving maps.

$(\underbrace{X, \mathcal{A}}_{\text{std. Borel meas. space}}, \underbrace{\rho}_{\text{prob. measure}})$

$(X, \mathcal{A}, \rho) \xrightarrow{\pi} (Y, \mathcal{B}, q)$ measurable,

and $\forall B \in \mathcal{B}, q(B) = \rho(\pi^{-1}(B))$.

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$$\text{and } \forall B \in \mathcal{B}, \quad q(B) = \rho(\pi^{-1}(B)).$$

Definition. The category **Krn** has:

- As objects, std. Borel prob. spaces
- As morphisms, measure-preserving Markov kernels.

$$(X, \mathcal{A}, \rho) \xrightarrow{k} (Y, \mathcal{B}, q)$$

$$\text{s.t. } \forall B \in \mathcal{B}: \quad q(B) = \int_X k(B|x) \rho(dx)$$

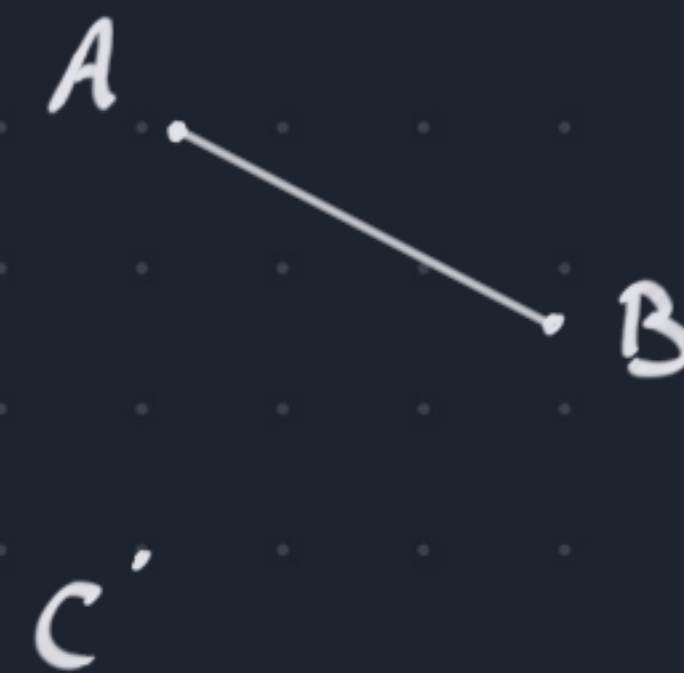
3. SHEAVES IN PROBABILITY · $\dagger$ -categories

Idea.

Dagger categories
categories

=

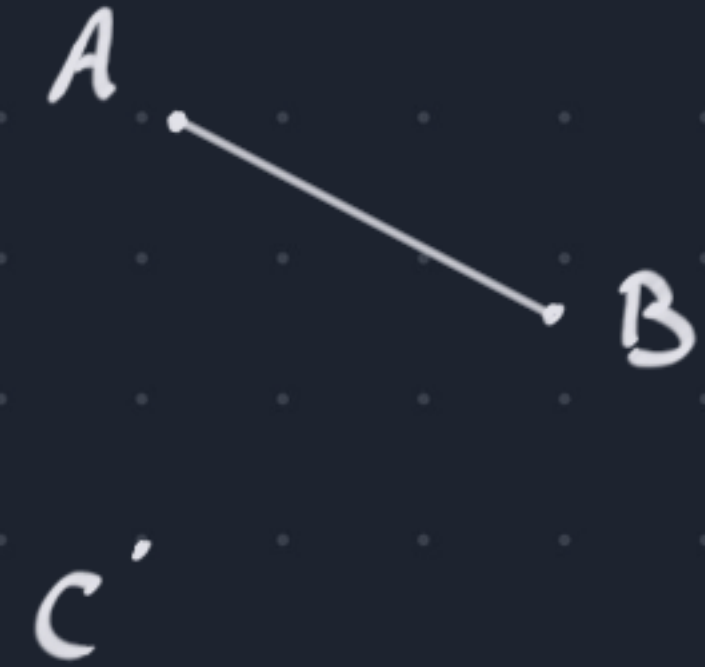
undirected graphs
directed graphs



3. SHEAVES IN PROBABILITY · $\dagger$ -categories

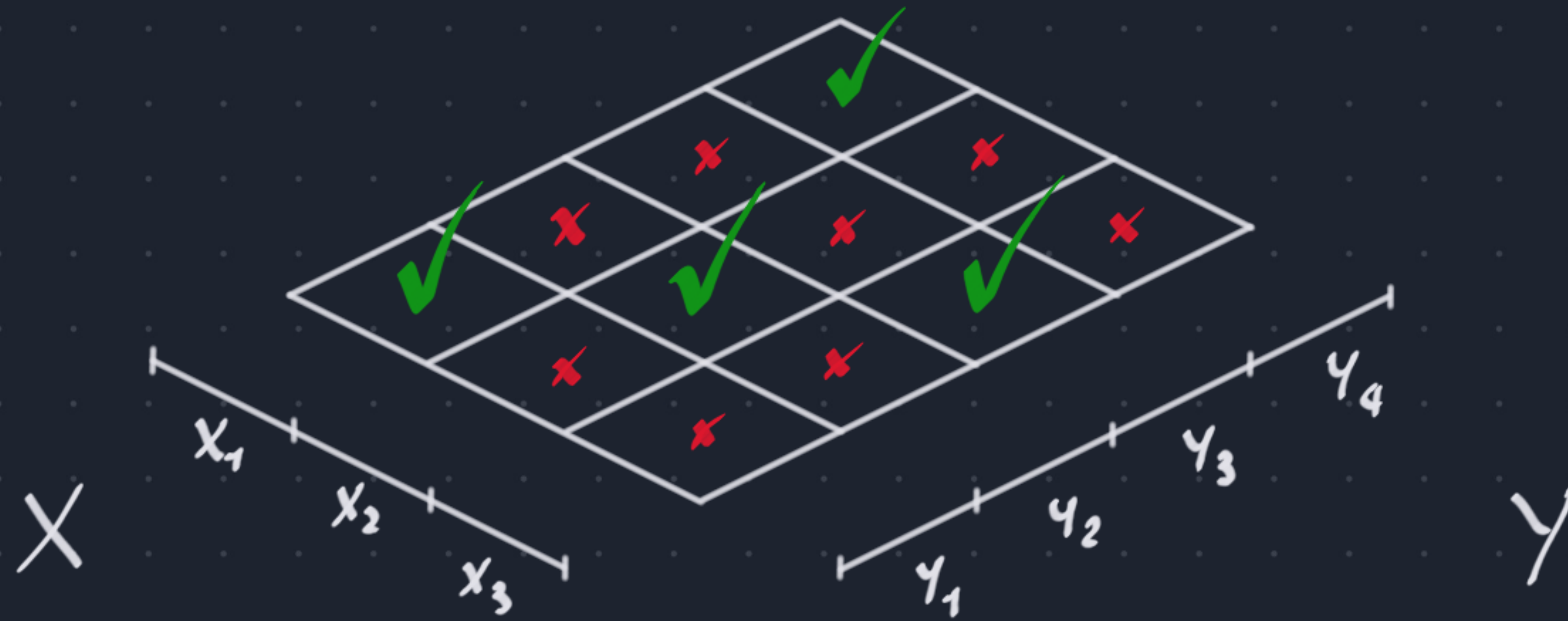
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$$\frac{\text{Dagger categories}}{\text{categories}} = \frac{\text{undirected graphs}}{\text{directed graphs}}$$



Examples.

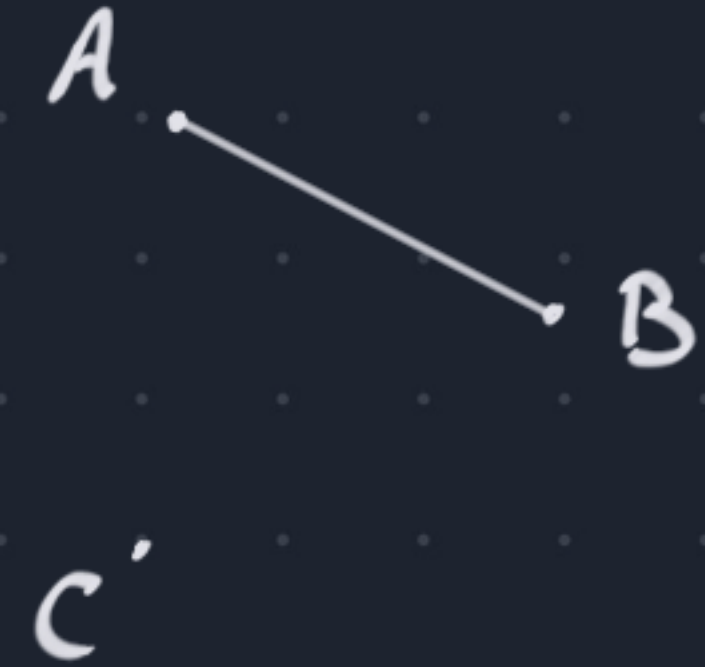
Rel



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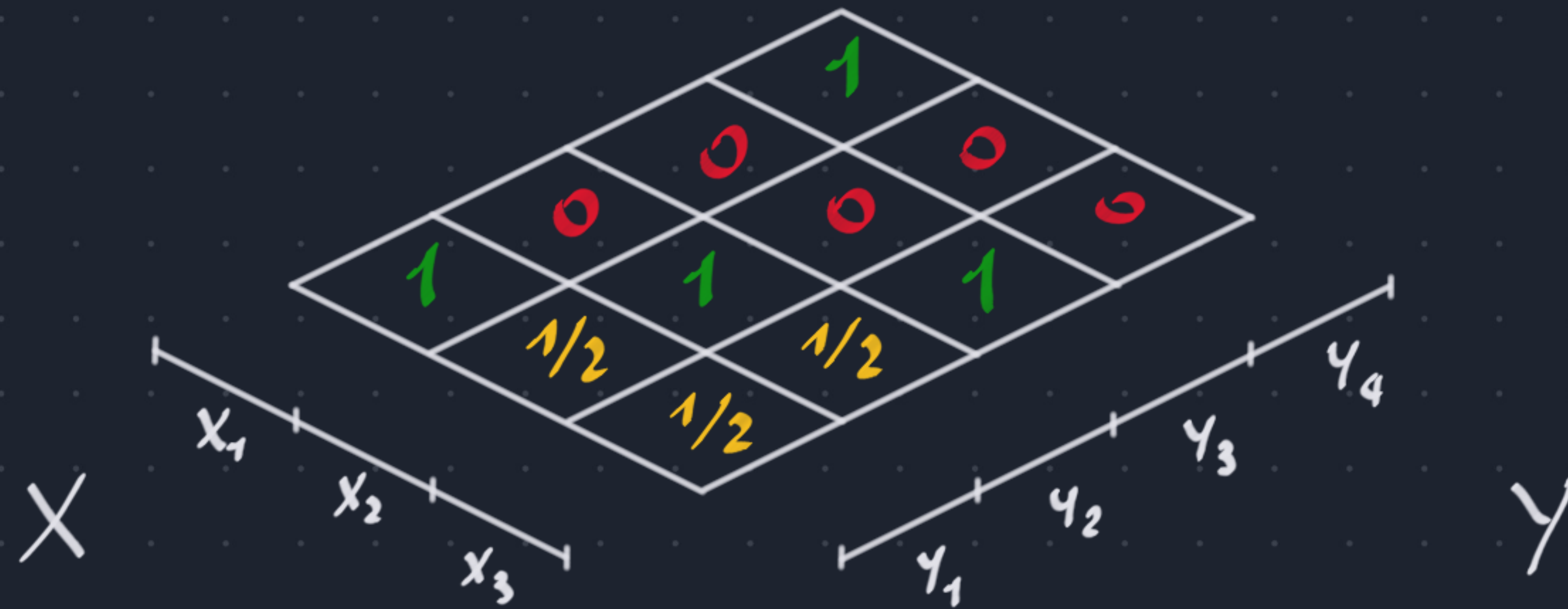
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Examples.

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Mat



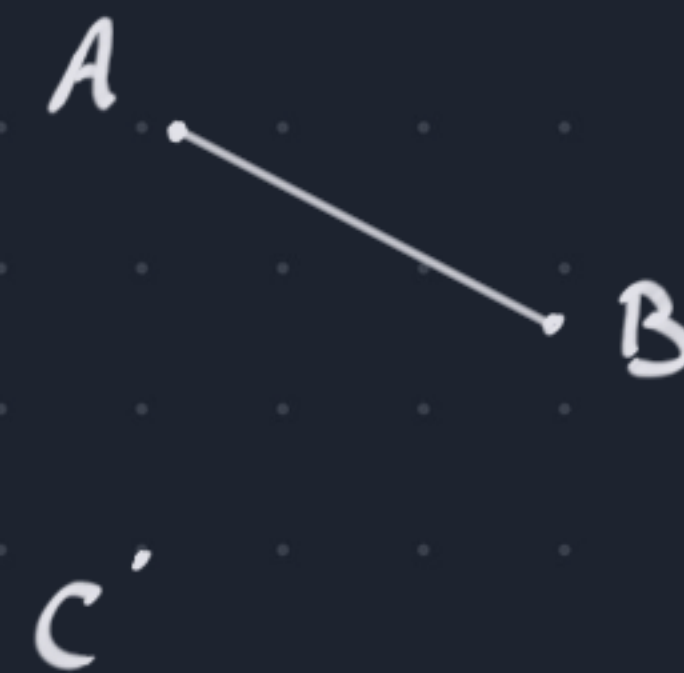
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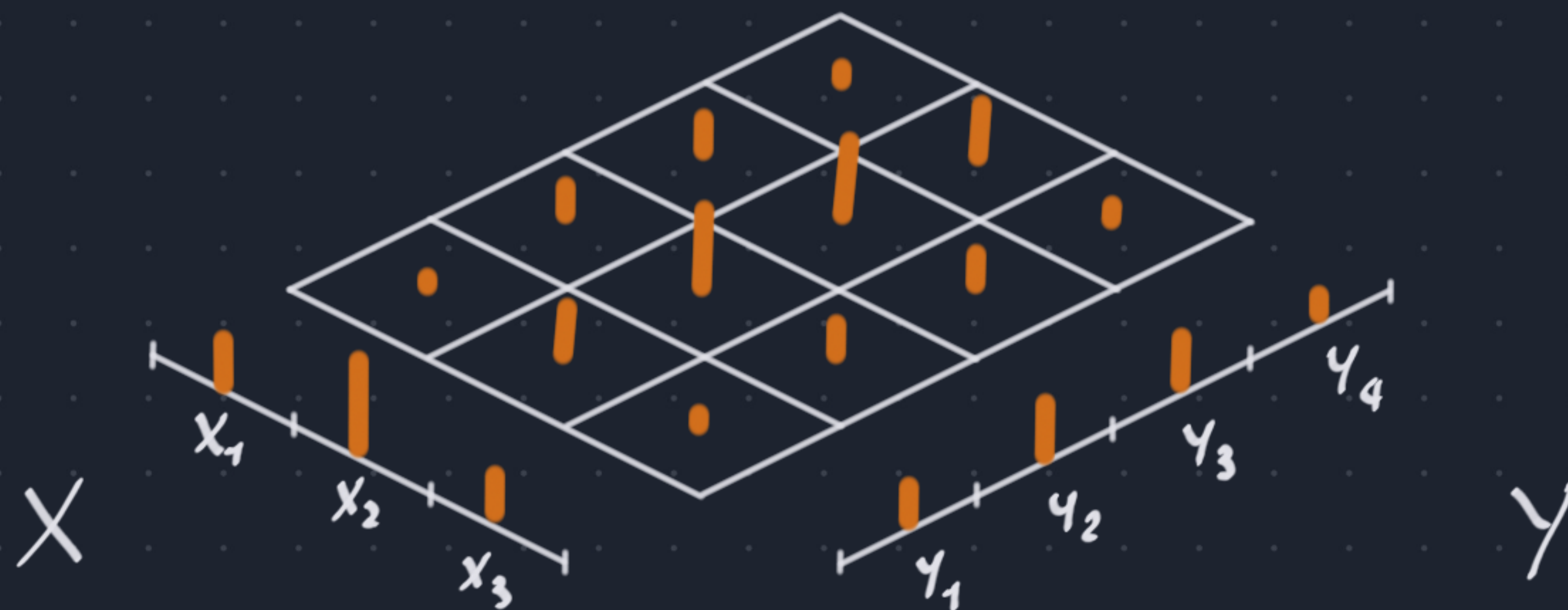


Examples.

Rel

Mat

Kru



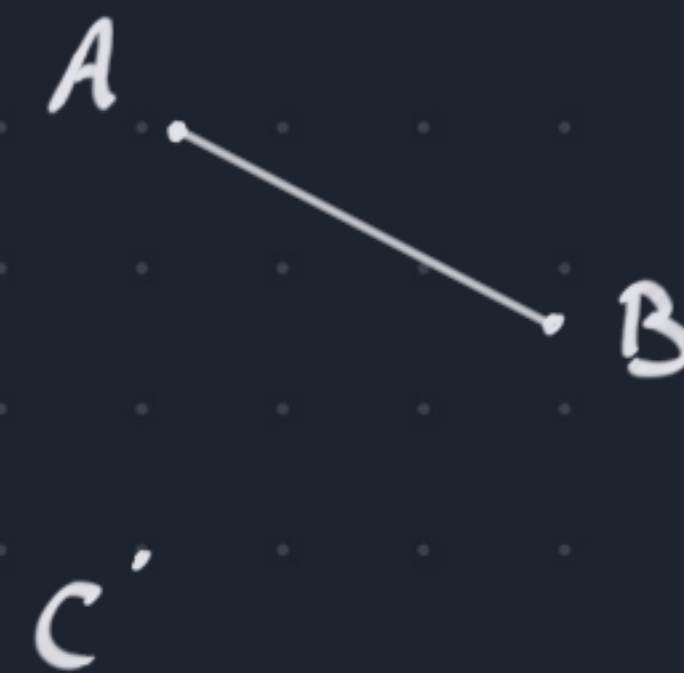
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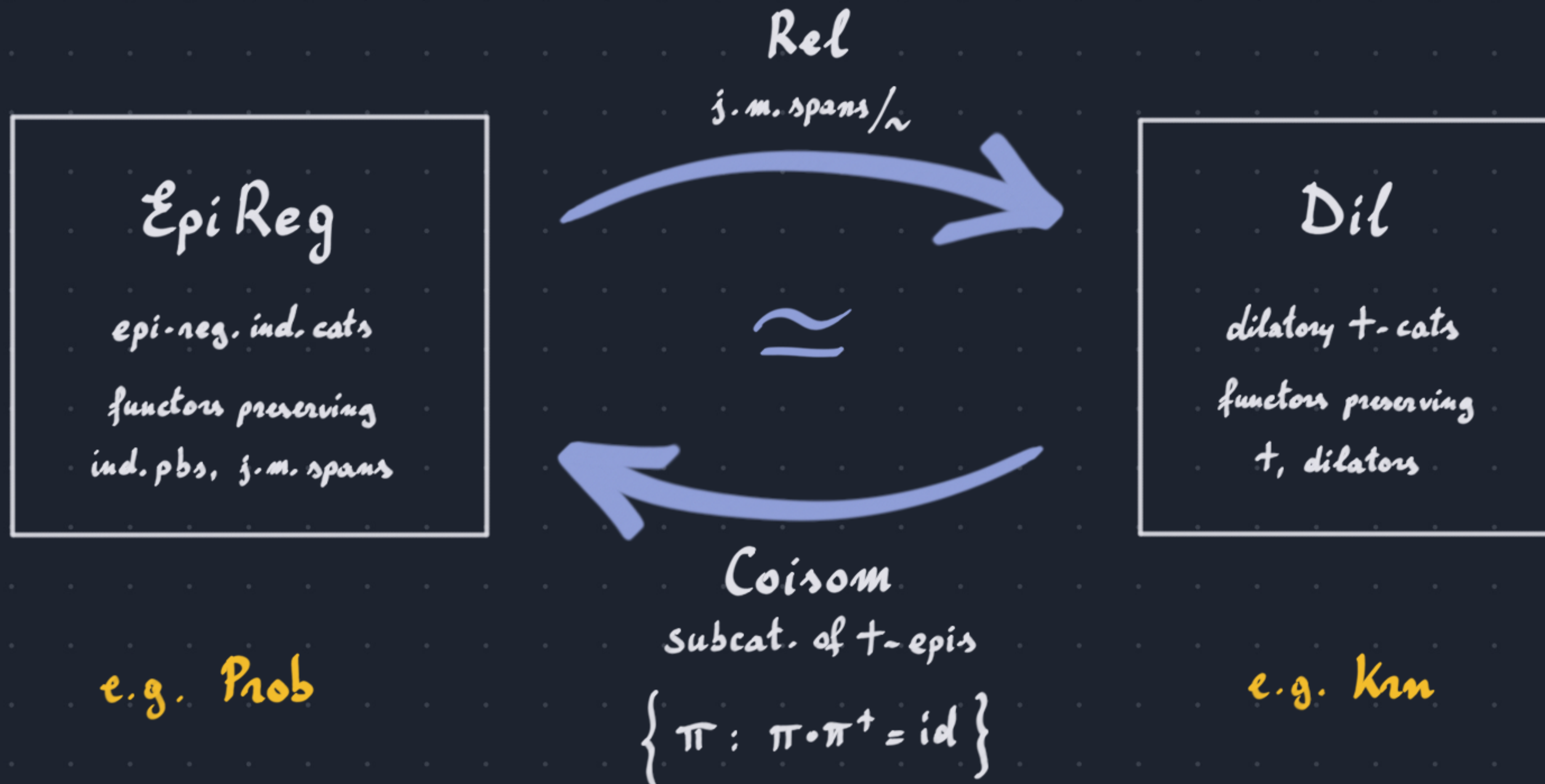
Definition.

A dagger structure on $\mathcal{C}$ is a functor $\mathcal{C}^{\text{op}} \xrightarrow[\cong]{\dagger} \mathcal{C}$ such that:

- On objects, $X^{\dagger} = X$;
- On morphisms, $f^{\dagger\dagger} = f$.

3. SHEAVES IN PROBABILITY · $\dagger$ -categories

Theorem. We have an adjoint equivalence of categories



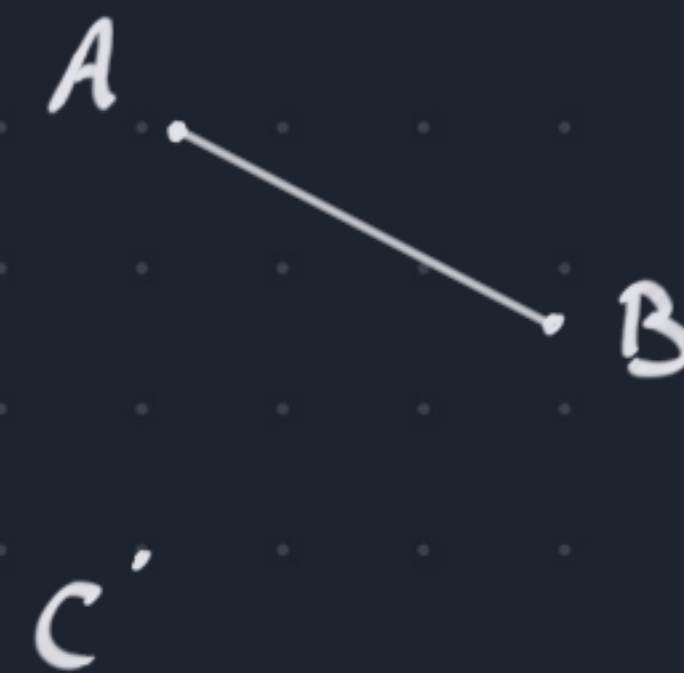
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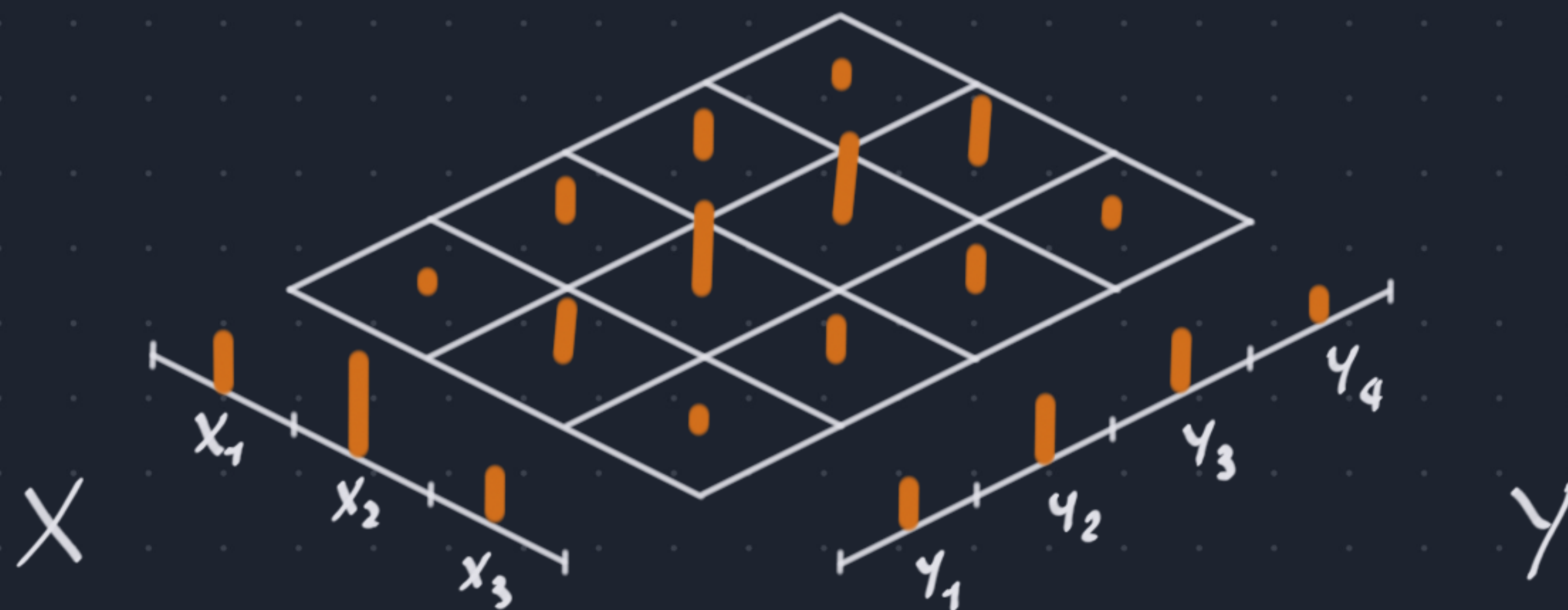


Examples.

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3. SHEAVES IN PROBABILITY · Presheaves of random variables

Definition. The L^p -space functors are constructed as follows.

$$\mathbf{Prob}^{\text{op}} \rightarrow \mathbf{Set}$$

$$X \longmapsto L^p(X) = \{ \text{eq. classes of } p\text{-integrable RVs} \}$$

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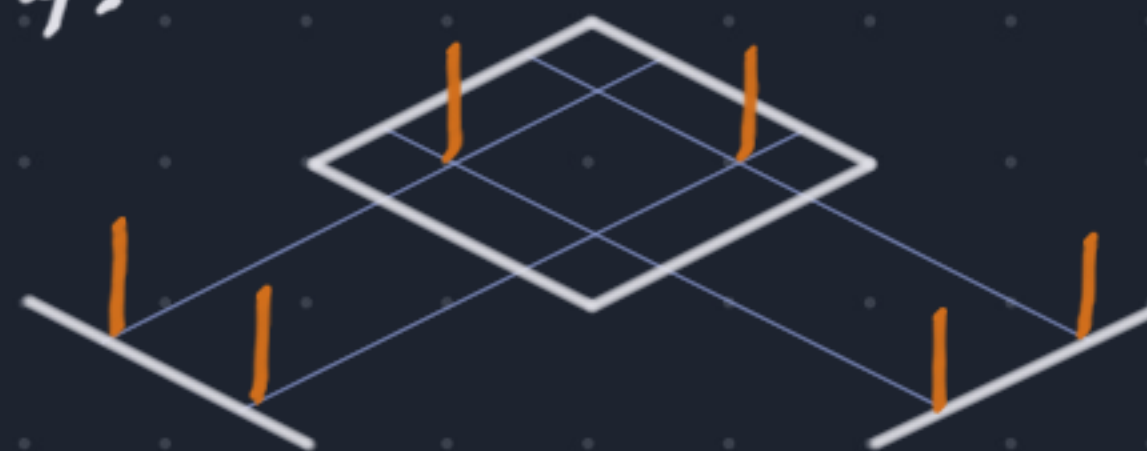
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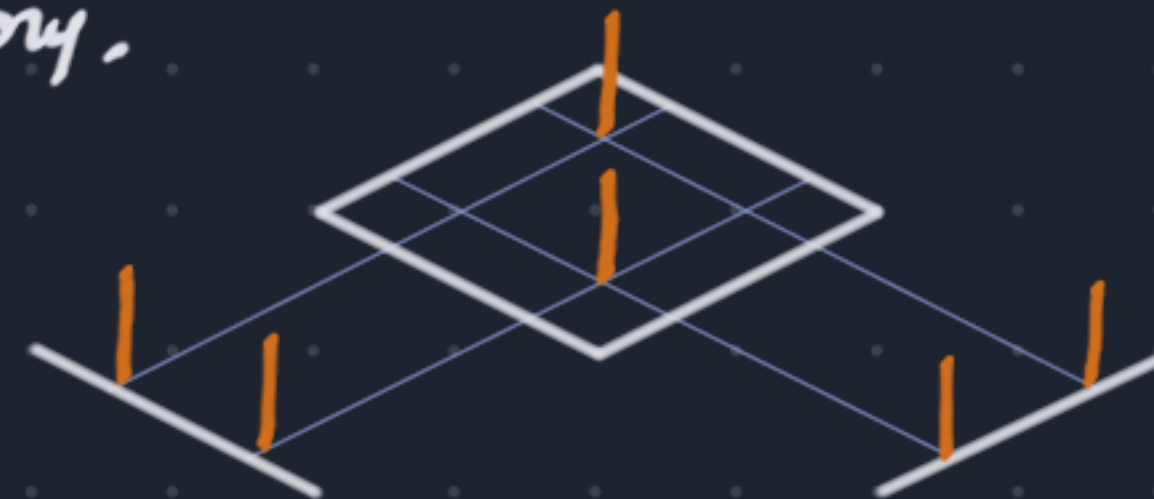
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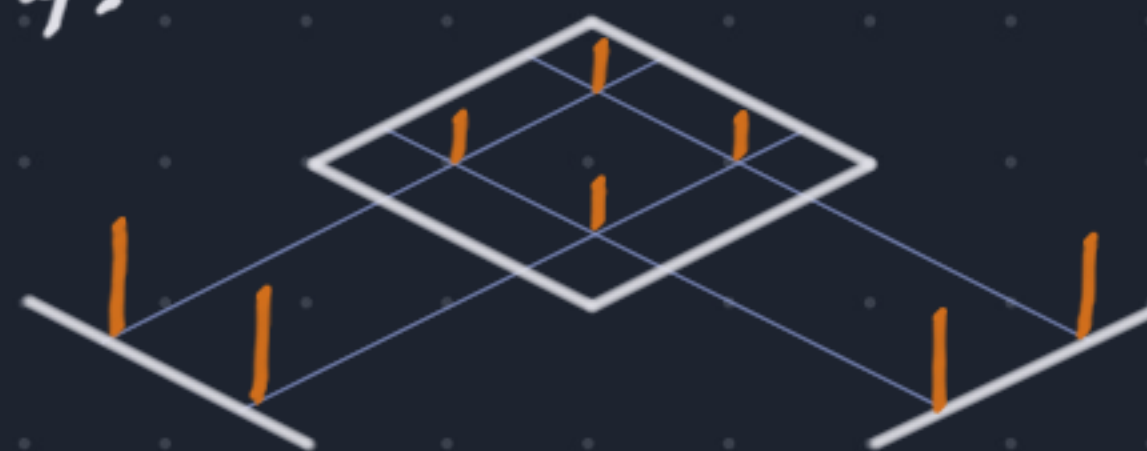
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3. SHEAVES IN PROBABILITY · Independent squares

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For $D \neq 0$: rel. orthogonal



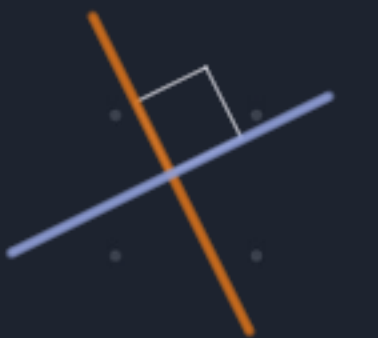
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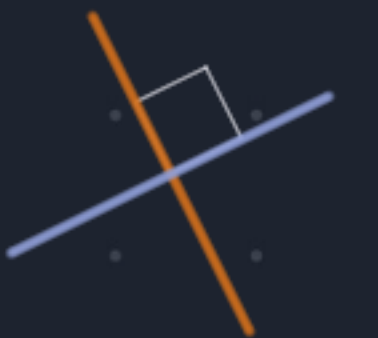
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For $D \neq 0$: $f \perp g \mid s \cdot f (= t \cdot g)$

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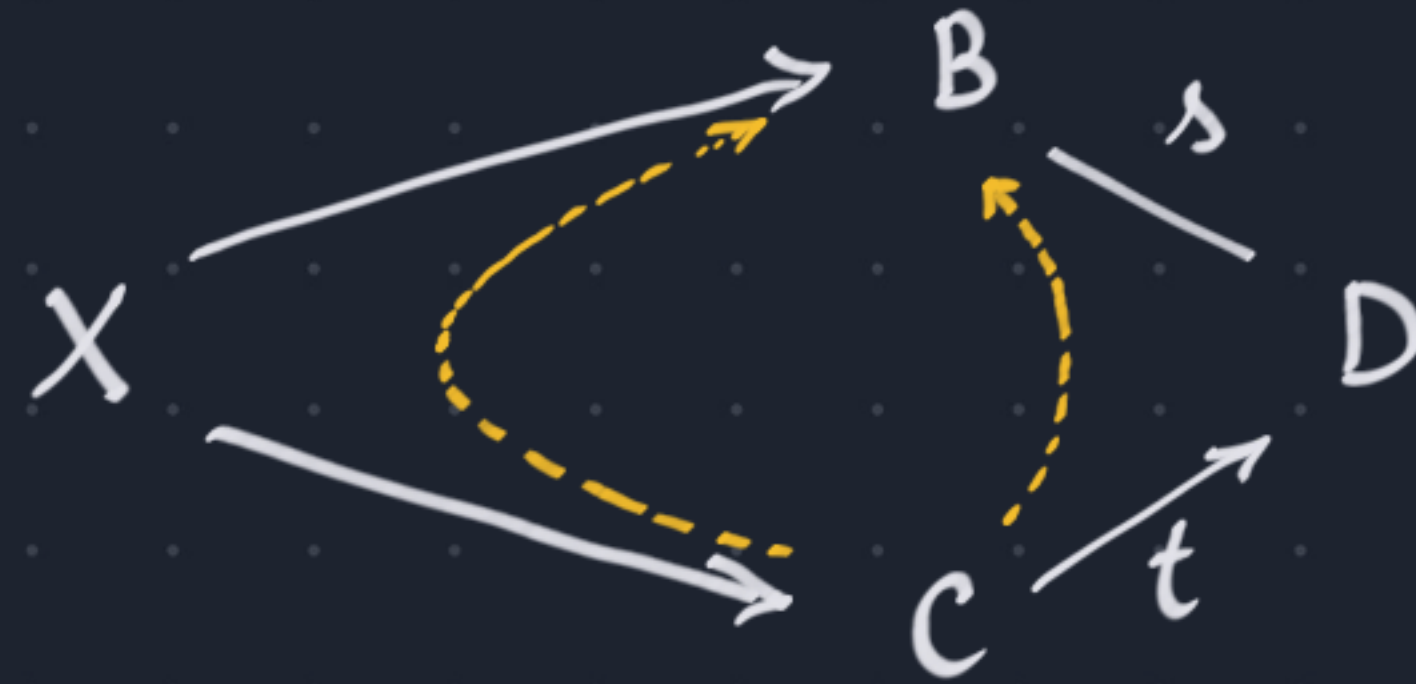
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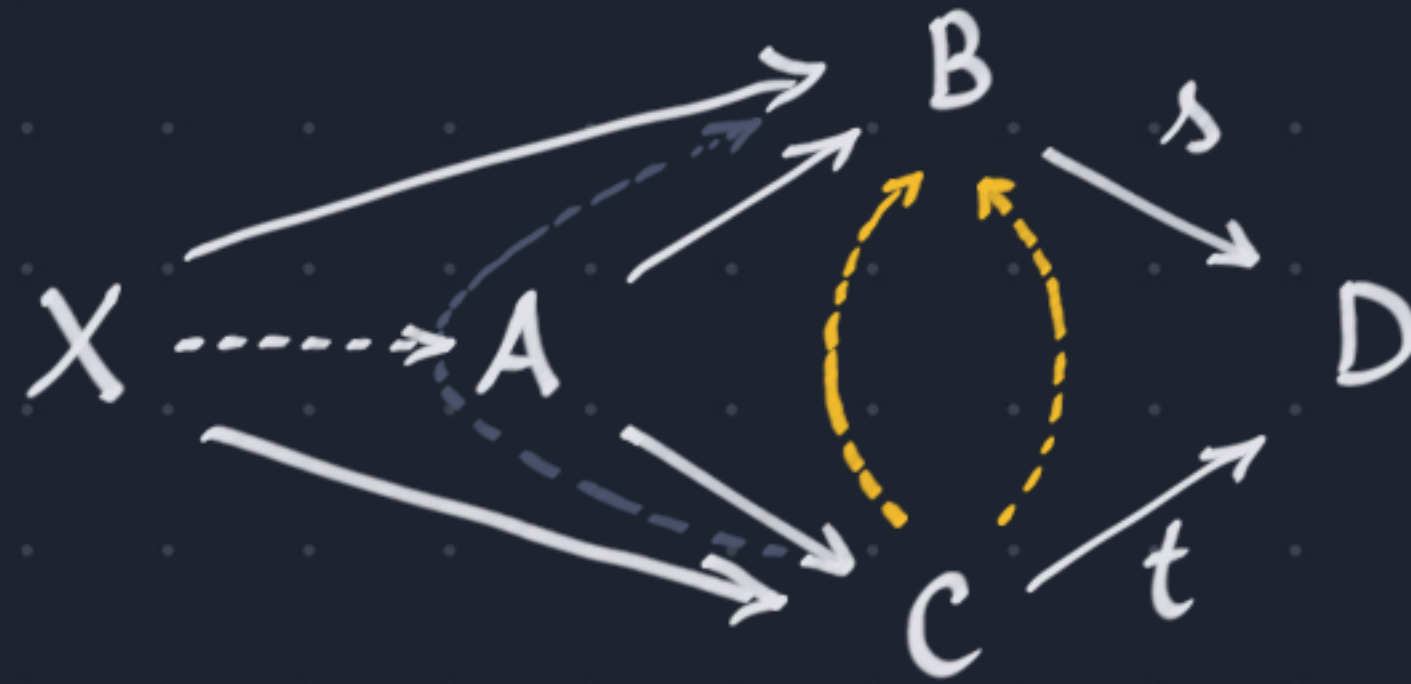
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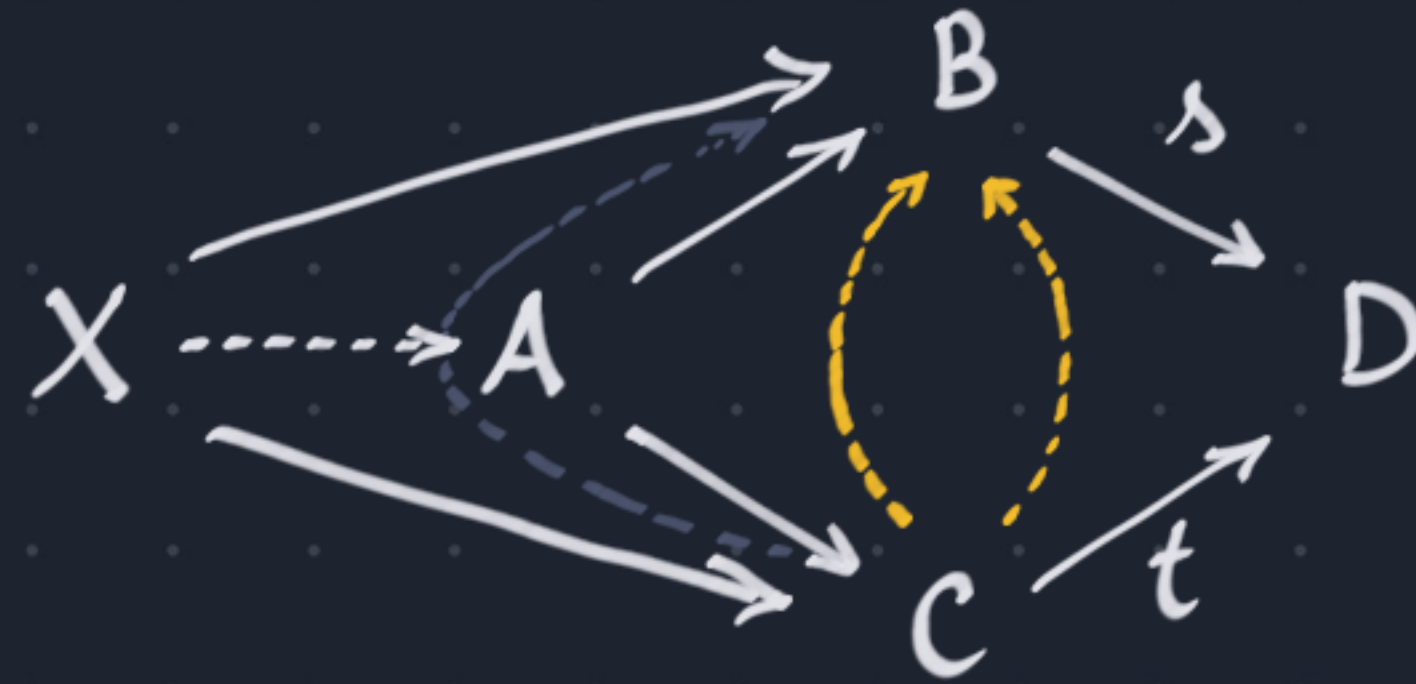
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Example. If $s = t$: independent kernel pair (not always a kernel pair!)

3. SHEAVES IN PROBABILITY · Independent squares



Property	Regular kernel pair	Independent kernel pair
Reflexivity $\text{id}_X \Rightarrow \pi_2 \cdot \pi_1^+$	✓	✗
Symmetry $\pi_2 \cdot \pi_1^+ = \pi_1 \cdot \pi_2^+$	✓	✓
Transitivity $\pi_2 \cdot \pi_1^+ \cdot \pi_2 \cdot \pi_1^+ \Rightarrow \pi_2 \cdot \pi_1^+$	✓	✓

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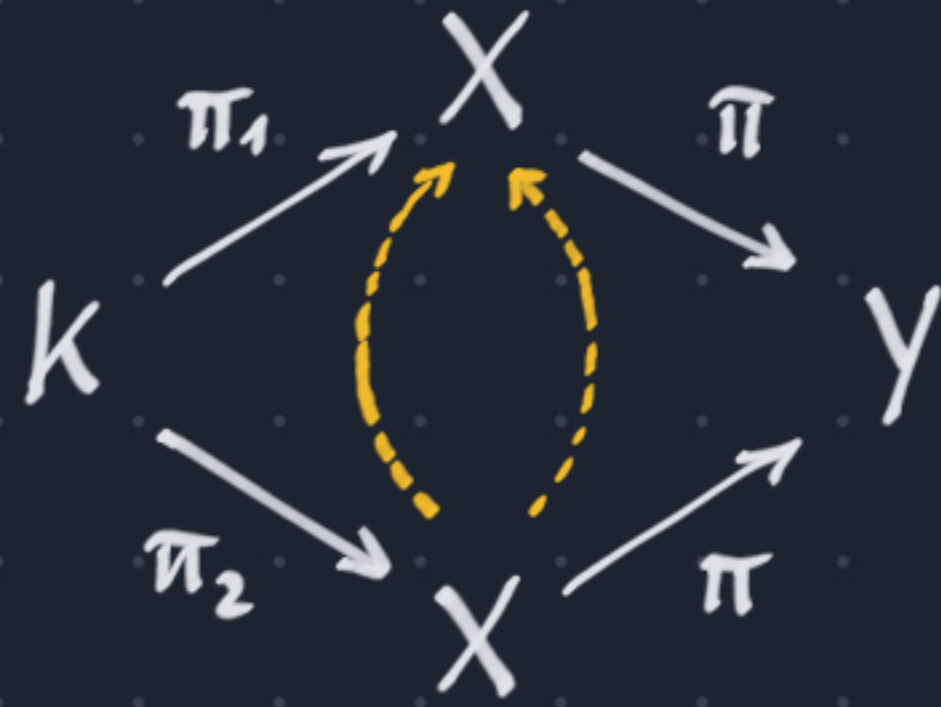
3. SHEAVES IN PROBABILITY · Conditional expectation



$$\pi_2 \cdot \pi_1^+ = \pi^+ \cdot \pi$$

In K_{un} : for all $A \in \mathcal{A}$, $\pi^+ \cdot \pi(A | -) = \mathbb{P}[A | \pi]$

3. SHEAVES IN PROBABILITY · Conditional expectation

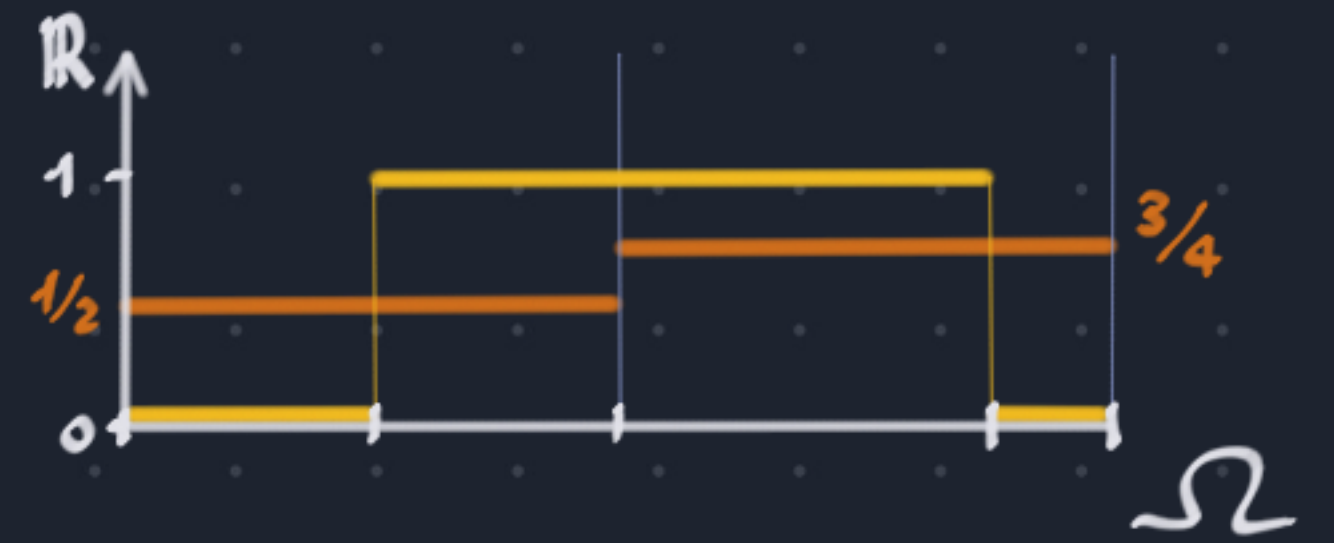


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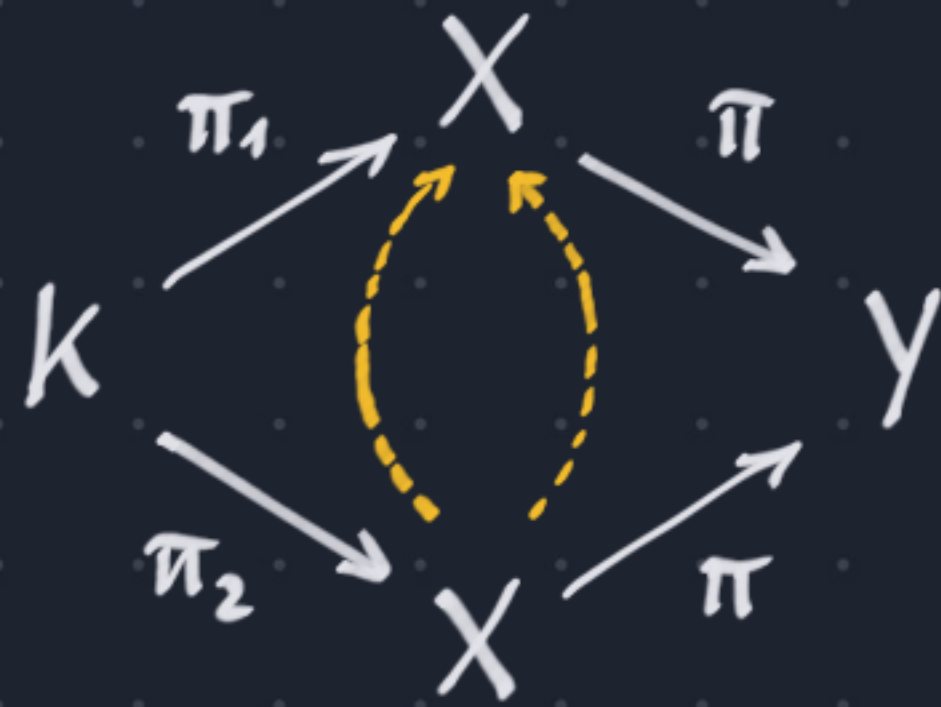
In Km : for all $A \in \mathcal{A}$, $\pi^+ \cdot \pi(A | -) = \mathbb{P}[A | \pi]$

Descent data (π, L^p) : RVs $f \in L^p(X)$ s.t. $(\pi^+ \cdot \pi)^* f \simeq f \Leftrightarrow E[f | \pi] \simeq f$.

Intuition: the "equivalence relation" is the idempotent $\pi^+ \cdot \pi$, "having the same local averages".



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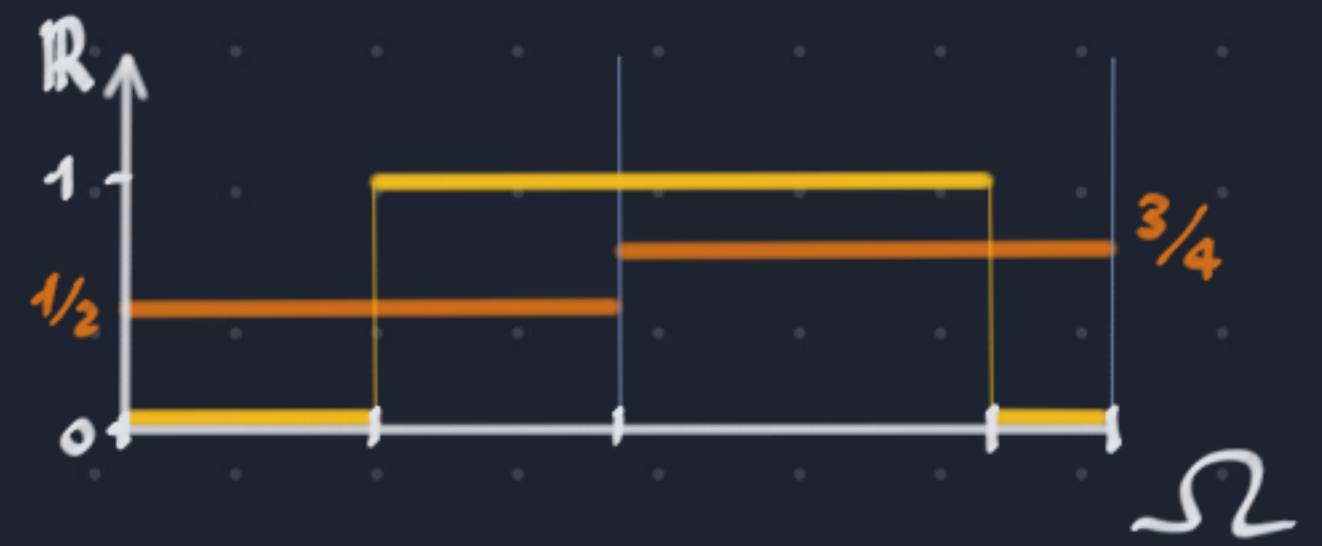
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Intuition: the "equivalence relation" is the idempotent $\pi^+ \cdot \pi$, "having the same local averages".

Theorem. Let $\Phi: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ be a presheaf on a dagger category with dilators.

Then $\Phi|_{\text{dm}}$ is always an atomic sheaf.



3. SHEAVES IN PROBABILITY · Presheaves of random variables

Definition. The L^p -space functors are constructed as follows.

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 \\
 X \longmapsto L^p(X) = \{ \text{eq. classes of } p\text{-integrable RVs} \} & & \\
 \\
 \begin{array}{ccccc}
 X & L^p(X) & x \xrightarrow{\pi} y \xrightarrow{f} \mathbb{R} & X & L^p(X) & k^*f: X \rightarrow \mathbb{R} \mid k^*f(x) := \int_Y f(y) k(dy|x) \\
 \pi \downarrow & \uparrow \pi & \uparrow & \downarrow k & \uparrow k^* & \uparrow \\
 Y & L^p(Y) & y \xrightarrow{f} \mathbb{R} & Y & L^p(Y) & f: Y \rightarrow \mathbb{R}
 \end{array}
 \end{array}$$

Therefore L^p is an atomic sheaf on Prob .

1. THE IDEA OF DESCENT
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3. SHEAVES IN PROBABILITY
4. STATISTICAL EXPERIMENTS

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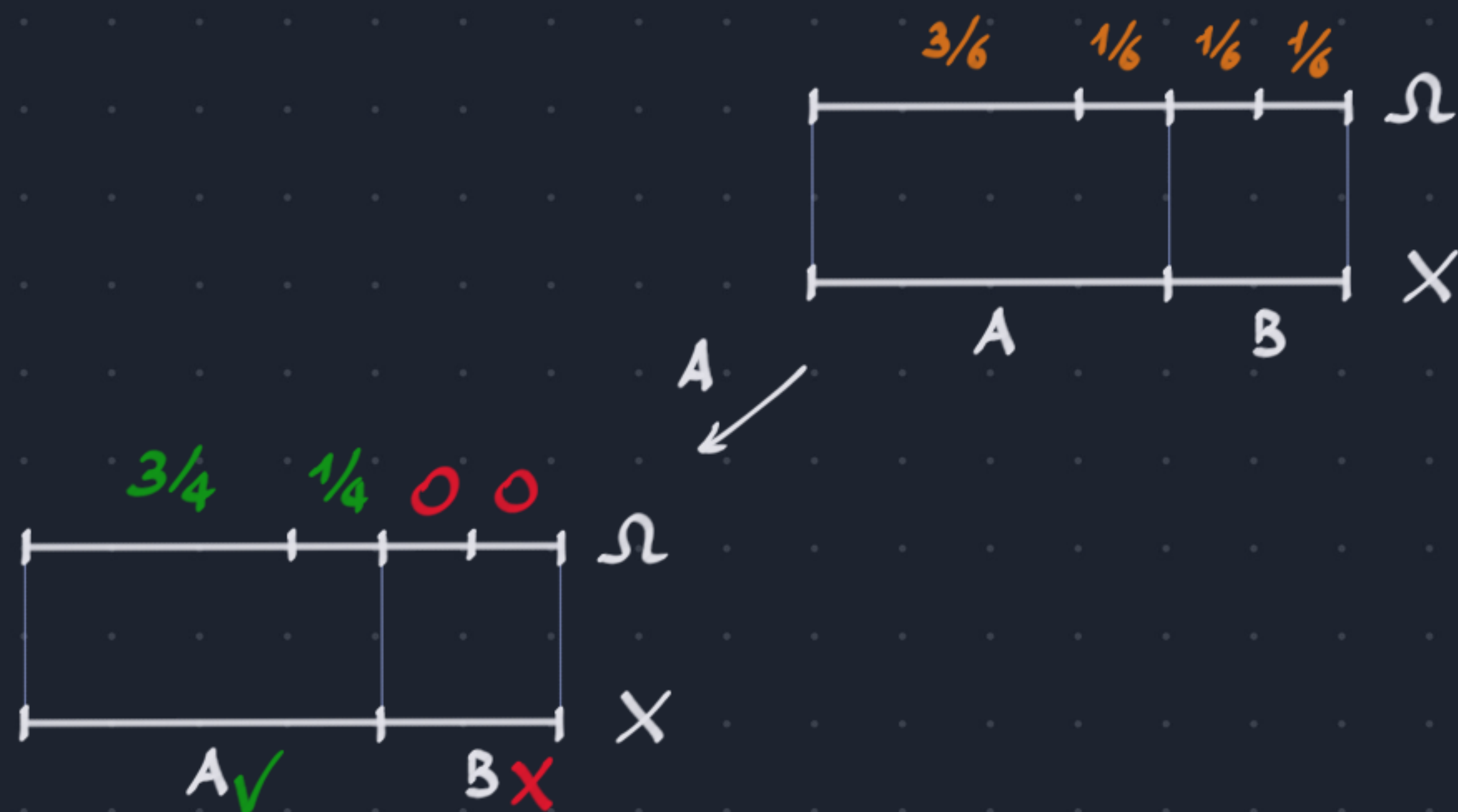


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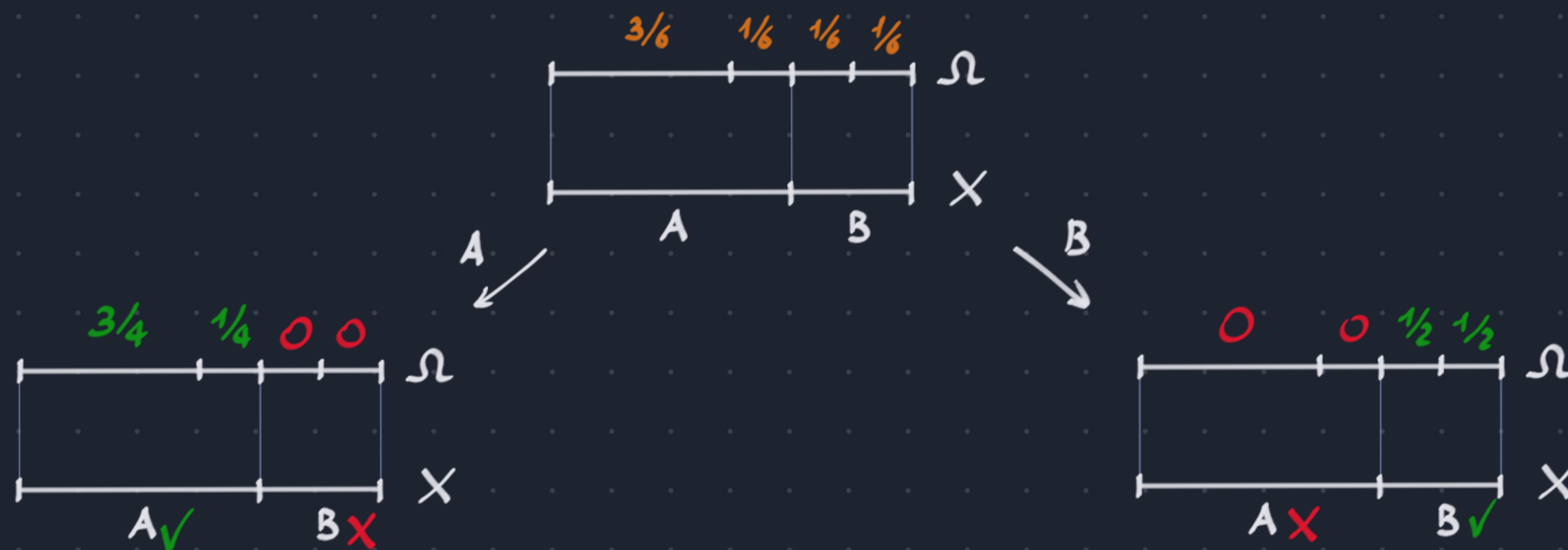


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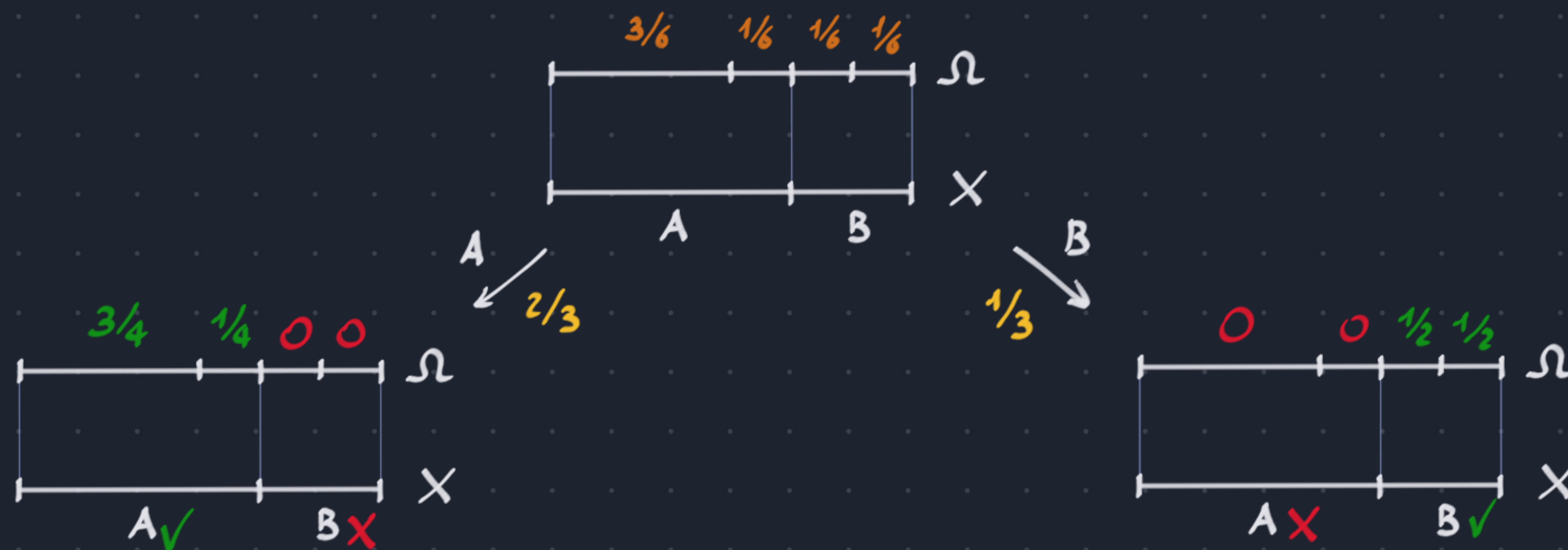


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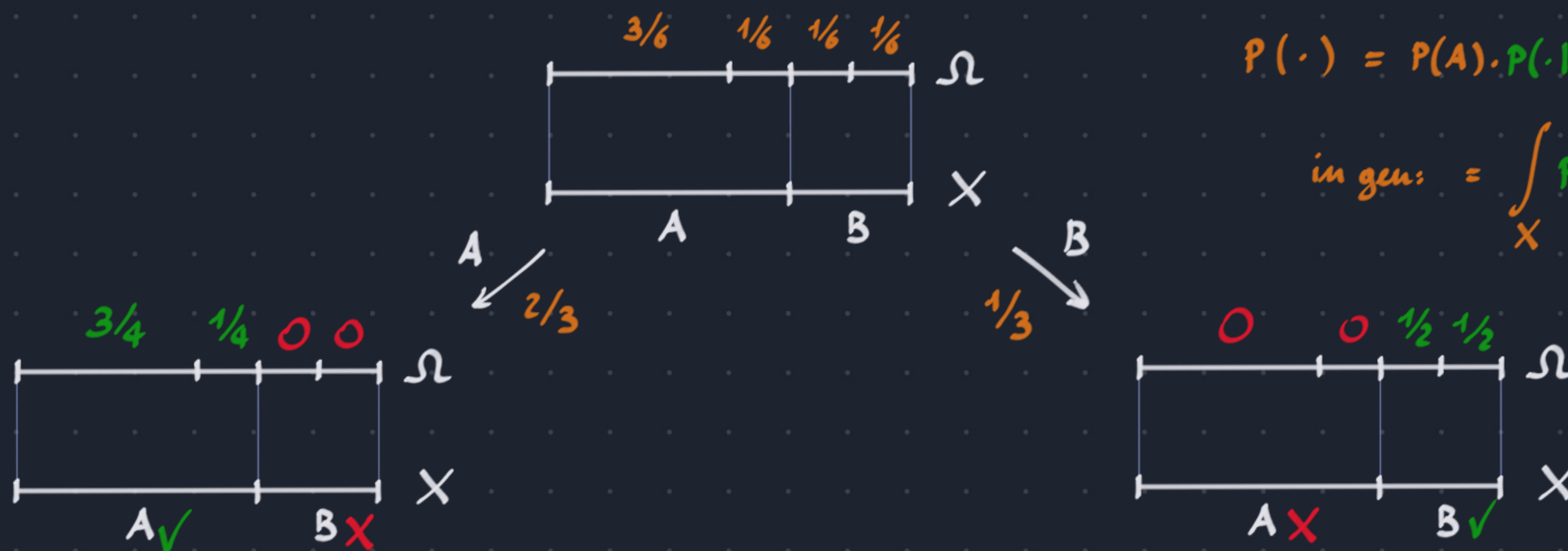


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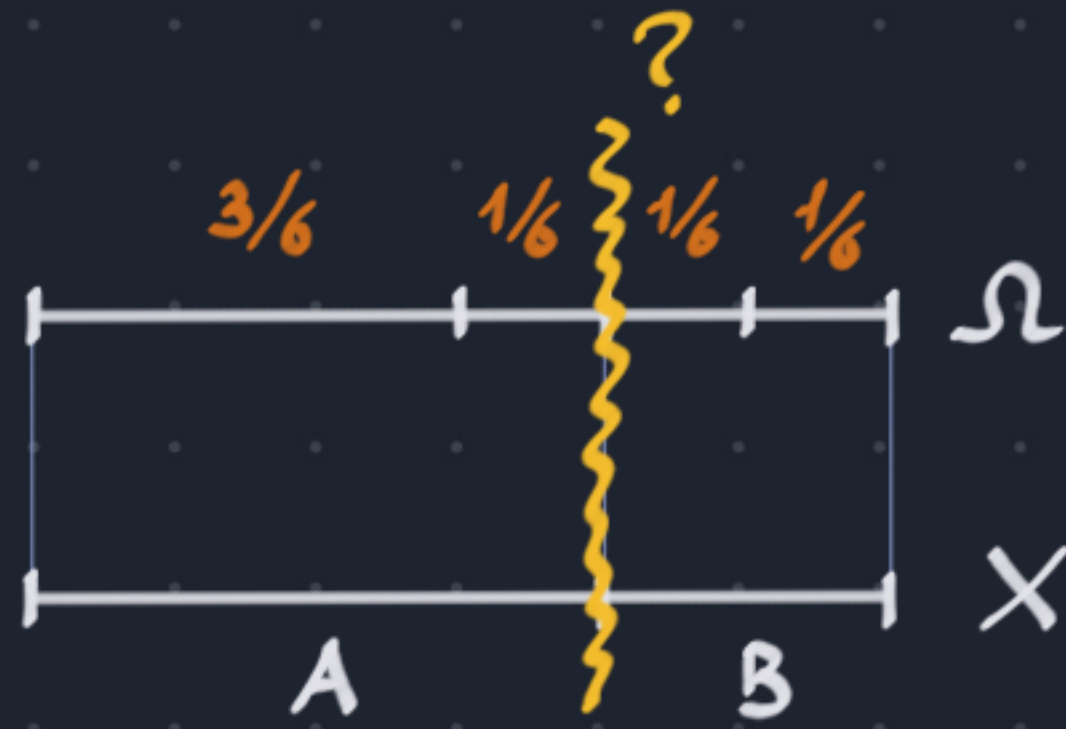
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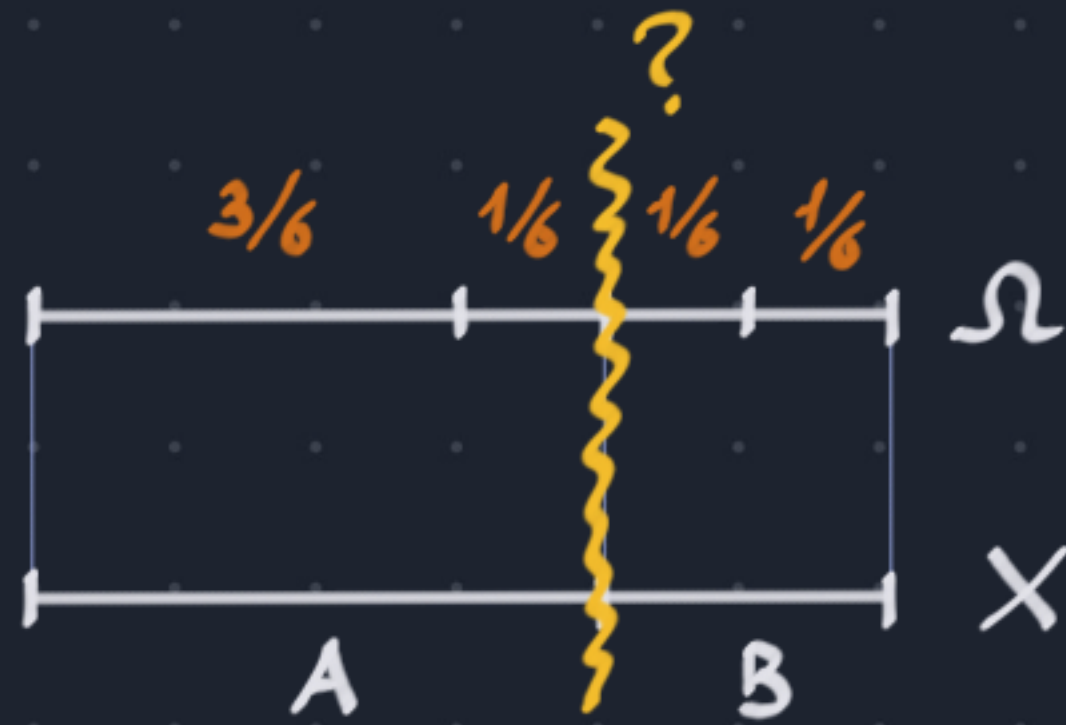
$$P(\cdot) = P(A) \cdot P(\cdot|A) + P(B) \cdot P(\cdot|B)$$

$$\text{in gen: } = \int_X P(\cdot|x) p(dx)$$

4. STATISTICAL EXPERIMENTS · Decomposition of measures



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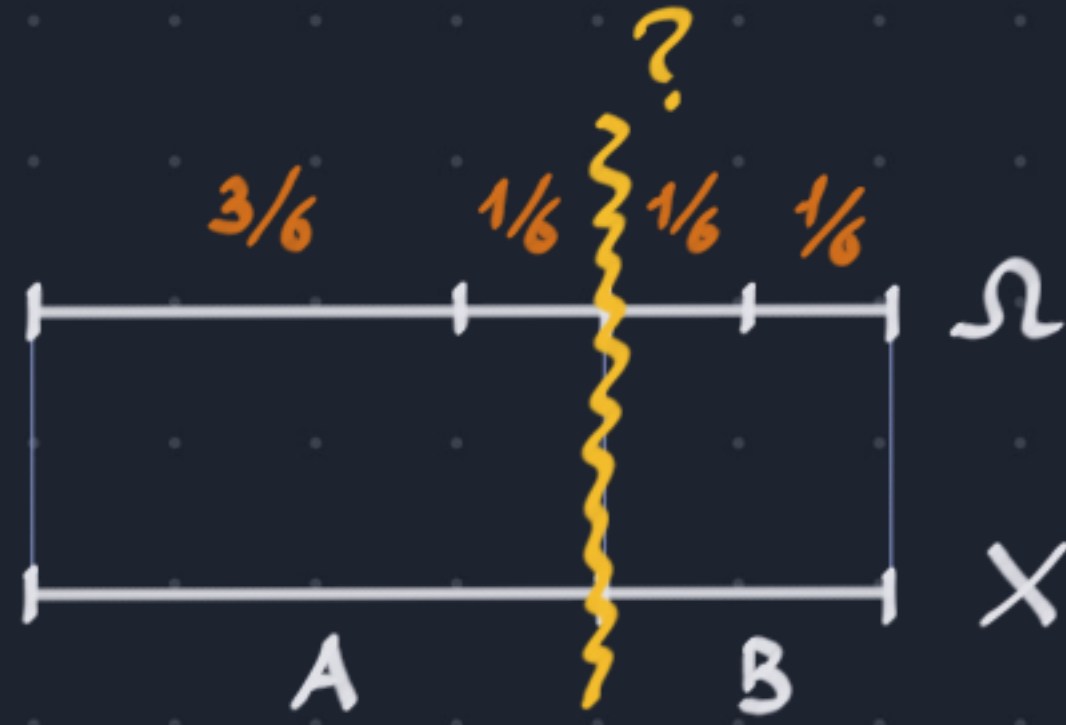


Idea. We use the theory of monads.

Example.

$$\begin{array}{c} 1 + 2 + 3 \\ \downarrow \text{evaluate} \\ 6 \end{array}$$

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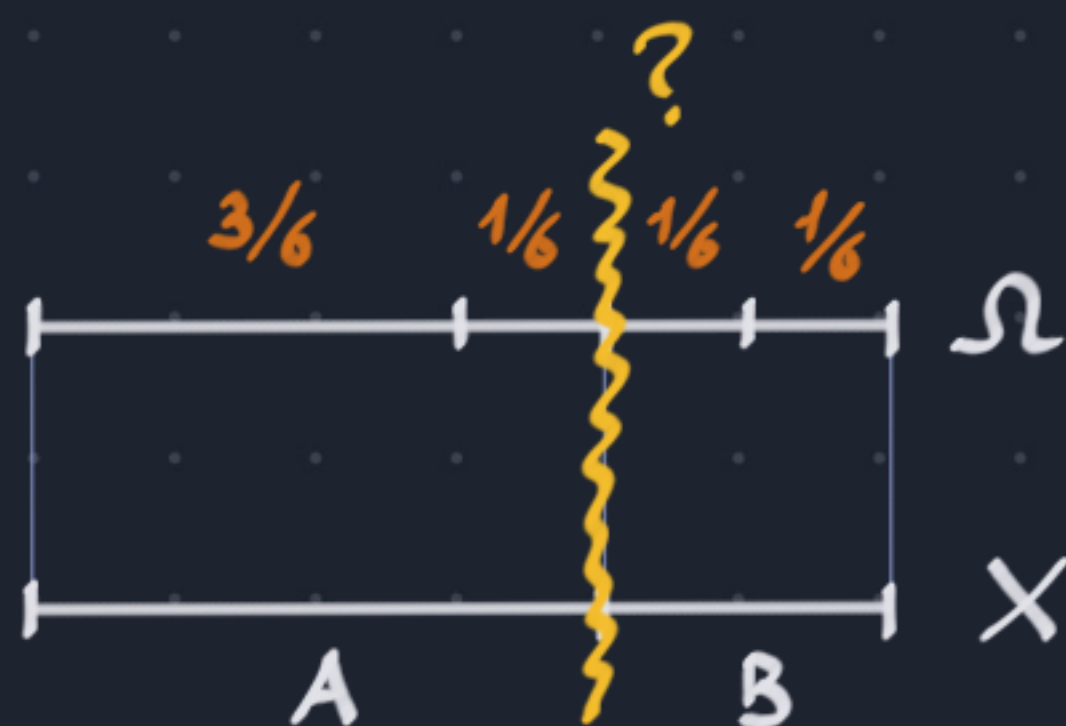
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For us: *giry monad*, free algebras.

$$\begin{array}{c} \mathbb{P}\mathbb{P}\mathbb{X} \\ \downarrow \mu \\ \mathbb{P}\mathbb{X} \end{array}$$

e.g. mixture models:

$$\text{bell curve} = \frac{1}{2} \text{bell curve} + \frac{1}{2} \text{bell curve}$$

4. STATISTICAL EXPERIMENTS · Descent condition

Definition. The functor $\text{Prob}^{\text{or}} \xrightarrow{\text{D}} \text{Set}$ is constructed as follows.

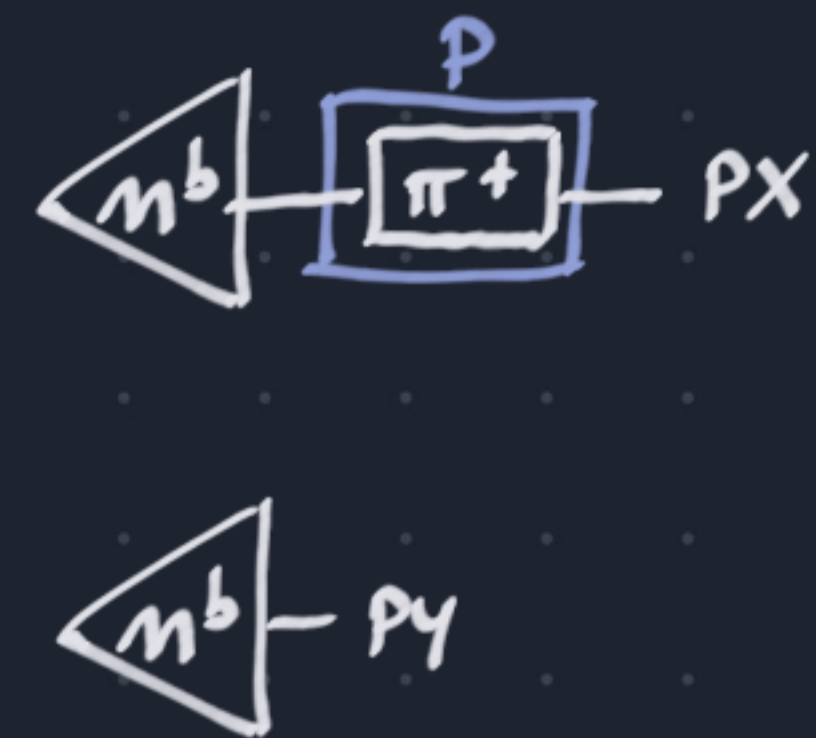
$$(X, \mathcal{A}, p) \mapsto \{m \in \text{PPX} \mid \mu(m) = p\}$$

$$\pi \downarrow$$

$$(Y, \mathcal{B}, q) \mapsto \{n \in \text{PPY} \mid \mu(n) = q\}$$

$$(\pi^+)_* m$$

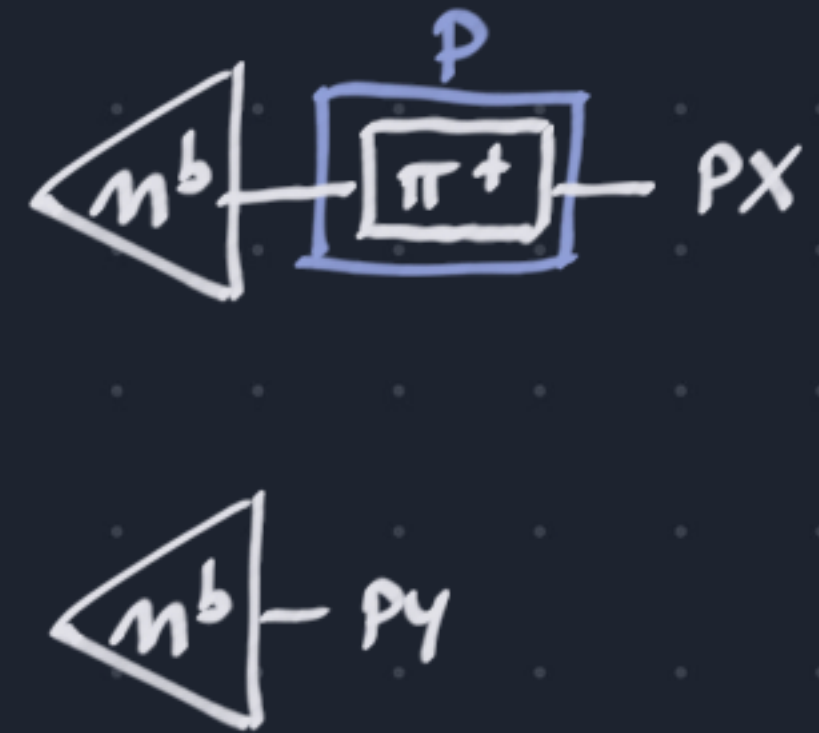
$$\uparrow n$$



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$$\begin{array}{ccc}
 (X, \mathcal{A}, p) & \mapsto & \{m \in \text{PPX} \mid \mu(m) = p\} \\
 \pi \downarrow & & \uparrow (\pi^+)_* \\
 (Y, \mathcal{B}, q) & \mapsto & \{n \in \text{PPY} \mid \mu(n) = q\}
 \end{array}$$



Theorem. Let $\Phi: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ be a presheaf on a dagger category with dilators.
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 (X, \mathcal{A}, p) \mapsto \{m \in \text{PPX} \mid \mu(m) = p\} & (\pi^+)_* m & \begin{array}{c} \text{p} \\ \triangleleft m^b \text{---} \boxed{\pi^+} \text{---} p_X \end{array} \\
 \pi \downarrow & \uparrow & \\
 (Y, \mathcal{B}, q) \mapsto \{n \in \text{PPY} \mid \mu(n) = q\} & n & \triangleleft m^b \text{---} p_Y
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Theorem. Let $\Phi: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ be a presheaf on a dagger category with dilators.
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Therefore, $\mathcal{D}$ is a sheaf.

Intuition: an experiment decomposes the probability according to its "resolution".

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References

- M. Di Meglio, C. Heunen, JS Lemay, P. Perrone, D. Stein, Dagger Categories of Relations. The equivalence of dilatory dagger categories and epi-regular independence categories. [arxiv: 2508.01146](#)
- M. Bohinen, P. Perrone, Categorical algebra of conditional probability. [arxiv: 2502.14941](#)
- D. Stein, Random Variables, Conditional Independence, and Categories of Abstract Sample Spaces. LICS 2025. [arxiv: 2503.02477](#)
- A. Simpson, Category-Theoretic Structure for Independence & Conditional Independence. ENTCS 2018.