Grothendieck homotopy theory and polynomial monads

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Grothendieck homotopy theory toolbox

1. Grothendieck construction and slice categories.
2. Quillen Theorem A.
3. Thomason theorem on homotopy colimits.
4. Aspherical functors.
5. Exact squares, smooth and proper functors.
6. Locally constant presheaves and Cisinski localisation.

GOAL.

Develop an extension of these fundamental constructions replacing Cat by $\mathbf{PolyMon}_f(\mathbf{Set})$ and presheaves categories by categories of algebras over finitary polynomial monads.

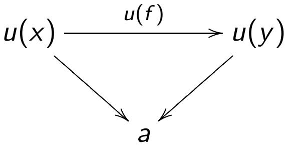
Grothendieck construction

Recollection

For a small category A and a functor $F : A \to \mathbf{Cat}$ its Grothendieck construction $\int F$ is a category, whose objects are pairs (a, x) where $x \in F(a)$ and whose morphisms are pairs $(f, \phi) : (a, x) \to (b, y)$ where $f : a \to b$ and $\phi : F(f)(x) \to y$. It comes with a projection $\int F \to A$ and this correspondence is completed to a 2-functor

$$\int : [A, \mathbf{Cat}] \to \mathbf{Cat}/A.$$

It has a left 2-adjoint given by slicing. For a functor $u : B \to A$ it associates a presheaf of categories on A given by $a \mapsto u/a$. The slice category u/a has arrows $u(x) \to a$ as objects and commutative triangles as morphisms:



Grothendieck construction for polynomial monads

Let T be a finitary polynomial monad

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I$$

and

$$F \in Alg_T(\mathbf{Cat}).$$

The polynomial Grothendieck construction $\int F$ has its set of objects the set of pairs (i, a) where $a \in F(i)$. An operation consists of:

1. An element $b \in B$;
2. For each element $e \in p^{-1}(b)$ an object $a_e \in F(s(e))$;
3. An object $y \in F(t(b))$;
4. A morphism $f_{(b,\sigma)} : m_{(b,\sigma)}(a_{\sigma(1)}, \ldots, a_{\sigma(k)}) \to y$ in $F(t(b))$ for each bijection $\sigma : \{1, \ldots, k\} \to p^{-1}(b)$.

This Grothendieck construction comes with a cartesian morphism of polynomial monads

$$p : \int F \to T.$$

Grothendieck construction for polynomial monads

Theorem (B – De Leger)

The Grothendieck construction is a 2-functor:

$$\int(-) : Alg_T(\mathbf{Cat}) \to \mathbf{PolyMon}/T.$$

which has a left 2-adjoint:

$$T^{(-)} : \mathbf{PolyMon}/T \to Alg_T(\mathbf{Cat}).$$

Example. Let M be the free monoid monad. And $F = (F, \otimes, I)$ be a strict monoidal category (that is an algebra of M in **Cat**). Then $\int F$ has the same objects as F. Multimorphisms are defined as follows:

$$F(a_1 \otimes \ldots \otimes a_n; a).$$

If $T \to M$ is a polynomial monad over M (that is a nonsymmetric operad) then M^T is the strict monoidal category associated to T.

Internal algebras classifiers

Definition

The value $T^S \in Alg_T(\mathbf{Cat})$ on a cartesian morphism $\phi : S \to T$ is called the classifier of internal S-algebras inside the categorical T-algebras.

A cartesian morphism $\phi : S \to T$ of polynomial monads induces a restriction 2-finctor $\Phi^* : Alg_T(\mathbf{Cat}) \to Alg_S(\mathbf{Cat})$.

Definition

For a categorical T-algebra A the category $Int_S(A)$ of internal S-algebras in A is the category of lax-morphisms of S-algebras

$$1 \to \Phi^*(A).$$

Example. Let $id : M \to M$ be the identity functor for free monoid monad and let $A \in Alg_M(\mathbf{Cat})$ be a strict monoidal category. Then an internal M-algebras in A

$$1 \xrightarrow{\text{lax-monoidal}} A$$

is the same thing as a monoid in A. and $Int_M(A) = Mon(A)$.

Internal algebras classifiers

Classifiers as representing objects.

Theorem (B)

An internal algebra classifier T^S of the monad morphism $\phi : S \to T$ is the representing object for the 2-functor

$$Int_S : Alg_T(\mathbf{Cat}) \to \mathbf{Cat}.$$

Scketch of a proof for $\phi = id$. Let $A \in Alg_T(\mathbf{Cat})$. One observe that the category of internal T-algebras in A is isomorphic to the category of sections of the Grothendieck construction $\int A \to T$. That is

$$Int_T(A) \cong \mathbf{PolyMon}/T(T, \int A) \cong Alg_T(\mathbf{Cat})(T^T, A).$$

Internal algebras classifiers

Classifiers as codescent objects.

Theorem (B)

- *The classifier T^T is the codescent object of a truncated simplicial categorical T-algebras:*

$$T1 \;\begin{array}{c} \xleftarrow{\;\;\mu_1\;\;} \\ \xrightarrow{\;\;T\eta_1\;\;} \\ \xleftarrow[\;\;T\tau\;\;]{} \end{array}\; T(T1) \;\begin{array}{c} \xleftarrow{\;\;T\mu_1\;\;} \\ \xleftarrow{\;\;\mu_{T1}\;\;} \\ \xleftarrow[\;\;T^2\tau\;\;]{} \end{array}\; T(T^21)$$

- *More generally the classifier T^S of the monad morphism $\phi : S \to T$ is the codescent object of a truncated simplicial categorical T-algebras:*

$$T(\phi_!(1)) \;\begin{array}{c} \xleftarrow{\;\;\mu_1\;\;} \\ \xrightarrow{\;\;T\eta_1\;\;} \\ \xleftarrow[\;\;T!\;\;]{} \end{array}\; T(\phi_!(S1)) \;\begin{array}{c} \xleftarrow{\;\;T\mu_1\;\;} \\ \xleftarrow{\;\;\mu_{T1}\;\;} \\ \xleftarrow[\;\;T^2!\;\;]{} \end{array}\; T(\phi_!(S^21))$$

Internal algebras classifiers

Examples

Functors between small categories. Let $u : S \to T$ be a map of linear polynomial monads that is a functor between small categories. $Alg_T(\mathbf{Cat}) = [T, \mathbf{Cat}]$. Then the categorical T-algebra T^S is a presheaf of slice categories:

$$t \mapsto u/t.$$

Monoids. Let M be the free monoid monad. A M-algebra in **Cat** is a (strict) monoidal category A. The category of internal M-algebras in A is the category of monoids $Mon(A)$ in A. The classifier M^M is the category of all finite ordinals Δ_+. The universal property means that

$$Int_M(A) = Mon(A) = MonCat_{\mathrm{strict}}(\Delta_+, A).$$

Internal algebras classifiers

Examples

Pointed sets and monoids Let $Id_\bullet$ be the monad for pointed sets. The classifier $Id_\bullet^{Id_\bullet}$ is the pointed arrow category $0 \to \mathbf{1}$.

There is a canonical morphism $Id_\bullet \to M$ of polynomial monads. The classifier $M^{Id_\bullet}$ is the monoidal subcategory $\Delta_+^{inj} \hookrightarrow \Delta_+$ of injective maps. It classifies pointed objects in a monoidal category.

Nonsymmetric operads. Let NOp be the polynomial monad for nonsymmetric operads. Then NOp^{NOp} is a nonsymmetric operad in Cat whose objects in degree n are planar rooted trees with n leaves. The morphisms are generated by contractions of internal edges and introduction of a new vertex of valency 2.

Symmetric operads. Let SOp be the polynomial monad for symmetric operads. The classifier SOp^{SOp} is the categorical operad of all rooted trees.

Classifiers from homotopy theory point of view

For any polynomial monad T the category of simplicial T-algebras (that is $Alg_T(SSet)$) has a projective model structure transferred from the category of collections $SSet/I$ along the forgetful functor $Alg_T(SSet) \to SSet/I$.

Theorem (B – Berger)

Let $\phi : S \to T$ be a cartesian map of polynomial monads.

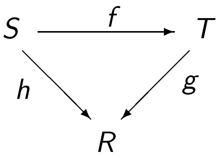
1. *The simplicial S-algebra $N(S^S)$ is a cofibrant replacement of the terminal S-algebra* 1.
2. *The left adjoint*
$$\phi_! : Alg_S(SSet) \to Alg_T(SSet)$$
to the restriction $\phi^* : Alg_T(SSet) \to Alg_S(SSet)$ is left Quillen and
$$\mathbb{L}\phi_!(1) \sim N(T^S).$$

Classical Quillen Theorem A

Recall that a functor between small categories is a Thomason equivalence if it induces a weak equivalence between nerves of categories.

Theorem (Quillen Theorem A)

If in a commutative triangle in **Cat**

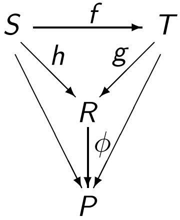


f induces a Thomason equivalence $h/r \to g/r$ for any object $r \in R$ then $f : S \to T$ is a Thomason equivalence.

Quillen Theorem A for polynomial monads

Theorem (B – De Leger)

For a commutative tetrtahedron of cartesian morphisms of polynomial monads



if $R^f : R^S \to R^T$ *is a pointwise Thomason equivalence then* $P^f : P^S \to P^T$ *is a pointwise Thomason equivalence.*

Remark. Classical Quillen Theorem A can be obtained if we put $\phi : R \to 1$, where 1 is the terminal category.

Thomason theorem I

Thomason theorem (1979)

For a presheaf $F : A \to \mathbf{Cat}$ there is a natural weak equivalence:

$$N(\int F) \to \mathrm{hocolim}_A N(F),$$

where $N(F)$ is a simplicial presheaf $N(F)(a) = N(F(a))$.

Polynomial version. Let $F \in Alg_A(\mathbf{Cat})$, for a polynomial monad A and let $\phi : A \to B$ be a cartesian polynomial monad morphism. Form a composite $\int F \to A \to B$.

Theorem (B – De Leger)

There is a weak equivalence of simplicial B-algebras:

$$N(B^{\int F}) \to \mathbb{L}\phi_!(N(F)).$$

Twisted Boardman-Vogt Tensor product

Let $F : A \to \mathbf{PolyMon}$ be a presheaf of polynomial monads on a small category A. Then there is a second version of Grothendieck construction

$$\oint F$$

which we call the twisted Boardman-Vogt tensor product. This is the lax-colimit of F in the 2-category **PolyMon** which can be described very explicitly.

Example 1. If F takes values in **Cat** the polynomial monad $\oint F = \int F$ is the classical Grothendieck construction.

Example 2. If F is a constant functor with $F(a) = D$, then

$$\oint F = A \otimes_{BV} D,$$

where the right hand side is the Boardman-Vogt tensor product of A and D as symmetric operads.

Thomason theorem II

Let $F : A \to \mathbf{PolyMon}$ be a presheaf of polynomial monads over a polynomial monad D. It means that for each a we have a morphism of polynomial monads: $F(a) \to D$ but also an induced morphism of polynomial monads $\oint F \to D$.

Theorem (B – De Leger)

There is a natural weak equivalence of simplicial D-algebras:

$$N(D^{\oint F}) \to \mathrm{hocolim}_A N(D^{F(a)}).$$

Aspherical morphisms of polynomial monads

Definition

A cartesian map $f : S \to T$ between polynomial monads is called $\mathcal{W}_\infty$-aspherical if T^S is $\mathcal{W}_\infty$-aspherical, that is $T^S \to 1$ is a pointwise Thomason equivalence.

Theorem (B – De Leger)

A cartesian map between polynomial monads $f : S \to T$ is $\mathcal{W}_\infty$-aspherical if and only if for any simplicial T-algebra X it induces a weak equivalence of derived mapping spaces:

$$Map_{Alg_S}(1, f^*(X)) \to Map_{Alg_T}(1, X).$$

Here 1 means the terminal algebra in the corresponding category of algebras and Map_{Alg} is the derived mapping space in the projective model structure on simplicial algebras.

Aspherical morphisms of polynomial monads

Examples.

1. A functor between small categories is $\mathcal{W}_\infty$-aspherical if it is left homotopically cofinal in the sense of Hirschhorn. The theorem above is a generalisation of Hirschhorn's theorem stating that a functor is left homotopy cofinal if and only if the restriction along it preserves homotopy limits.
2. Let NOp_{**} be a polynomial monad whose algebras are double multiplicative nonsymmeteric operads, i.e. nonsymmeteic operad X equipped with two maps of operads $r, l : Ass \to X$. Let $Bimod_\bullet$ be a polynomial monad whose algebras are Ass-bimodules with a distinguished point in the degree 1 space.

Theorem (B–De Leger)

There are $\mathcal{W}_\infty$-aspherical morphisms of polynomial monads

$$f : Bimod_{\bullet_1} \to NOp_{**} \quad and \quad g : InBimod_{\bullet_0} \to Bimod_{**}$$

Aspherical diagrams of polynomial monads

Theorem (B – De Leger)

Let $F : A \to \mathbf{PolyMon}/D$ be a presheaf of polynomial monads over a polynomial monad D. The following conditions are equivalent

1. *$\oint F \to D$ is a $\mathcal{W}_\infty$-aspherical map of polynomial monads;*
2. *For any map of polynomial monads $D \to R$ a natural map*

$$\mathrm{hocolim}_A^{\mathcal{W}} N(R^{F(a)}) \to N(R^D)$$

is a weak equivalence of simplicial R-algebras.

Definition

A diagram $F : A \to \mathbf{PolyMon}/D$ is called $\mathcal{W}_\infty$-aspherical if it satisfies the above equivalent conditions.

Aspherical squares of polynomial monads

A commutative square of polynomial monads

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle v}\downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is called $\mathcal{W}_\infty$-aspherical if it represents a $\mathcal{W}_\infty$-aspherical diagram.

Example. Let a commutative square above be a diagram in **Cat**. Then it is $\mathcal{W}_\infty$-aspherical if and only if the square of simplicial sets

$$\begin{array}{ccc} N(A) & \xrightarrow{u} & N(B) \\ {\scriptstyle v}\downarrow & & \downarrow \\ N(C) & \longrightarrow & N(D) \end{array}$$

is a homotopy pushout.

Delooping of mapping spaces between pointed algebras

Let P be a polynomial monad. Let P_* be a monad for pointed P-algebras that is $1/Alg_P \cong Alg_{P_*}$. Here 1 is the terminal P-algebra. There is a map of monads $u : P \to P_*$ such that the restriction functor $u^* : Alg_{P_*} \to Alg_P$ 'forgets the point'. Let P_{**} be the category of double pointed algebras, that is the category $1 \coprod 1/Alg_P$. We have a pushout of monads

$$\begin{array}{ccc} P & \xrightarrow{u} & P_* \\ {\scriptstyle u}\downarrow & & \downarrow \\ P_* & \longrightarrow & P_{**} \end{array}$$

The identity $id : P_* \to P_*$ induces a map of monads $U : P_{**} \to P_*$.

Delooping of mapping spaces between pointed algebras

Theorem (B – De Leger)

Let P be a polynomial monad such that P_* and P_{**} are also polynomial monads and the square from the previous slide is $\mathcal{W}_\infty$-aspherical. Then for a pointed simplicial P-algebra X there is a weak equivalence of simplicial sets:

$$\Omega Map_{Alg_P}(1, u^*X) \sim Map_{Alg_{P_{**}}}(1, U^*X)$$

where $\Omega Map_{Alg_P}(1, u^*X)$ is the loop space with the base point given by the point $1 \to X$ in the P-algebra X.

Delooping of mapping spaces between pointed algebras

Examples of monads satisfying delooping theorem.

1. Monad NOp for nonsymmetric operads.
2. Monad $LMod_{\mathcal{O}}$ ($RMod_{\mathcal{O}}$) for left (right) modules over a nonsymmetric operad $\mathcal{O}$ (in **Set**).
3. Monad $Bimod_{\mathcal{O}}$ of bimodules over a nonsymmetric operad $\mathcal{O}$ (in **Set**).
4. The Baez-Dolan plus-construction P^+ for any polynomial monad P.
5. Monads for left, right and bimodules over P^+.
6. Polynomial monads over P^+.

Remark

The monad $NOp = M^+$ *and* $M = Id^+$.

Delooping of space of long knots

B–DL delooping theorem together the $\mathcal{W}_\infty$-asphericity of the morphisms $f : Bimod_{\bullet_1} \to NOp_{**}$ and $g : InBimod_{\bullet_0} \to Bimod_{**}$ (and a result of Sinha about a totalisation of Kontsevich operad) immediately imply the following spectacular theorem for the space of 'long knots' conjectured by Kontsevich and proved independently by Dwyer-Hess and Turchin.

Let us denote $\overline{Emb}(\mathbb{R}^1, \mathbb{R}^n)$ the homotopy fiber of the map

$$Emb(\mathbb{R}^1, \mathbb{R}^n) \to Imm(\mathbb{R}^1, \mathbb{R}^n).$$

Theorem (Dwyer – Hess, Turchin)

For $n > 3$ there is a weak equivalence of spaces

$$\overline{Emb}(\mathbb{R}^1, \mathbb{R}^n) \sim \Omega^2 Map_{\mathrm{SOp}}(\mathcal{D}_1, \mathcal{D}_n),$$

where $\mathcal{D}_n$ is the little n-disks operad and the mapping space is taken in the category of symmetric operads.

$\mathcal{W}$-Locally constant presheaves

Let $\mathcal{W}$ be a Grothendieck fundamental localiser. A small category A is called $\mathcal{W}$-aspherical if $A \to 1$ is belongs to $\mathcal{W}$. Let A be a small category and let $\mathbb{V}$ be a model category. Let $Ho[A, \mathbb{V}]$ be the localisation of the category of covariant presheaves $[A, \mathbb{V}]$ with respect to levelwise weak equivalences.

Definition (Cisinski)

A presheaf $F : A \to \mathbb{V}$, is called $\mathcal{W}$-locally constant if for any $\mathcal{W}$-aspherical small category A' and any functor $u : A' \to A$ the presheaf $u^*(F) : A' \to \mathbb{V}$ is isomorphic to a constant presheaf in $Ho[A', \mathbb{V}]$

Example. A presheaf $F : A \to \mathbb{V}$ is $\mathcal{W}_\infty$-locally constant if and only if for any $f : a \to b$ in A the value $F(f)$ is a weak equivalence in $\mathbb{V}$.

Cisinski localisation

Theorem (Cisinski, B– White)

Let $\mathcal{W}$ be a proper fundamental localiser and $\mathbb{V}$ a combinatorial model category. Then:

1. *For $A \in$ **Cat** there exists a left Bousfield localisation of the projective model structure $[A, \mathbb{V}]^{\mathcal{W}}_{proj}$ such that its local objects are levelwise fibrant and $\mathcal{W}$-locally constant presheaves.*
2. *For a $\mathcal{W}$-weak equivalence $u : A \to B$ between small categories, the restriction functor*

$$u^* : [B, \mathbb{V}]^{\mathcal{W}}_{proj} \to [A, \mathbb{V}]^{\mathcal{W}}_{proj}$$

is a right Quillen equivalence.

Remark. Last statement is even if and only if for $\mathcal{W} = \mathcal{W}_\infty$ (Cisinski).

$\mathcal{W}$-locally constant algebras

Let P be a polynomial monad equipped with an identity on objects morphism $\eta : A \to P$, where A is a small category.

Definition

A P-algebra X is called $\mathcal{W}$-locally constant if its underlying presheaf $\eta^*(X) : A \to \mathbb{V}$ is a $\mathcal{W}$-locally constant presheaf.

Examples. 1. Let P be a small category, $\eta : A \to P$ is its subcategory. Then $\mathcal{W}_\infty$-locally constant P-algebras are covariant presheaves $F : P \to \mathbb{V}$ such $\eta^* F(f)$ is a weak equivalence for each $f \in A$.

2. $\mathcal{W}_\infty$-locally constant n-operads are higher braided operads. Here we consider an inclusion $Q_n^{op} \to Op_n$, where Q_n is the category of n-ordinals and their quasibijections and Op_n is the polynomial monad for n-operads. The nerve $N(Q_n^{op})$ has the homotopy type of the configuration space of unordered points in $\mathbb{R}^n$.

Localisation of algebras

Theorem (B – White)

Let $\mathcal{W}$ be a proper fundamental localiser then

1. *There exists a left (semimodel) Bousfield localisation of $Alg_P^{\mathcal{W}}(\mathbb{V})$ of the projective model structure whose fibrant objects are exactly fibrant $\mathcal{W}$-locally constant P-algebras.*
2. *This structure coincides with the transferred (semimodel) structure along the restriction functor $\eta^* : Alg_P(\mathbb{V}) \to [A, \mathbb{V}]_{proj}^{\mathcal{W}}$ if this transferred structure exists.*
3. *If $(f, g) : (P, A) \to (Q, B)$ is a Beck-Chevalley morphism of polynomial monads, such that (2) is satisfied and g is a $\mathcal{W}$-equivalence then $f_! : Alg_P^{\mathcal{W}}(\mathbb{V}) \to Alg_Q^{\mathcal{W}}(\mathbb{V})$ is a left Quillen equivalence.*

Stabilisation for locally constant n-operads

Theorem (B – White)

Let $\mathbb{V}$ be a combinatorial symmetric monoidal model category with the cofibrant tensor unit. For all $n \geq 3$ and $2 \leq k+1 \leq n < m \leq \infty$ the left Quillen functors

$$sym_n : \mathrm{Op}_n^{\mathcal{W}_k}(\mathbb{V}) \to \mathrm{SOp}(\mathbb{V}),\ \Sigma_! : \mathrm{Op}_n^{\mathcal{W}_k}(\mathbb{V}) \to \mathrm{Op}_m^{\mathcal{W}_k}(\mathbb{V})$$

are left Quillen equivalences. For $1 \leq k \leq \infty$ the functor

$$brd : \mathrm{Op}_2^{\mathcal{W}_k}(\mathbb{V}) \to \mathrm{BOp}(\mathbb{V})$$

is a left Quillen equivalence. Here $\mathrm{BOp}(\mathbb{V})$ is the model category of braided operads in $\mathbb{V}$.

Remark This stabilisation theorem is a consequence of the fact that the morphism of polynomial monads $Op_n \to SOp$ is $\mathcal{W}_{n-2}$-aspherical. Baez-Dolan stabilization hypothesis for higher categories and classical Freudenthal stabilisation Theorem follow from the above theorem immediately.

Further generalisation

Finatary polynomial monads in **Set** are equivalent to Σ-free operads.

How to extend this theory to all operads?

Answer. Use Mark Weber approach to operad! An operad in Weber's theory is a map $P \to Sm$ of polynomial monads in **Cat**, where Sm is the monad for strict symmetric monoidal categories.

There exists a corresponding Grothendieck construction with its left adjoint also called internal algebra classifier. For example, the classifier Sm^{Sm} is the symmetric monoidal category of finite sets **FinSet** (with coproduct as the tensor product).

Remark. More generally, the classifier Sm^P has a canonical **Feynman category** structure in the sense of Kaufman and Ward. Moreover, up to equivalence all Feynman categories are such classifiers (B – Kock – Weber).

Further generalisation

Even more interesting is to extend this theory to the analytic monads as developed by Gepner, Kock and Haugseng or higher operads and operadic categories.

THANK YOU!