

POLYNOMIALS IN CATEGORIES WITH PULLBACKS

1) The operational view / extensive view

Originally, a poly. functor $\underline{\text{Set}} \rightarrow \underline{\text{Set}}$ is one that's in the closure of id under $\times, +$; more generally, under \prod, Σ .

More generally, a multivariate poly functor $\text{Set}^I \rightarrow \text{Set}$ is one in the closure of the projection fns $\text{Set}^I \rightarrow \text{Set}$ under \prod, Σ .

① Even more generally, a poly functor $\text{Set}^I \rightarrow \text{Set}^J$ is a J -indexed family of poly functors $\text{Set}^I \rightarrow \text{Set}$.

There are many other views on poly functors:

② Functors $\text{Set}/_I \rightarrow \text{Set}/_J$ which are composites of:

$$\text{Set}/_I \xrightarrow[\Delta f \downarrow]{\Sigma f} \text{Set}/_J \quad f: J \rightarrow I$$

← Σf
← $\Delta f \downarrow$
← $\prod f$

$\Delta f =$ pullback along f

$\Sigma f =$ postcompose with f

$\prod f =$ "dependent product"

$$\prod f: \text{Set}^J \longrightarrow \text{Set}^I$$

$$(A_{ij} : i \in I, j \in J_i) \mapsto (\prod_{j \in J_i} A_{ij} : i \in I)$$

where think of $f: J \rightarrow I$ as giving family of sets $(J_i : i \in I)$ with $J_i = f^{-1}(i)$

③ Functors $\text{Set}/I \xrightarrow{F} \text{Set}/J$ of form $\text{Set}/I \xrightarrow{\Delta_t} \text{Set}/E \xrightarrow{\Pi_p} \text{Set}/B \xrightarrow{\Sigma_s} \text{Set}/J$

④ Functors $\text{Set}/I \xrightarrow{F} \text{Set}/J$ which preserve connected limits and are small.
(Girard's normal functors (88))

⑤ with a left multiadjoint (Diers '78)

⑥ whose slices F/x are right adjoints
(local right adjoint: Street '00)

What happens if I generalise these from Set to \mathcal{E} ?

Typically:

④ = ⑤ = ⑥ if, say, \mathcal{E} is locally presentable.

③ \subseteq ② : but note here that $\Pi f: \mathcal{E}/I \rightarrow \mathcal{E}/J$ may not exist for all f : but we can restrict to the class of exponentiable f ; by defn these are the f for which Πf exists.

① not very much to do with rest; except ① \subseteq ② if \mathcal{E} extensive.

Now ② \subseteq ④ and typically ② \subsetneq ④.

④ = ⑤ \neq ⑥ typically super interesting:

- \mathcal{E} = prestack caty, have pre. functors that David spoke of
- \mathcal{E} = variety, $\dots \rightarrow$ corings + plethories + friends (Toll-Wraith, Bergman-Hauskrocht, Joyal, Boyer, ...)
- $\mathcal{E} = \text{Sh}(X)$, X Stone space $\dots \rightarrow$ topological dynamics + non-comm. geometry.

However, (2) and (3) give a much more uniform theory, which is the theory of polynomial functors in \mathcal{E} .

Magical fact: in any caty \mathcal{E} with pbs, (2) and (3) coincide.
 Why? Things in (2) have normal forms in (3).

(Gambino, Koch 2013; Weber 2015)

Idea:

(1) If we have $\mathcal{E}/I \xrightarrow{\Sigma f} \mathcal{E}/J \xrightarrow{\Delta g} \mathcal{E}/K$,
 I can form pb to the right
 and now the 2-cell

$$\begin{array}{ccc} \mathcal{E}/J & \xrightarrow{\Delta g} & \mathcal{E}/K \\ \Delta f \downarrow \cong & & \downarrow \Delta u \\ \mathcal{E}/I & \xrightarrow{\Delta v} & \mathcal{E}/L \end{array} \quad \text{transposes to one} \quad \begin{array}{ccc} \mathcal{E}/I & \xrightarrow{\Delta v} & \mathcal{E}/L \\ \Sigma f \downarrow & & \downarrow \Sigma u \\ \mathcal{E}/J & \xrightarrow{\Delta g} & \mathcal{E}/K \end{array}$$

which turns out to be invertible (Beck-Chevalley).

So now

$$\begin{array}{ccc} \mathcal{E}/I & \xrightarrow{\Sigma f} & \mathcal{E}/J & \xrightarrow{\Delta g} & \mathcal{E}/K \\ & & \Downarrow & & \\ & & \mathcal{E}/L & \xrightarrow{\Sigma u} & \mathcal{E}/K \end{array}$$

(2) Similarly, if we have $\mathcal{E}/I \xrightarrow{\Pi f} \mathcal{E}/J \xrightarrow{\Delta g} \mathcal{E}/K$
 can form pullback and the canonical 2-cell

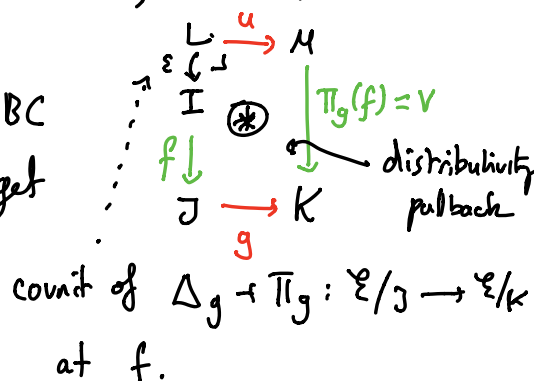
$$\begin{array}{ccc} \mathcal{E}/I & \xrightarrow{\Delta v} & \mathcal{E}/L \\ \Pi f \downarrow \Rightarrow & & \downarrow \Pi u \\ \mathcal{E}/J & \xrightarrow{\Delta g} & \mathcal{E}/K \end{array} \quad \text{is again invertible (Beck-Chevalley)} \\ \text{so can rewrite.}$$

$$\begin{array}{ccc} L & \xrightarrow{v} & I \\ \downarrow u & & \downarrow f \\ K & \xrightarrow{g} & J \end{array}$$

$$\begin{array}{ccc} L & \xrightarrow{v} & I \\ \downarrow u & & \downarrow f \\ K & \xrightarrow{g} & J \end{array}$$

③ If we have $\mathcal{E}/I \xrightarrow{\Sigma f} \mathcal{E}/J \xrightarrow{\Pi g} \mathcal{E}/K$, can form

By messing around with adjoints, using BC isom. for Π 's in this pb square, get a canonical 2-cell:



$$\begin{array}{ccc} \mathcal{E}/I & \xrightarrow{\Sigma f} & \mathcal{E}/J \\ \Delta \epsilon \downarrow & & \downarrow \Pi g \\ \mathcal{E}/L & \Rightarrow & \\ \Pi u \downarrow & & \\ \mathcal{E}/M & \xrightarrow{\Sigma v} & \mathcal{E}/K \end{array}$$

which turns out to be invertible.

In Set, this invertibility expresses the isom.

$$\prod_{a \in A} \sum_{b \in B_a} C_{ab} = \sum_{f \in \prod_{a \in A} B_a} \prod_{a \in A} C_{a, f(a)}$$

called type theoretic axiom of choice, or complete distributivity.

Using the invertible $(*)$, we can rewrite $\begin{array}{ccc} \xrightarrow{\text{green}} & \xrightarrow{\text{red}} & \\ & \xrightarrow{\text{red}} & \xrightarrow{\text{green}} \end{array}$ as

Using these three rewrites, we can turn any ② into a ③.

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REMARK So typically, have

$$\textcircled{2} = \textcircled{3} \subset \textcircled{4} = \textcircled{5} = \textcircled{6}$$

However, if we interpret $\textcircled{4} = \textcircled{5} \neq \textcircled{6}$ less naively in \mathcal{E} ,

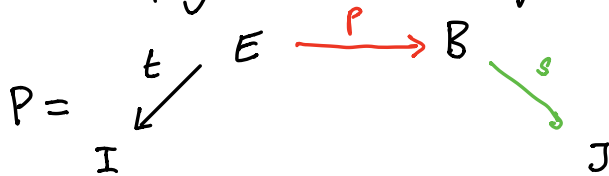
we get another equivalent formulation of $(2) \Leftrightarrow (3)$. Namely, if we look at indexed functors between indexed slice cats of \mathcal{E} , which in a suitable indexed sense are local right adjoint, then we get polynomial functors. (Kock and Kock 2013).

Talk 2

2) THE COMBINATORIAL VIEW (OR INTENSIVE VIEW)

The "normal form" of a poly functor $\mathcal{E}/I \rightarrow \mathcal{E}/J$ ($\rightarrow \rightarrow \rightarrow$) gives us a more compact way of viewing them.

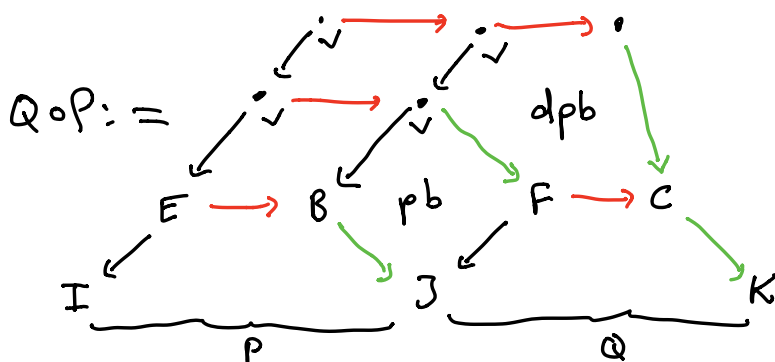
DEFN A polynomial ^{from I to J} in a caty w/ pullbacks \mathcal{E} is a diagram



The associated poly. functor of P is

$$F_P := \mathcal{E}/I \xrightarrow{\Delta \epsilon} \mathcal{E}/E \xrightarrow{\text{Tip } P} \mathcal{E}/B \xrightarrow{\Sigma s} \mathcal{E}/J.$$

To compose two polys, we compose the associated poly functor and take the normal form:



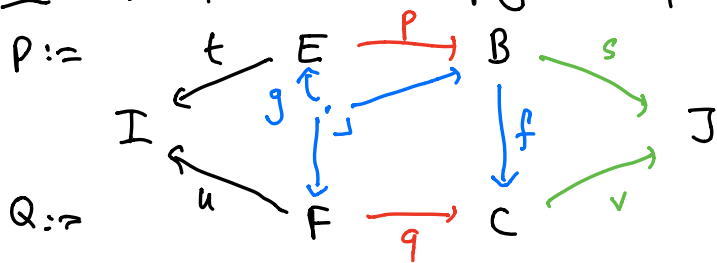
By construction, $F_{Q \circ P} \cong F_Q \circ F_P$.

3) MAPS OF POLYNOMIALS

We'd like to see polys from I to J as 1-cells in a bicategory Poly_2 . So what are 2-cells?

To motivate this, let's return to the Set case. If $F, G: \text{Set}^I \rightarrow \text{Set}^J$ are poly functors, then a map between them is just a nat xfun. Since poly functors in Set preserve cona. limits, they are pointwise coproducts of representables, and so we can use Yoneda to get a representation of such nat xfun in terms of the poly. normal form.

Prop If P, Q are set-polynomials from I to J :



Then to give $\alpha: F_P \Rightarrow F_Q$ is equally to give the blue maps f, g above.

Proof. We write $(B_j : j \in J)$ for family of fibres of s .

• We write $(E_{i,b} : i \in I, b \in B)$ for fibres of $(t, p): E \rightarrow I \times B$.

In these terms, $F_p: \text{Set}^I \rightarrow \text{Set}^J$

$$(X_i: i \in I) \mapsto \left(\sum_{b \in B_j} \prod_{i \in I} X_i^{E_{ib}} : j \in J \right)$$

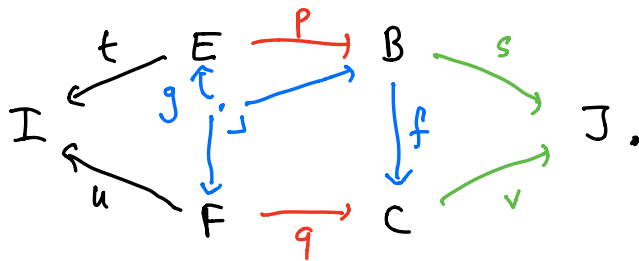
So now α is $\left(\sum_{b \in B_j} \prod_{i \in I} (-)^{E_{ib}} \xrightarrow{\alpha_j} \sum_{c \in C_j} \prod_{i \in I} (-)^{F_{ic}} : j \in J \right)$

rep. functor
 $\text{Set}^I \rightarrow \text{Set}$:
 rep^t by
 $(E_{ib}: i \in I)$

$$\left(\prod_{i \in I} (-)^{E_{ib}} \xrightarrow{\alpha_{jb}} \sum_{c \in C_j} \prod_{i \in I} (-)^{F_{ic}} : j \in J, b \in B_j \right)$$

$$\left(\alpha_{jb}(\lambda_i \cdot 1) =: \tilde{\alpha}_{jb} \in \sum_{c \in C_j} \prod_{i \in I} E_{ib}^{F_{ic}} \right)$$

If we write $\tilde{\alpha}_{jb}$ as $(f(b) \in C_j, (g_{ib}: F_{i, f(b)} \rightarrow E_{i,b})_{i \in I})$
 then we get



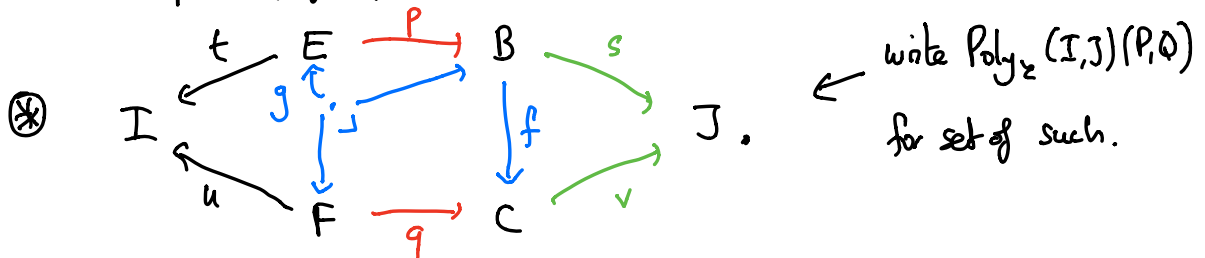
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What happens in an arbitrary caty \mathcal{E} with pullbacks?

The naive thing doesn't work: if we define a map between polys from I to J to be a nat transformation between P_f and P_g , we get nowhere. The reason is that the P_f 's are no longer precise sums of representables, so we can't apply Yoneda.

However... I said last time we can view poly functors $\mathcal{E}/I \rightarrow \mathcal{E}/J$ as indexed functors (over \mathcal{E}); now as indexed functors they are pointwise coprods of representables, and so the "same" argument applies. So what we have is:-

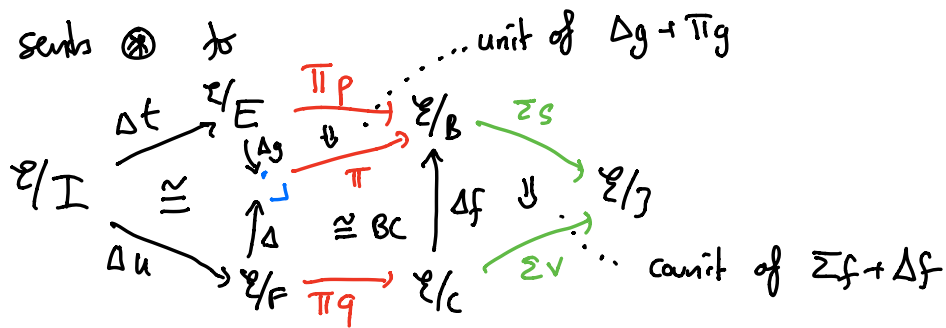
Defn A map of polys from I to J in \mathcal{E} is



Prop There's an assignment \dots (Abbott 2003, Gombro-Koch 2013)

$$\text{Poly}_{\mathcal{E}}(I, J)(P, Q) \longrightarrow \text{IdxNat}(\mathcal{E}/I, \mathcal{E}/J)(F_P, F_Q)$$

which sends ⊗ to



This is an isomorphism, and so we get a caty $\text{Poly}_{\mathcal{E}}(I, J)$ with a f.f. functor $\text{Poly}_{\mathcal{E}}(I, J) \longrightarrow \text{IdxNat}(\mathcal{E}/I, \mathcal{E}/J)$.

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So finally, we have:

Defn The 2-caty of polynomial functors in \mathcal{E} has:

- obs: those of \mathcal{E}
- 1-cells $I \rightarrow J$: poly functors $\mathcal{E}/I \rightarrow \mathcal{E}/J$
- 2-cells: indexed transformations $\alpha: F \Rightarrow G: \mathcal{E}/I \rightarrow \mathcal{E}/J$

The bicat of polynomials in \mathcal{E} has:

- obs those of \mathcal{E}
- 1-cells $I \rightarrow J$: polys $I \rightarrow J$
- 2-cells: maps of polynomials,

with remaining data forced by the requirement that there be a locally fully faithful homomorphism of bicategories

$$\begin{array}{ccc} \text{Poly}_{\mathcal{E}} & \longrightarrow & \text{Idx Cat}_{\mathcal{E}} \\ I & \longleftarrow & \mathcal{E}/I \end{array}$$

and given on homs by

$$\text{Poly}_{\mathcal{E}}(I, J) \longrightarrow \text{Idx Cat}_{\mathcal{E}}(\mathcal{E}/I, \mathcal{E}/J)$$

as in previous proposition.

Talk 3

4) UNIVERSAL PROPERTY OF $\mathcal{E} \mapsto \text{Poly}_{\mathcal{E}}$.

We'll assume in this section: \mathcal{E} is locally cartesian closed, i.e., all maps f are exponentiable (ie Πf exists).

Poly. functors involve $\Sigma + \Delta + \Pi$ satisfying B.C. and distributivity axioms. We can make sense of these adjunctions + coherence data in any bicat - motivating the following universal property.

Warm-up: Span \mathcal{E} .

Defn Let \mathcal{E} be a caty w/ pullbacks, \mathcal{K} a bicatg.

1) A Δ -functor $F: \mathcal{E}^{op} \rightarrow \mathcal{K}$ is just a pseudofunctor, with action written as:

$$X \longmapsto FX$$

$$f: X \rightarrow Y \longmapsto F_{\Delta}f: FY \rightarrow FX$$

2) A $\Sigma\Delta$ -functor $F: \mathcal{E}^{op} \rightarrow \mathcal{K}$ is a Δ -functor st:

- each $F_{\Delta}f$ has a left adjoint $F_{\Sigma}f: FX \rightarrow FY$ in \mathcal{K} .

- for each pb square $D \xrightarrow{v} C$ in \mathcal{E} , the canonical

$$\begin{array}{ccc} D & \xrightarrow{v} & C \\ \downarrow u & & \downarrow g \\ B & \xrightarrow{f} & A \end{array} \quad \text{BC-2-cell}$$

$$\begin{array}{ccc} FC & \xrightarrow{F_{\Delta}v} & FD \\ F_{\Sigma}g \downarrow & \Downarrow & \downarrow F_{\Sigma}g \\ FA & \xrightarrow{F_{\Delta}f} & FB \end{array} \quad \text{is invertible.}$$

Examples

a) A Δ -functor $\mathcal{E}^{op} \rightarrow \text{Cat}$ is an \mathcal{E} -indexed caty

• $\Sigma\Delta$ • • • • • with sums

In phic., there's a $\Sigma\Delta$ -functor $\mathcal{E}^{op} \rightarrow \text{Cat}$

$$\begin{array}{l} X \longmapsto \mathcal{E}/X \\ f \longmapsto F_{\Delta}f = \Delta f. \end{array}$$

b) There's a $\Sigma\Delta$ -functor $\eta: \mathcal{E}^{op} \rightarrow \text{Span}_{\mathcal{E}}$
 $X \mapsto X$

with $F_{\Delta}(f) = \begin{array}{ccc} & X & \\ f \swarrow & & \searrow \\ Y & & X \end{array}$, $F_{\Sigma}(f) = \begin{array}{ccc} & X & \\ & \swarrow & \searrow f \\ X & & Y \end{array}$

Thm (Dawson, Paré, Poiret 2004) $\eta: \mathcal{E}^{op} \rightarrow \text{Span}_{\mathcal{E}}$ is the universal $\Sigma\Delta$ -functor out of \mathcal{E} . ie, for any bicat \mathcal{K} , composition with η induces a biequivalence

$$\text{Hom}(\text{Span}_{\mathcal{E}}, \mathcal{K}) \longrightarrow \Sigma\Delta\text{-Funct}(\mathcal{E}^{op}, \mathcal{K}).$$

Now let's do $\text{Poly}_{\mathcal{E}}$! As mentioned above, let's take \mathcal{E} lccc.

Defn • A $\Delta\Pi$ -functor $F: \mathcal{E}^{op} \rightarrow \mathcal{K}$ is a Δ -functor st. each $F_{\Delta}f$ has a right adjoint $F_{\Pi}f$ such that the canonical bic. 2-cell associated to any pb square in \mathcal{E} is invertible.

• A $\Sigma\Delta\Pi$ -functor $F: \mathcal{E}^{op} \rightarrow \mathcal{K}$ is a Δ -functor which is both a $\Sigma\Delta$ -functor and a $\Delta\Pi$ -functor, and such that, for any distributivity pullback

$$\begin{array}{ccc} E & \xrightarrow{u} & D \\ \varepsilon \downarrow & \lrcorner & \downarrow \Pi_g f = v \\ A & & C \\ f \downarrow & & \downarrow g \\ B & & C \end{array}$$

the can. 2-cell
$$\begin{array}{ccc} FE & \xrightarrow{F_{\Pi}u} & FD \\ F_{\Delta}\varepsilon \uparrow & \Uparrow & \downarrow F_{\Sigma}v \\ FA & \Uparrow & \\ F_{\Sigma}f \downarrow & & \downarrow F_{\Sigma}g \\ FB & \xrightarrow{F_{\Delta}g} & FC \end{array}$$
 is invertible.

Examples

a) A $\Delta\Pi$ -functor $\mathcal{E}^{\text{op}} \rightarrow \text{Cat}$ is an indexed category with products

$\cdot \Sigma\Delta\Pi \dots \dots \dots$ sums, products, distributivity

For example: have $F: \mathcal{E}^{\text{op}} \longrightarrow \text{Cat}$
 $X \longmapsto \mathcal{E}/X$

with $F_{\Delta}f = \Delta f$, $F_{\Pi}f = \Pi f$, $F_{\Sigma}f = \Sigma f$.

For example: let \mathcal{C} be any locally small category.

Have a Δ -functor $F: \text{Set}^{\text{op}} \longrightarrow \text{CAT}$
 $I \longmapsto \mathcal{C}^I$

- This is:
- a $\Sigma\Delta$ functor if \mathcal{C} has small coproducts
 - a $\Delta\Pi$ functor if \mathcal{C} has small products
 - a $\Sigma\Delta\Pi$ functor if \mathcal{C} has small products, small coproducts and infinite distributivity:

$$\prod_{j \in J} \sum_{k \in K_j} X_{jk} \cong \sum_{f: \prod_{j \in J} K_j} \prod_{j \in J} X_{j(fj)}.$$

For example: let \mathcal{E} be a topos. Have a Δ -functor

$F: \mathcal{E}^{\text{op}} \longrightarrow \text{Poset}$
 $X \longmapsto \text{Sub}(X)$

This is always a $\Sigma\Delta$ - and a $\Delta\Pi$ -functor.

It's a $\Sigma\Delta\Pi$ -functor if we have in \mathcal{E} that

$$\forall j \in J. \exists k \in K_j. \mathcal{G}(j, k) \Rightarrow \exists f: \prod_{j \in J} K_j. \forall j \in J. \mathcal{G}(j, h).$$

b) There's a $\Sigma\Delta\Pi$ -functor $\eta: \mathcal{E}^{op} \rightarrow \text{Poly}_{\mathcal{E}}$ and with

$$F_{\Delta}(f) = \begin{array}{c} x = x \\ \swarrow \quad \searrow \\ y \quad \quad x \end{array} \quad F_{\Sigma}(f) = \begin{array}{c} x = x \\ \swarrow \quad \searrow \\ x \quad \quad y \end{array} \quad F_{\Pi}(f) = \begin{array}{c} x \rightarrow y \\ \swarrow \quad \searrow \\ x \quad \quad y \end{array}$$

THM (Walker, 2019) $\eta: \mathcal{E}^{op} \rightarrow \text{Poly}_{\mathcal{E}}$ is the universal $\Sigma\Delta\Pi$ -functor out of \mathcal{E} :

$$\text{Hom}(\text{Poly}_{\mathcal{E}}, K) \simeq \Sigma\Delta\Pi\text{-Funct}(\mathcal{E}^{op}, K).$$

In particular, the $\Sigma\Delta\Pi$ -functor $\mathcal{E}^{op} \rightarrow \text{Cat}$

$$x \longmapsto \mathcal{E}/x$$

induces the canonical homomorphism $\text{Poly}_{\mathcal{E}} \rightarrow \text{Cat}$.

$$x \longmapsto \mathcal{E}/x$$

5) THE KLEISLI VIEW

(after von Glehn)

Let's define $\text{Id}_x \text{Cat}(\mathcal{E}) := \Delta\text{-Funct}(\mathcal{E}^{op}, \text{Cat})$

$$\text{Id}_x \text{Cat}_{\Pi}(\mathcal{E}) := \Delta\Pi \dots \dots \dots$$

$$\text{Id}_x \text{Cat}_{\Sigma}(\mathcal{E}) := \Sigma\Delta \dots \dots \dots$$

We have $\text{Id}_x \text{Cat}_{\Sigma}(\mathcal{E}) \xrightarrow{\dashv} \text{Id}_x \text{Cat}(\mathcal{E}) \xleftarrow{\dashv} \text{Id}_x \text{Cat}_{\Pi}(\mathcal{E})$ psmonadic.

So writing T_Σ, T_Π for induced pseudomonads on $\text{IdxCat}(\mathcal{E})$,
have

$$\text{IdxCat}_\Sigma(\mathcal{E}) \simeq T_\Sigma\text{-alg} \quad \text{and same for } \Pi.$$

FACT: there's a ps-distributive law $T_\Pi T_\Sigma \Rightarrow T_\Sigma T_\Pi$,
and algs for composite pseudomonad are idxCats with
sums, products + distributivity. $\sim T_\Sigma T_\Pi$

Defn The Omy theory $\tilde{\Sigma}$ of $T_\Sigma T_\Pi$ is the full sub-bicategory
of $\text{Kl}(T_\Sigma T_\Pi)$ on the representables $y \in \text{Hom}(\mathcal{E}^{\text{op}}, \text{Cat})$.

$$\text{ThM } \tilde{\Sigma} \simeq \text{Poly}_{\mathcal{E}}^{\text{op}}.$$

"Proof" Fact: $T_\Sigma T_\Pi$ is a cocontinuous pseudomonad. So $T_\Sigma T_\Pi\text{-alg}$
is biequivalent to $\text{Hom}(\tilde{\Sigma}^{\text{op}}, \text{Cat})$. But we know that $T_\Sigma T_\Pi\text{-alg}$
is the bicat $\Sigma \Delta \Pi\text{-Fun}(\mathcal{E}^{\text{op}}, \text{Cat}) \simeq \text{Hom}(\text{Poly}_{\mathcal{E}}, \text{Cat})$.

"So" $\tilde{\Sigma} \simeq \text{Poly}_{\mathcal{E}}^{\text{op}}$. □