POLYNOMIALS IN CATEGORIES WITH PULLBACKS

1) The operational view / extensive view

Originally, a poly. functor $\underline{Set} \to \underline{Set}$ is one that's in the closure of id under $\times$, $+$; more generally, under Π, Σ.

More generally, a <u>multivariate</u> poly functor $Set^I \longrightarrow Set$ is one in the closure of the projection fns $Set^I \to Set$ under Π, Σ.

(1) Even more generally, a poly functor $Set^I \longrightarrow Set^J$ is a J-indexed family of poly functors $Set^I \longrightarrow Set$.

There are <u>many</u> other views on poly functors:

(2) Functors $Set/_I \longrightarrow Set/_J$ which are composites of:

$$\begin{tikzcd} Set/_I \arrow[r, "\Delta f"] & Set/_J \arrow[l, bend right, "\Sigma f"'] \arrow[l, bend left, "\Pi f"] \end{tikzcd} \qquad f : J \longrightarrow I$$

Δf = pullback along f

Σf = postcompose with f

Πf = "dependent product"

$$\Pi_f : Set^J \longrightarrow Set^I$$

$$(A_{ij} : i \in I, j \in J_i) \longmapsto \left(\prod_{j \in J_i} A_{ij} : i \in I\right)$$

where think of $f : J \longrightarrow I$ as giving family of sets $(J_i : i \in I)$ with $J_i = f^{-1}(i)$

③ Functors $\mathrm{Set}/_I \xrightarrow{F} \mathrm{Set}/_J$ of form $\mathrm{Set}/_I \xrightarrow{\Delta t} \mathrm{Set}/_E \xrightarrow{\Pi_p} \mathrm{Set}/_B \xrightarrow{\Sigma_s} \mathrm{Set}/_J$

④ Functors $\mathrm{Set}/_I \xrightarrow{F} \mathrm{Set}/_J$ which preserve connected limits and are small.
(Girard's normal functors (88))

⑤ . . . . with a left multiadjoint (Diers '78)

⑥ . . . . whose slices $F/_x$ are right adjoints
(local right adjoint: Street '00)

What happens if I generalise these from Set to $\mathcal{E}$?

Typically:

④ = ⑤ = ⑥ if, say, $\mathcal{E}$ is locally presentable.

③ ⊆ ② : but note here that $\Pi f : \mathcal{E}/_I \to \mathcal{E}/_J$ may not exist for all f: but we can restrict to the class of <u>exponentiable</u> f; by defn these are the f for which Πf exists.

① not very much to do with rest; except ① ⊆ ② if $\mathcal{E}$ lextensive.

Now ② ⊆ ④ and typically ② ⊊ ④.

④ = ⑤ = ⑥ typically super interesting:

- $\mathcal{E}$ = presheaf caty, have pra. functors that David spoke of
- $\mathcal{E}$ = variety, ⇢ corings + plethories + friends (Tall-Wraith, Bergman-Hausknecht, Joyal, Borger, ...)
- $\mathcal{E} = \mathrm{Sh}(X)$, X Stone space ⇢ topological dynamics + non-comm. geometry.

However, ② and ③ give a much more uniform theory, which is the theory of <u>polynomial functors</u> in $\mathcal{E}$.

Magical fact: in any catg $\mathcal{E}$ with pb's, ② and ③ coincide.
Why? Things in ② have normal forms in ③.
(Gambino, Kock 2013; Weber 2015)

<u>Idea</u>:

① If we have $\mathcal{E}/_I \xrightarrow{\Sigma f} \mathcal{E}/_J \xrightarrow{\Delta g} \mathcal{E}/_K$, I can form pb to the right

$$\begin{tikzcd} L \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, "f"] \\ K \arrow[r, "g"'] & J \end{tikzcd}$$

and now the 2-cell

$$\begin{tikzcd} \mathcal{E}/J \arrow[r, "\Delta g"] \arrow[d, "\Delta f"'] \arrow[dr, phantom, "\cong"] & \mathcal{E}/K \arrow[d, "\Delta u"] \\ \mathcal{E}/I \arrow[r, "\Delta v"'] & \mathcal{E}/L \end{tikzcd}$$

transposes to one

$$\begin{tikzcd} \mathcal{E}/I \arrow[r, "\Delta v"] \arrow[d, "\Sigma f"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{E}/L \arrow[d, "\Sigma u"] \\ \mathcal{E}/J \arrow[r, "\Delta g"'] & \mathcal{E}/K \end{tikzcd}$$

which turns out to be <u>invertible</u> (Beck-Chevalley).

So now

$$\begin{tikzcd} & \mathcal{E}/J \arrow[dr, "\Delta g"] & \\ \mathcal{E}/I \arrow[ur, "\Sigma f"] \arrow[dr, "\Delta v"'] & \cong & \mathcal{E}/K \\ & \mathcal{E}/L \arrow[ur, "\Sigma u"'] & \end{tikzcd}$$

② Similarly, if we have $\mathcal{E}/_I \xrightarrow{\Pi f} \mathcal{E}/_J \xrightarrow{\Delta g} \mathcal{E}/_K$ can form pullback and the canonical 2-cell

$$\begin{tikzcd} L \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, "f"] \\ K \arrow[r, "g"'] & J \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E}/I \arrow[r, "\Delta v"] \arrow[d, "\Pi f"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{E}/L \arrow[d, "\Pi u"] \\ \mathcal{E}/J \arrow[r, "\Delta g"'] & \mathcal{E}/K \end{tikzcd}$$

is again invertible (Beck-Chevalley) so can rewrite.

③ If we have $\mathcal{E}/I \xrightarrow{\Sigma f} \mathcal{E}/J \xrightarrow{\Pi g} \mathcal{E}/K$, can form

$$\begin{tikzcd} L \arrow[r, "u"] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "\circledast"] & M \arrow[d, "\Pi_g(f) = v"] \\ I \arrow[d, "f"'] & \\ J \arrow[r, "g"'] & K \end{tikzcd}$$

distributivity pullback

By messing around with adjoints, using BC isom. for Π's in this pb square, get a canonical 2-cell:

counit of $\Delta_g \dashv \Pi_g : \mathcal{E}/J \to \mathcal{E}/K$ at f.

$$\begin{tikzcd} \mathcal{E}/I \arrow[r, "\Sigma f"] \arrow[d, "\Delta\varepsilon"'] & \mathcal{E}/J \arrow[dd, "\Pi g"] \\ \mathcal{E}/L \arrow[d, "\Pi u"'] \arrow[r, phantom, "\Rightarrow"] & \\ \mathcal{E}/M \arrow[r, "\Sigma v"'] & \mathcal{E}/K \end{tikzcd}$$

which turns out to be invertible.

In Set, this invertibility expresses the isom.

$$\prod_{a \in A} \sum_{b \in B_a} C_{ab} = \sum_{f \in \prod_{a \in A} B_a} \prod_{a \in A} C_{a, f(a)}$$

called type-theoretic axiom of choice, or complete distributivity.

Using the invertible ⊛, we can rewrite $\longrightarrow \longrightarrow$ as

$\longrightarrow \longrightarrow \longrightarrow$

Using these three rewrites, we can turn any ② into a ③.

REMARK So typically, have

② = ③ $\subsetneq$ ④ = ⑤ $\neq$ ⑥

However, if we interpret ④ = ⑤ $\neq$ ⑥ less naively in $\mathcal{E}$, we get another equivalent formulation of ② = ③. Namely, if we look at <u>indexed</u> functors between <u>indexed</u> slice cats of $\mathcal{E}$, which in a suitable indexed sense are local right adjoint, then we get polynomial functors. (Koch and Kock 2013).

Talk 2

2) <u>THE COMBINATORIAL VIEW (OR INTENSIVE VIEW)</u>

The "normal form" of a poly functor $\mathcal{E}/_I \to \mathcal{E}/_J$ $(\to \to \to)$ gives us a more compact way of viewing them.

<u>DEFN</u> A <u>polynomial</u> from I to J in a caty w/ pullbacks $\mathcal{E}$ is a diagram

$$P = \begin{tikzcd} & E \arrow[dl, "t"'] \arrow[r, "p"] & B \arrow[dr, "s"] & \\ I & & & J \end{tikzcd}$$

The <u>associated poly. functor</u> of P is

$$F_p := \mathcal{E}/_I \xrightarrow{\Delta t} \mathcal{E}/_E \xrightarrow{\Pi p} \mathcal{E}/_B \xrightarrow{\Sigma s} \mathcal{E}/_J .$$

To compose two polys, we compose the associated poly functors and take the normal form:

$$Q \circ P := \begin{tikzcd} & & & \cdot \arrow[r] \arrow[dl] & \cdot \arrow[r] \arrow[dl] & \cdot \arrow[dd] & \\ & & \cdot \arrow[r] \arrow[dl] & \cdot \arrow[dl] \arrow[dr] & \text{dpb} & & \\ & E \arrow[r] \arrow[dl] & B \arrow[dr] & \text{pb} & F \arrow[r] \arrow[dl] & C \arrow[dr] & \\ I & & & J & & & K \\ & & P & & & Q & \end{tikzcd}$$

By construction, $F_{Q \circ P} \cong F_Q \circ F_P$.

3) Maps of polynomials

We'd like to see polys from I to J as 1-cells in a bicategory $\mathrm{Poly}_{\mathcal{E}}$. So what are 2-cells?

To motivate this, let's return to the $\underline{\mathrm{Set}}$ case. If $F, G : \mathrm{Set}^I \longrightarrow \mathrm{Set}^J$ are poly functors, then a $\underline{\text{map}}$ between them is just a nat xfm. Since poly functors in $\underline{\mathrm{Set}}$ preserve conn. limits, they are pointwise coproducts of representables, and so we can use Yoneda to get a representation of such nat xfms in terms of the poly. normal form.

<u>Prop</u> If P, Q are set-polynomials from I to J:

$$\begin{tikzcd}
P := & & E \arrow[dl, "t"'] \arrow[r, "p"] & B \arrow[dr, "s"] \arrow[dd, "f"] & \\
& I & \cdot \arrow[u, "g"] \arrow[d] \arrow[ur] & & J \\
Q := & & F \arrow[ul, "u"] \arrow[r, "q"'] & C \arrow[ur, "v"'] &
\end{tikzcd}$$

Then to give $\alpha : F_P \Rightarrow F_Q$ is equally to give the blue maps f, g above.

<u>Proof</u>. • We write $(B_j : j \in J)$ for family of fibres of s.
• We write $(E_{ib} : i \in I, b \in B)$ for fibres of $(t, p) : E \to I \times B$.

In these terms, $F_p : \mathbf{Set}^I \longrightarrow \mathbf{Set}^J$

$$(X_i : i\in I) \longmapsto \Big(\sum_{b\in B_j}\prod_{i\in I} X_i^{E_{ib}} : j\in J\Big)$$

So now α is $\Big(\sum_{b\in B_j}\prod_{i\in I}(-)_i^{E_{ib}} \overset{\alpha_j}{\Longrightarrow} \sum_{c\in C_j}\prod_{i\in I}(-)_i^{F_{ic}} : j\in J\Big)$

$$=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=$$

rep. functor $\mathbf{Set}^I \to \mathbf{Set}$: rep^d by $(E_{ib} : i\in I)$

$$\Big(\prod_{i\in I}(-)_i^{E_{ib}} \overset{\alpha_{jb}}{\Longrightarrow} \sum_{c\in C_j}\prod_{i\in I}(-)_i^{F_{ic}} : j\in J,\ b\in B_j\Big)$$

$$=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=$$

$$\Big(\alpha_{jb}(\lambda_i.1) =: \tilde{\alpha}_{jb} \in \sum_{c\in C_j}\prod_{i\in I} E_{ib}^{F_{ic}}\Big)$$

If we write $\tilde{\alpha}_{jb}$ as $\big(f(b)\in C_j,\ (g_{ib} : F_{i,f(b)} \to E_{i,b})_{i\in I}\big)$
then we get

$$\begin{tikzcd}
 & E \arrow[dl, "t"'] \arrow[rr, "p"] & & B \arrow[dd, "f"] \arrow[dr, "s"] & \\
I & & \cdot \arrow[ul, "g"] \arrow[dl] \arrow[ur] \arrow[dr, phantom, "\lrcorner", very near start] & & J. \\
 & F \arrow[ul, "u"] \arrow[rr, "q"'] & & C \arrow[ur, "v"'] &
\end{tikzcd}$$

$=$

What happens in an arbitrary caty $\mathcal{E}$ with pullbacks?
The naive thing doesn't work: if we define a map between polys from I to J to be a nat transformation between P_F and P_G, we get nowhere. The reason is that the P_F's are no longer pure sums of representables, so we can't apply Yoneda.

However... I said last time we can view poly functors $\mathcal{E}/I \to \mathcal{E}/J$ as indexed functors (over $\mathcal{E}$); now as indexed functors they are pointwise coprods of representables, and so the "same" argument applies.
So what we have is:-

Defn A map of polys from I to J in $\mathcal{E}$ is

$$(*)\quad \begin{tikzcd}
& E \arrow[dl, "t"'] \arrow[r, "p"] & B \arrow[dr, "s"] \arrow[dd, "f"] & \\
I & \cdot \arrow[u, "g"] \arrow[d] \arrow[ur] \arrow[dr, phantom, "\lrcorner"] & & J. \\
& F \arrow[ul, "u"] \arrow[r, "q"'] & C \arrow[ur, "v"'] &
\end{tikzcd}$$

← write $\mathrm{Poly}_{\mathcal{E}}(I,J)(P,Q)$ for set of such.

Prop There's an assignment (Abbott 2003, Gambino–Kock 2013)

$$\mathrm{Poly}_{\mathcal{E}}(I,J)(P,Q) \longrightarrow \mathrm{IdxNat}(\mathcal{E}/I, \mathcal{E}/J)(F_P, F_Q)$$

which sends $(*)$ to

$$\begin{tikzcd}
& \mathcal{E}/E \arrow[r, "\Pi p"] \arrow[d, "\Delta g"] \arrow[dr, phantom, "\Downarrow"] & \mathcal{E}/B \arrow[dr, "\Sigma s"] & \\
\mathcal{E}/I \arrow[ur, "\Delta t"] \arrow[dr, "\Delta u"'] \arrow[r, phantom, "\cong"] & \mathcal{E}/\cdot \arrow[ur, "\pi"'] \arrow[dr, phantom, "\cong BC"] & & \mathcal{E}/J \arrow[l, phantom, "\Downarrow"] \\
& \mathcal{E}/F \arrow[u, "\Delta"] \arrow[r, "\Pi q"'] & \mathcal{E}/C \arrow[uu, "\Delta f"'] \arrow[ur, "\Sigma v"'] &
\end{tikzcd}$$

...unit of $\Delta g \dashv \Pi g$

... counit of $\Sigma f \dashv \Delta f$

This is an isomorphism, and so we get a caty $\mathrm{Poly}_{\mathcal{E}}(I,J)$ with a f.f. functor $\mathrm{Poly}_{\mathcal{E}}(I,J) \longrightarrow \mathrm{IdxNat}(\mathcal{E}/I, \mathcal{E}/J)$.

So finally, we have:

Defn The 2-caty of polynomial functors in $\mathcal{E}$ has:

- obs: those of $\mathcal{E}$
- 1-cells $I \to J$: poly functors $\mathcal{E}/I \to \mathcal{E}/J$
- 2-cells: indexed transformations $\alpha: F \Rightarrow G: \mathcal{E}/I \to \mathcal{E}/J$

The bicaty of polynomials in $\mathcal{E}$ $\mathrm{Poly}_{\mathcal{E}}$ has:

- obs those of $\mathcal{E}$
- 1-cells $I \to J$: polys $I \to J$
- 2-cells: maps of polynomials,

with remaining data forced by the requirement that there be a locally fully faithful homomorphism of bicategories

$$\begin{array}{ccc} \mathrm{Poly}_{\mathcal{E}} & \longrightarrow & \mathrm{Idx\,Cat}_{\mathcal{E}} \\ I & \longmapsto & \mathcal{E}/I \end{array}$$

and given on homs by

$$\mathrm{Poly}_{\mathcal{E}}(I, J) \longrightarrow \mathrm{Idx\,Cat}_{\mathcal{E}}(\mathcal{E}/I, \mathcal{E}/J)$$

as in previous proposition.

Talk 3

4) UNIVERSAL PROPERTY OF $\mathcal{E} \mapsto \mathrm{Poly}_{\mathcal{E}}$.

We'll assume in this section: $\mathcal{E}$ is locally cartesian closed, i.e, all maps f are exponentiable (ie Πf exists).

Poly. functors involve $\Sigma \dashv \Delta \dashv \Pi$ satisfying B.C. and distributivity axioms. We can make sense of these adjunctions + coherence data in any bicaty — motivating the following universal property.

Warm-up: $\mathrm{Span}_{\mathcal{E}}$.

Defn Let $\mathcal{E}$ be a caty w/ pullbacks, $\mathcal{K}$ a bicaty.

1) A **Δ-functor** $F: \mathcal{E}^{op} \longrightarrow \mathcal{K}$ is just a pseudofunctor, with action written as:

$$X \longmapsto FX$$
$$f: X \to Y \longmapsto F_\Delta f : FY \longrightarrow FX$$

2) A **$\Sigma\Delta$-functor** $F: \mathcal{E}^{op} \longrightarrow \mathcal{K}$ is a Δ-functor st:

- each $F_\Delta f$ has a left adjoint $F_\Sigma f : FX \to FY$ in $\mathcal{K}$.
- for each pb square
$$\begin{tikzcd} D \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$
in $\mathcal{E}$, the canonical B.C. 2-cell
$$\begin{tikzcd} FC \arrow[r, "F_\Delta v"] \arrow[d, "F_\Sigma g"'] \arrow[dr, phantom, "\Downarrow"] & FD \arrow[d, "F_\Sigma u"] \\ FA \arrow[r, "F_\Delta f"'] & FB \end{tikzcd}$$
is invertible.

Examples

a) A Δ-functor $\mathcal{E}^{op} \longrightarrow \mathrm{Cat}$ is an $\mathcal{E}$-indexed caty

. $\Sigma\Delta$ . . . . with sums

In ptic., there's a $\Sigma\Delta$-functor $\mathcal{E}^{op} \longrightarrow \mathrm{Cat}$
$$X \longmapsto \mathcal{E}/X$$
$$f \longmapsto F_\Delta f = \Delta f.$$

b) There's a $\Sigma\Delta$-functor $\eta : \mathcal{E}^{op} \longrightarrow \mathrm{Span}_{\mathcal{E}}$

$$X \longmapsto X$$

with $F_\Delta(f) = \begin{tikzcd}[column sep=small] & X \arrow[dl, "f"'] \arrow[dr, "1"] & \\ Y & & X \end{tikzcd}$, $\quad F_\Sigma(f) = \begin{tikzcd}[column sep=small] & X \arrow[dl, "1"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$

<u>THM</u> (Dawson, Paré, Pronk 2004) $\eta : \mathcal{E}^{op} \to \mathrm{Span}_{\mathcal{E}}$ is the universal $\Sigma\Delta$-functor out of $\mathcal{E}$. ie, for any bicategory K, composition with η induces a biequivalence

$$\mathrm{Hom}(\mathrm{Span}_{\mathcal{E}}, K) \longrightarrow \Sigma\Delta\text{-Funct}(\mathcal{E}^{op}, K).$$

Now let's do $\mathrm{Poly}_{\mathcal{E}}$! As mentioned above, let's take $\mathcal{E}$ lccc.

<u>Defn</u> • A $\Delta\Pi$-functor $F : \mathcal{E}^{op} \to K$ is a Δ-functor st. each $F_\Delta f$ has a right adjoint $F_\Pi f$ such that the canonical B.C. 2-cell associated to any pb square in $\mathcal{E}$ is invertible.

• A $\Sigma\Delta\Pi$-functor $F : \mathcal{E}^{op} \to K$ is a Δ-functor which is both a $\Sigma\Delta$-functor and a $\Delta\Pi$-functor, <u>and</u> such that, for any distributivity pullback

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[dd, "\Pi_g f = v"] \\ A \arrow[d, "f"'] & \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

the can. 2-cell

$$\begin{tikzcd} FE \arrow[r, "F_\Pi u"] & FD \arrow[dd, "F_\Sigma v"] \\ FA \arrow[u, "F_\Delta \varepsilon"] \arrow[d, "F_\Sigma f"'] \arrow[r, phantom, "\Uparrow"] & \\ FB \arrow[r, "F_\Pi g"'] & FC \end{tikzcd}$$

is invertible.

Examples

a) A $\Delta\Pi$-functor $\mathcal{E}^{op} \to \mathrm{Cat}$ is an indexed caty with products
· $\Sigma\Delta\Pi$ · · · · · · · · · · · · sums, products, distributivity

For example: have
$$F: \mathcal{E}^{op} \longrightarrow \mathrm{Cat}$$
$$X \longmapsto \mathcal{E}/X$$
with $F_\Delta f = \Delta f$, $F_\Pi f = \Pi f$, $F_\Sigma f = \Sigma f$.

For example: let $\mathcal{C}$ be any locally small caty.
Have a Δ-functor
$$F: \mathrm{Set}^{op} \longrightarrow \mathrm{CAT}$$
$$I \longmapsto \mathcal{C}^I$$
This is:
- a $\Sigma\Delta$ functor if $\mathcal{C}$ has small coproducts
- a $\Delta\Pi$ functor if $\mathcal{C}$ has small products
- a $\Sigma\Delta\Pi$ functor if $\mathcal{C}$ has small products, small coproducts and infinite distributivity:

$$\prod_{j \in J} \sum_{k \in K_j} X_{jk} \cong \sum_{f : \prod_{j \in J} K_j} \prod_{j \in J} X_{j, f(j)} .$$

For example: let $\mathcal{E}$ be a topos. Have a Δ-functor
$$F: \mathcal{E}^{op} \longrightarrow \mathrm{Poset}$$
$$X \longmapsto \mathrm{Sub}(X)$$
This is always a $\Sigma\Delta$- and a $\Delta\Pi$-functor.

It's a $\Sigma\Delta\Pi$-functor if we have in $\mathcal{E}$ that

$$\forall j\in J.\ \exists k\in K_j.\ \varphi(j,k) \Rightarrow \exists f:\prod_{j\in J} K_j.\ \forall j\in J.\ \varphi(j,h).$$

b) There's a $\Sigma\Delta\Pi$-functor $\begin{array}{l} \eta: \mathcal{E}^{op} \longrightarrow \mathrm{Poly}_{\mathcal{E}} \\ \quad X \longmapsto X \end{array}$ and with

$$F_\Delta(f) = \begin{tikzcd}[column sep=small] & X \arrow[dl, "f"'] \arrow[r, equal] & X \arrow[dr, equal] & \\ Y & & & X \end{tikzcd} \qquad F_\Sigma(f) = \begin{tikzcd}[column sep=small] & X \arrow[dl, equal] \arrow[r, equal] & X \arrow[dr, "f"] & \\ X & & & Y \end{tikzcd} \qquad F_\Pi(f) = \begin{tikzcd}[column sep=small] & X \arrow[dl, equal] \arrow[r] & Y \arrow[dr, equal] & \\ X & & & Y \end{tikzcd}$$

THM (Walther, 2019) $\eta: \mathcal{E}^{op} \to \mathrm{Poly}_{\mathcal{E}}$ is the universal $\Sigma\Delta\Pi$-functor out of $\mathcal{E}$:

$$\mathrm{Hom}(\mathrm{Poly}_{\mathcal{E}}, K) \simeq \Sigma\Delta\Pi\text{-}\mathrm{Funct}(\mathcal{E}^{op}, K).$$

In particular, the $\Sigma\Delta\Pi$-functor $\begin{array}{l} \mathcal{E}^{op} \longrightarrow \mathrm{Cat} \\ X \longmapsto \mathcal{E}/X \end{array}$

induces the canonical homomorphism $\begin{array}{l} \mathrm{Poly}_{\mathcal{E}} \longrightarrow \mathrm{Cat} \\ \quad X \longmapsto \mathcal{E}/X \end{array}$.

5) THE KLEISLI VIEW

(after von Glehn)

Let's define

$$\begin{aligned} \mathrm{IdxCat}(\mathcal{E}) &:= \Delta\text{-}\mathrm{Funct}(\mathcal{E}^{op}, \mathrm{Cat}) \\ \mathrm{IdxCat}_\Pi(\mathcal{E}) &:= \Delta\Pi \ \cdots\cdots \\ \mathrm{IdxCat}_\Sigma(\mathcal{E}) &:= \Sigma\Delta \ \cdots\cdots \end{aligned}$$

We have $\mathrm{IdxCat}_\Sigma(\mathcal{E}) \underset{\longrightarrow}{\overset{\leftarrow}{\perp}} \mathrm{IdxCat}(\mathcal{E}) \underset{\longleftarrow}{\overset{\rightarrow}{\perp}} \mathrm{IdxCat}_\Pi(\mathcal{E})$ psmonadic.

So writing T_Σ, T_Π for induced psmonads on $\mathrm{IdxCat}(\mathcal{E})$, have

$$\mathrm{IdxCat}_\Sigma(\mathcal{E}) \simeq T_\Sigma\text{-alg} \quad \text{and same for } \Pi.$$

<u>FACT</u>: there's a ps. distributive law $T_\Pi T_\Sigma \Rightarrow T_\Sigma T_\Pi$, and algs for composite psmonad are idx cats with sums, products + distributivity. $\rightsquigarrow T_\Sigma T_\Pi$

<u>Defn</u> The <u>0-ary theory</u> $\mathcal{T}$ of $T_\Sigma T_\Pi$ is the full sub-bicategory of $\mathrm{Kl}(T_\Sigma T_\Pi)$ on the representables $yE \in \mathrm{Hom}(\mathcal{E}^{op}, \mathrm{Cat})$.

<u>THM</u> $\mathcal{T} \simeq \mathrm{Poly}_{\mathcal{E}}^{op}$.

"<u>Proof</u>" Fact: $T_\Sigma T_\Pi$ is a cocontinuous pseudomonad. So $T_\Sigma T_\Pi$-alg is biequivalent to $\mathrm{Hom}(\mathcal{T}^{op}, \mathrm{Cat})$. But we know that $T_\Sigma T_\Pi$-alg is the bicaty $\Sigma\Delta\Pi\text{-Fun}(\mathcal{E}^{op}, \mathrm{Cat}) \simeq \mathrm{Hom}(\mathrm{Poly}_{\mathcal{E}}, \mathrm{Cat})$.
"So" $\mathcal{T} \simeq \mathrm{Poly}_{\mathcal{E}}^{op}$. $\square$