Coalgebras and their Modal Logics: Polynomial Functors and Beyond

Part 2: Coalgebraic Modal Logic

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Workshop on Polynomial Functors, 17 March 2021

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Introduction

Modal Logic

Origin: Philosophical logic, reasoning about:

- modalities of truth (φ is necessarily true, φ is possibly true,...)
- deontic, temporal, epistemic, doxastic notions.

Applications in CS (formal verification):

- program logics: Hennessy-Milner logic, PDL
- databases: XPath
- knowledge representation: description logics
- game logics: Coalition Logic, Game Logic
- temporal logics: LTL, CTL, CTL*, ATL, ATL*
- fixpoint logic: modal μ-calculus

Nice properties: good trade-off between

- expressiveness (of relevant properties), and
- complexity (often decidability in PSPACE, with fixpoints: EXPTIME); suitable for automated verification

Big Picture: Algebra vs Coalgebra

Algebra

- construction
- congruence
- compositionality
- universal algebra
- parametric in signature and equations

Coalgebra

- destruction/observation
- bisimulation
- abstraction
- universal coalgebra
- parametric in transitions and observations

$$\frac{\text{Equational Logic}}{\text{Algebra}} \qquad \frac{\text{Modal Logic}}{\text{Coalgebra}}$$

"Modal logics are coalgebraic" [Cirstea et al.'11]

Overview of Today

Part 2:

1. Introduction
2. Basic Modal Logic
3. Coalgebraic Modal Logic
   - via Predicate Liftings
   - via Relation Lifting
   - Extensions and Uniform Theorems
4. Concluding Part 2

Basic Modal Logic

Basic Modal Logic

Syntax: The language of basic modal logic over a set Prop of atomic propositions, is Boolean propositional logic plus modalities:

$$\varphi ::= p \in \mathsf{Prop} \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi$$

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Def. A Kripke model (X, R, v) consists of:

- a set X (of worlds),
- an accessibility relation $R \subseteq X \times X$ on X,
- a valuation $v \colon \mathsf{Prop} \to \mathcal{P}(X)$ of atomic propositions.

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Kripke semantics

Truth in a Kripke model $\mathbb{M} = (X, R, v)$ is defined by:

$\mathbb{M}, x \models p$	iff	$x \in v(p)$	for $p \in \mathsf{Prop}$
$\mathbb{M}, x \models \neg\varphi$	iff	not $\mathbb{M}, x \models \varphi$	
$\mathbb{M}, x \models \varphi \wedge \psi$	iff	$\mathbb{M}, x \models \varphi$ and $\mathbb{M}, x \models \psi$	
$\mathbb{M}, x \models \varphi \vee \psi$	iff	$\mathbb{M}, x \models \varphi$ or $\mathbb{M}, x \models \psi$	
$\mathbb{M}, x \models \Box\varphi$	iff	for all $y \in X : R(x, y)$ implies $\mathbb{M}, y \models \varphi$	
$\mathbb{M}, x \models \Diamond\varphi$	iff	there exists $y \in X : R(x, y)$ and $\mathbb{M}, y \models \varphi$	

Kripke Bisimulation

Let $\mathbb{M}_1 = (X_1, R_1, v_1)$ and $\mathbb{M}_2 = (X_2, R_2, v_2)$ be Kripke models.

Def. A bisimulation between $\mathbb{M}_1$ and $\mathbb{M}_2$ is a relation $Z \subseteq X_1 \times X_2$ such that for all $(x_1, x_2) \in Z$:
(prop) for all $p \in \mathsf{Prop}$: $x_1 \in v(p)$ iff $x_2 \in v(p)$.
(forth) for all $y_1 \in R_1(x_1)$ there is $y_2 \in R_2(x_2)$ such that $(y_1, y_2) \in Z$.
(back) for all $y_2 \in R_2(x_2)$ there is $y_1 \in R_1(x_1)$ such that $(y_1, y_2) \in Z$.

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Def. A bounded morphism $f\colon \mathbb{M}_1 \to \mathbb{M}_2$ is a functional bisimulation between $\mathbb{M}_1$ and $\mathbb{M}_2$.

Notation: for $x_1 \in \mathbb{M}_1$ and $x_2 \in \mathbb{M}_2$, we write:
$x_1 \leftrightarroweq x_2$ if x_1 and x_2 are linked by some bisimulation.
$x_1 \equiv x_2$ if x_1 and x_2 satisfy the same modal formulas, i.e., for all modal formulas φ: $\mathbb{M}_1, x_1 \models \varphi$ iff $\mathbb{M}_2, x_2 \models \varphi$.

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Modal truth is bisimulation invariant:
Theorem If $x_1 \leftrightarroweq x_2$ then $x_1 \equiv x_2$. (Proof by struct. induction on φ.)

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$$\begin{aligned} st_v(p) &= P(v) \\ st_v(\neg\varphi) &= \neg st_v(\varphi) \\ &\vdots \\ st_v(\Box\varphi) &= \forall u.R(v,u) \to st_u(\varphi) \\ st_v(\Diamond\varphi) &= \exists u.R(v,u) \land st_u(\varphi) \end{aligned}$$

Theorem: For all Kripke models $\mathbb{M}$ and all modal formulas φ, $\mathbb{M}, x \models \varphi$ iff $\mathbb{M}^1 \models st_v(\varphi)[v \mapsto x]$

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Theorem (Van Benthem)

Modal logic is the bisimulation invariant fragment of first-order logic. In particular, every FO formula that is invariant for bisimulation is equivalent to the translation of a modal formula.

cf. [Van Benthem'76]

Kripke Frames are $\mathcal{P}$-Coalgebras

Let X, Y be sets and $f\colon X \to Y$ a function

- Covariant powerset functor $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$

$$\begin{aligned} \mathcal{P}(X) &= \text{powerset of } X \\ \mathcal{P}(f) &= f[-]\colon \mathcal{P}(X) \to \mathcal{P}(Y) \quad \text{(direct image)} \end{aligned}$$

- Relation $R \subseteq X \times X \leftrightsquigarrow$ map $R(-)\colon X \to \mathcal{P}(X)$ where $R(x) = \{y \in X \mid R(x, y)\}$.
- Kripke bisimulation $= \mathcal{P}$-bisimulation
- Bounded morphism $= \mathcal{P}$-coalgebra morphism:

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle R(-)}\downarrow & & \downarrow{\scriptstyle S(-)} \\ \mathcal{P}(X) & \xrightarrow{\;\mathcal{P}(f)\;} & \mathcal{P}(Y) \end{array}$$

i.e. $\forall x \in X, y \in Y:$

$$y \in f[R(x)] \iff y \in S(f(x))$$

Note: $\mathcal{P}$ preserves weak pullbacks, so over $\mathcal{P}$-coalgebras, behavioral equivalence coincides with bisimilarity.

Neighbourhood Semantics

Sometimes, Kripke semantics is not suitable.

E.g. Game Logic (Parikh): reasoning about strategic ability in determined of 2-player games.

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- Kripke valid: $\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$, but not valid wrt intended interpretation (strategies for φ and ψ may be conflicting).
- Only monotonicity holds:

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Solution: Interpret in neighbourhood model (N assigns to each state a collection of neighbourhoods):

$$(X, N \colon X \rightarrow \mathcal{P}(\mathcal{P}(X)), v : \mathsf{Prop} \rightarrow \mathcal{P}(X))$$

Modal semantics: $\mathbb{M}, x \models \Box\varphi$ iff $[\![\varphi]\!] \in N(x)$.

Neighbourhood Structures are Coalgebras

- Contravariant powerset functor $\mathcal{Q}\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$

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\mathcal{Q}(X) &= \text{powerset of } X \\
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- Neighbourhood frames are $\mathcal{N}$-coalgebras where

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\mathcal{N}(X) &= \mathcal{Q}(\mathcal{Q}(X)) \\
\mathcal{N}(f) &= (f^{-1})^{-1}[-]\colon \mathcal{N}(X) \to \mathcal{N}(Y) \quad \text{(double inverse image)} \\
& \quad U \in \mathcal{N}(f)(H) \text{ iff } f^{-1}[U] \in H
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- Monotone neighbourhood frames are coalgebras for

$$\begin{array}{rcl} \mathcal{M}(X) & = & \{H \in \mathcal{N}(X) \mid H \text{ closed under supersets}\} \\ \mathcal{M}(f) & = & (f^{-1})^{-1}[-]\colon \mathcal{M}(X) \to \mathcal{M}(Y) \quad \text{(double inverse image)} \end{array}$$

Note: $\mathcal{N}$ and $\mathcal{M}$ do not preserve weak pullbacks.

Bisimulations for Neighbourhood Frames

An application of coalgebra.

- Existing notions of bisimulation for labelled transition systems, Kripke frames, probabilistic systems … found ad hoc.
- Neighbourhood semantics: Segerberg (1971), Chellas (1980). Only little model theory (no notion of morphism and bisimulation).
- Bisimulation for monotonic neighbourhood frames: Van Benthem, Pauly (ca. 1999).
- Bisimulation for neighbourhood frames: H, Kupke, Pacuit (2009) using coalgebra.
  $\rightsquigarrow$ Hennessy-Milner Thm, Characterisation Thm, Craig Interpolation for classical modal logic.

Coalgebraic Modal Logic

General aim: Modal logics for T-coalgebras that are:

- developed uniformly, parametric in T.
- adequate wrt coalgebraic semantics: behaviorally equivalence implies modal equivalence.

Two approaches to modal logics for coalgebras:

- via relation lifting (Moss' ∇-logic)
- via predicate liftings (Pattinson, Rössiger, Jacobs)

Basic idea of Predicate Lifiting Approach

$$\frac{\text{Basic Modal Logic}}{\text{Kripke frames } X \to \mathcal{P}(X)} = \frac{\text{Coalgebraic Modal Logic}}{T\text{-coalgebras } X \to T(X)}$$

Coalgebraic modal logic means coalgebraic semantics of modal languages.

Syntax

Given a collection Λ of modal operators (with arities), and a set Prop of propositional variables, the set $\mathcal{L}_\Lambda$ of formulas over Λ is Boolean propositional logic plus modalities:

$$\mathcal{L}_\Lambda \ni \varphi ::= p \in \mathsf{Prop} \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \heartsuit(\varphi_1, \dots, \varphi_n), \quad \heartsuit \in \Lambda,\ n\text{-ary}$$

For notational simplicity, we focus on unary modalities from now on. Generalisation to n-ary modalities straightforward.

Coalgebraic semantics: We want to interpret formulas in T-coalgebra model $\mathbb{X} = (X, \gamma\colon X \to T(X), \upsilon\colon \mathsf{Prop} \to \mathcal{P}(X))$ which corresponds to $T \times \mathcal{P}(\mathsf{Prop})$-coalgebra $\langle \gamma : X \to TX, \hat{\upsilon} : X \to \mathcal{P}(\mathsf{Prop})\rangle$.
(We can take atomic props to be part of the structure.)

Kripke and Neighbourhood Semantics, Uniformly

In Kripke model $\mathbb{M} = (X, R\colon X \to \mathcal{P}(X), v\colon \mathsf{Prop} \to \mathcal{P}(X))$:

$$\begin{array}{llllll} \mathbb{M}, x \models \Box\varphi & \text{iff} & R(x) \subseteq [\![\varphi]\!] & \text{iff} & R(x) \in \{U \in \mathcal{P}(X) \mid U \subseteq [\![\varphi]\!]\} \\ \mathbb{M}, x \models \Diamond\varphi & \text{iff} & R(x) \cap [\![\varphi]\!] \neq \emptyset & \text{iff} & R(x) \in \{U \in \mathcal{P}(X) \mid U \cap [\![\varphi]\!] \neq \emptyset\} \end{array}$$

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where $[\![\varphi]\!] = \{x \in X \mid \mathbb{M}, x \models \varphi\}$ (truth-set of φ).

In neighbourhood model $\mathbb{M} = (X, N\colon X \to \mathcal{N}(X), v\colon \mathsf{Prop} \to \mathcal{P}(X))$:

$$\mathbb{M}, x \models \Box\varphi \quad \text{iff} \quad [\![\varphi]\!] \in N(x) \quad \text{iff} \quad N(x) \in \{H \in \mathcal{N}(X) \mid [\![\varphi]\!] \in H\}$$

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In coalgebraic model $\mathbb{X} = (X, \gamma\colon X \to T(X), v\colon \mathsf{Prop} \to \mathcal{P}(X))$

$$\mathbb{X}, x \models \heartsuit\varphi \quad \text{iff} \quad \gamma(x) \text{ satisfies condition involving } [\![\varphi]\!]$$

Predicate Liftings

T-coalgebraic semantics consists of:

- a functor $T\colon \mathsf{Set} \to \mathsf{Set}$
- for every modal operator $\heartsuit \in \Lambda$, a natural transformation

  $$[\![\heartsuit]\!] : \mathcal{Q} \Rightarrow \mathcal{Q}\,T \qquad (\mathcal{Q} \text{ is contravariant powerset fctr})$$

  i.e. $[\![\heartsuit]\!]$ is a family of set-indexed maps such that for all $f\colon X \to Y$,

  $$\begin{array}{ccc} \mathcal{Q}(X) & \xrightarrow{[\![\heartsuit]\!]_X} & \mathcal{Q}\,T(X) \\ {\scriptstyle \mathcal{Q}(f)}\uparrow & & \uparrow{\scriptstyle \mathcal{Q}\,T(f)} \\ \mathcal{Q}(Y) & \xrightarrow{[\![\heartsuit]\!]_Y} & \mathcal{Q}\,T(Y) \end{array}$$

- $[\![\heartsuit]\!]$ is called a predicate lifting:
  for all X, $[\![\heartsuit]\!]_X\colon\ \mathcal{Q}(X) \to \mathcal{Q}(T(X))$ lifts a predicate over X to a predicate over $T(X)$).

Remark: Predicate liftings for Kripke polynomial Set-functors T can be defined inductively over the structure of T (cf Bart Jacobs' talk).

Truth in T-model $\mathbb{X} = (X, \gamma : X \to T(X), v : \mathsf{Prop} \to \mathcal{P}(X))$

$$
\begin{array}{llll}
\mathbb{X}, x \models p & \text{iff} & x \in v(p) & \text{for } p \in \mathsf{Prop} \\
 & \vdots & & \\
\mathbb{X}, x \models \heartsuit\varphi & \text{iff} & \gamma(x) \in [\![\heartsuit]\!]_X([\![\varphi]\!]) & \text{where } [\![\varphi]\!] = \{y \mid \mathbb{X}, y \models \varphi\}
\end{array}
$$

Examples:

$$
\begin{array}{llll}
\text{Kripke box:} & [\![\heartsuit]\!]_X(U) & = & \{V \in \mathcal{P}(X) \mid V \subseteq U\}, \\
\text{Kripke diamond:} & [\![\heartsuit]\!]_X(U) & = & \{V \in \mathcal{P}(X) \mid V \cap U \neq \emptyset\} \\
\text{Neighbourhood modality:} & [\![\heartsuit]\!]_X(U) & = & \{H \in \mathcal{N}(X) \mid U \in H\}
\end{array}
$$

Adequacy: Truth Invariance

Proposition

For all T-coalgebra morphisms $f\colon (X,\gamma) \to (Y,\delta)$, $x \equiv f(x)$.
(Equivalently, for all φ: $[\![\varphi]\!]_X = f^{-1}[[\![\varphi]\!]_Y]$. It follows that:
$x \sim y \;\Rightarrow\; x \equiv y$.

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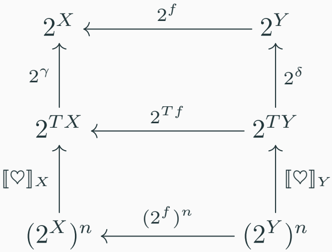
Proof By structural induction on φ. Induction step, modal case, use that (writing 2^X for $\mathcal{Q}X$):



which says:
for all $x \in X$, and all $U_1, \ldots, U_n$:

$$\gamma(x) \in [\![\heartsuit]\!]_X(f^{-1}[U_1], \ldots, f^{-1}[U_n])$$
iff
$$\delta(f(x)) \in [\![\heartsuit]\!]_Y(U_1, \ldots, U_n)$$

Yoneda Correspondence

Via Yoneda Lemma, 1-1 correspondence:

$$\frac{\text{predicate liftings } [\![\heartsuit]\!] \colon (2^{-})^{n} \Rightarrow 2^{T-}}{\text{subsets } C_{\heartsuit} \subseteq T(2^{n})}$$

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Alternative view on predicate lifting: “allowed 0-1 patterns”

$$\begin{array}{ccccc} X & \xrightarrow{\langle [\![\varphi_1]\!],\dots,[\![\varphi_n]\!]\rangle} & 2^{n} & & \\ {\scriptstyle\gamma}\big\downarrow & & & & \\ TX & \xrightarrow{T\langle [\![\varphi]\!]\dots,[\![\varphi_n]\!]\rangle} & T(2^{n}) & \xrightarrow{\chi_{C_\heartsuit}} & 2 \end{array}$$

where $\chi_{C_\heartsuit}$ is characteristic function that says which 0-1 patterns of T-structures are “allowed” by $\heartsuit$.

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where $\chi_{C_\heartsuit}$ is characteristic function that says which 0-1 patterns of T-structures are "allowed" by $\heartsuit$.

It also tells us how many predicate liftings, there are.
E.g. for $\mathcal{P}$: there are $2^{\mathcal{P}(2)} = 16$ unary predicate liftings.

cf. [Schröder'08],[Gumm]

Def. A logic $\mathcal{L}_\Lambda$ is expressive if $\mathbb{X}, x \equiv \mathbb{Y}, y$ implies $\mathbb{X}, x \sim \mathbb{Y}, y$.

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Def. The collection $[\![\Lambda]\!] = ([\![\heartsuit]\!])_{\heartsuit \in \Lambda}$ is separating (for T) if for all $t_1 \neq t_2$ in TX there is a $\heartsuit \in \Lambda$ (n-ary) and $(A_1, \ldots, A_n) \in (\mathcal{Q}X)^n$ such that $t_1 \in [\![\heartsuit]\!]_X(A_1, \ldots, A_n)$ and $t_2 \notin [\![\heartsuit]\!]_X(A_1, \ldots, A_n)$, or vice versa. [Pattinson'04]

Theorem If T is finitary and $[\![\Lambda]\!]$ is separating, then $\mathcal{L}_\Lambda$ is expressive.

Theorem [Schröder'08]

If T is finitary, then there is a separating set of (n-ary) predicate liftings for T (and hence an expressive modal logic).

Coalgebraic Modal Logic via Relation Lifing

Introduced by [Moss'oo].

Basic idea:

- Language has one "canonical" modality ∇ that takes elements from $T(\mathcal{L})$ as argument (where $\mathcal{L}$ is the set of formulas).
- Semantics of ∇ via lifting of satisfaction relation $\models \subseteq X \times \mathcal{L}$: For $\alpha \in T(\mathcal{L})$,

$$(X, \gamma), x \models \nabla\alpha \quad \text{iff} \quad (\gamma(x), \alpha) \in \overline{T}(\models)$$

where $\overline{T}$ is the so-called *Barr lifting*:

$$\overline{T}(R) = \{\langle T(\pi_1)(u), T(\pi_2)(u)\rangle \mid u \in T(R)\} \quad \subseteq T(X_1) \times T(X_2)$$

Remarks:

- To show adequacy, T needs to preserve weak pullbacks.
- ∇-logic is always expressive.
- Canonical language, but non-standard.

Example: ∇ for $\mathcal{P}$-coalgebras

Example: For $T = \mathcal{P}$, $\overline{\mathcal{P}}$ is also known as the Egli-Milner lifting

$$\begin{aligned}\overline{\mathcal{P}}(R) \quad = \quad & \{(U,V) \in \mathcal{P}X \times \mathcal{P}Y \mid \forall u \in U \exists v \in V : (u,v) \in R\} \cap \\ & \{(U,V) \in \mathcal{P}X \times \mathcal{P}Y \mid \forall v \in V \exists u \in U : (u,v) \in R\}\end{aligned}$$

That means, for a set $\Phi \in \mathcal{P}(\mathcal{L})$ of formulas

$x \models \nabla\Phi$ iff

- all R-successors of x satisfy some $\varphi \in \Phi$, and
- all $\varphi \in \Phi$ are satisfied by some R-successor of x.

In other words, $\nabla\Phi$ is equivalent with:

$$\Box \bigvee_{\varphi \in \Phi} \varphi \wedge \bigwedge_{\varphi \in \Phi} \Diamond \varphi$$

In general, ∇ can be expressed by predicate liftings and vice versa.
[Leal & Kurz]

Extensions

Extensions of basic coalgebraic modal logic:

- with fixpoints: coalgebraic μ-calculus (both ∇ and pred.lifts) [Venema, Kupke, Fontaine, Enqvist, Seifan,...]
- with temporal operators [Cirstea]
- coalgebraic dynamic logic (PDL) [H, Kupke]
- coalgebraic predicate logic [Litak, Sano, Pattinson, Schröder]

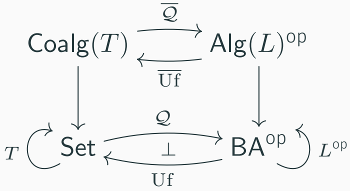
Uniform Theorems

Some coalgebraic generalisations of classic results

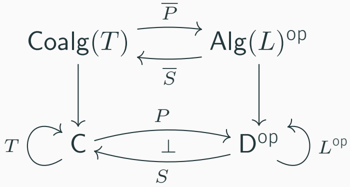
- Hennessy-Milner thm (Schröder),
- Van Benthem Characterisation thm: CML = FOL/ $\sim$ (Pattinson, Schröder, Litak, Sano)
- Janin-Walukiewics thm: μ-CML = MSOL/ $\sim$ (Enqvist,Seifan,Venema)
- Goldblatt-Thomason thm: modal analogue of Birkhoff Variety thm. (Kurz, Rosický)
- Completeness
  - coalgebraic canonical model construction (Pattinson, Schröder),
  - ∇-logic (Kupke, Kurz, Venema),
  - coalgebraic dynamic logics (H, Kupke)
- Decidability in PSPACE (Schröder, Pattinson)
- Uniform Interpolation (Marti, Enqvist, Seifan, Venema)

Modal Logic via Dual Adjunctions

Stone-type duality:



Generalise to non-classical base logic and other base categories



cf. [Kupke et al'04], [Bonsangue & Kurz'05], [Klin'07], [Jacobs & Sokolova'10], [Klin & Rot'16], [de Groot et al.'20] m.m. (cf. next talk)

Concluding Part 2

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- Universal coalgebra: unifying theory of state-based systems, parametric in T
- Coalgebraic modal logic: uniform development of modal logics for coalgebras.
- Modal logics are coalgebraic: fundamental relationship between modal expressiveness and behavioral equivalence/bisimilarity.
- Many theorems proved at level of T-coalgebras, by identifying conditions on the functor T etc.
- Polynomial functors are well-behaved (weak pullback preserving, ω^{op}-continuous): nice coalgebraic theory and modal logics.

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THANK YOU

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