Polynomial Monads and Segal Conditions

Workshop on Polynomial Functors
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Polynomial Monads

$\mathcal{S}$ = ∞-cat. of spaces/∞-groupoids/ht.py types

$$f : X \to Y \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{S}_{/X} \arrow[r, bend left=50, "f_!"] \arrow[r, bend right=50, "f_*"'] & \mathcal{S}_{/Y} \arrow[l, "f^*"'] \end{tikzcd} \qquad f_! \dashv f^* \dashv f_*$$

A polynomial functor is a composite of such functors

Propn (Gepner-H.-Kock): $F : \mathcal{S}_{/X} \to \mathcal{S}_{/Y}$ is polynomial iff

F is accessible and preserves weakly contractible limits [wide pullbacks]

Defn.: A polynomial monad on $\mathcal{S}_{/X}$ is a cartesian monad s.t. endofunctor is polynomial.

Example (Gepner-H.-Kock): ∞-operads corresponding to analytic monads (polynomial + preserves sifted colimits)

Characterization of polynomial functors/monads make sense as abstract defn.s more generally, in particular could consider for $\mathrm{Fun}(\mathcal{I}, \mathcal{S})$, $\mathcal{I}$ any small ∞-cat. (Note: $\mathcal{S}_{/X} \simeq \mathrm{Fun}(X, \mathcal{S})$)

Idea: Polynomial monads on presheaf ∞-cat.s are closely related to homotopy-coherent algebraic str.s described by Segal conditions.

Segal Conditions

Defn.: An *algebraic pattern* is an ∞-cat. $\mathcal{O}$ w/

- a factorization system $\quad (\mathcal{O}^{int}, \mathcal{O}^{act}$ subcat.s$)$

$$\begin{tikzcd} O \arrow[rr, "f"] \arrow[dr, rightarrowtail, "\text{``inert''}"'] & & O' \\ & \bullet \arrow[ur, squiggly, "\text{``active''}"'] & \end{tikzcd}$$

- a full subcat. $\mathcal{O}^{el} \subset \mathcal{O}^{int}$ of "elementary objects"

A <u>Segal $\mathcal{O}$-space</u> is $F : \mathcal{O} \to \mathcal{S}$ s.t. for $O \in \mathcal{O}$

$$F(O) \xrightarrow{\sim} \lim_{O \rightarrowtail E \in \mathcal{O}^{el}_{O/}} F(E)$$

$$\mathcal{O}^{el}_{O/} \simeq \mathcal{O}^{el} \times_{\mathcal{O}^{int}} \mathcal{O}^{int}_{O/}$$

(<u>Or</u>: $F|_{\mathcal{O}^{int}}$ is a RKE of $F|_{\mathcal{O}^{el}}$)

<u>Examples:</u>

(1) $\mathbb{F}_*$ = finite pointed sets $\langle n \rangle = (\{0,1,\dots,n\}, 0)$

$f : \langle n \rangle \to \langle m \rangle$ is <u>inert</u> if $|f^{-1}(i)| = 1$ if $i \neq 0$

<u>active</u> if $f^{-1}(0) = \{0\}$

$\mathbb{F}_*^{el} = \{\langle 1\rangle\}$

Segal $\mathbb{F}_*$-space: $F : \mathbb{F}_* \to \mathcal{S}$

$$F(\langle n\rangle) \xrightarrow{\sim} \prod_{i=1}^{n} F(\langle 1\rangle)$$

Segal's special Γ-space - model for comm. alg.s (E_∞-alg.s)

(2) Δ = non-empty finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$

$f : [n] \to [m]$ $\underline{\text{inert}}$: $f(i) = f(0) + i$

$\underline{\text{active}}$: $f(0) = 0,\ f(n) = m$

$\Delta^{op,el} = \{[1]\}$: Segal Δ^{op}-space

$$F : \Delta^{op} \to \mathcal{S} \qquad F([n]) \xrightarrow{\sim} \prod_{i=1}^{n} F([1])$$

associative alg.

$$\Delta^{op,el} = \{ [1] \rightrightarrows [0] \} \; : \quad F([n]) \xrightarrow{\sim} F([1]) \times_{F([0])} \cdots \times_{F([0])} F([1])$$

Rezk's Segal spaces — ∞-categories

(3) Θ_n^{op} (Joyal, Rezk, Berger)

— (∞, n)-categories

(4) Ω^{op} (Moerdijk–Weiss, Cisinski–Moerdijk)

— ∞-operads

(5) cat.s of graphs (Hackney - Robertson - Yau)

∞-properads, cyclic / modular ∞-operads

(6) any ∞-operad in Lurie's framework

Polynomial Monads from Patterns

$$\begin{tikzcd}
\mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \subset \mathrm{Fun}(\mathcal{O}, \mathcal{S}) \arrow[d, "j^*"'] \arrow[dd, bend right=60, "U_{\mathcal{O}}"'] \\
\mathrm{Seg}_{\mathcal{O}^{int}}(\mathcal{S}) \arrow[d, "i^*"', "\wr"] \\
\mathrm{Fun}(\mathcal{O}^{el}, \mathcal{S})
\end{tikzcd}$$

(inverse i_*)

full subcat. of Segal $\mathcal{O}$-spaces

$$\mathcal{O}^{el} \overset{i}{\hookrightarrow} \mathcal{O}^{int} \xrightarrow{j} \mathcal{O}$$

U_{Θ} is a monadic right adjoint w/ left adjoint F_{Θ}

$$\mathrm{Fun}(\Theta^{el}, \mathcal{S}) \xrightarrow[i_*]{\simeq} \mathrm{Seg}_{\Theta^{int}}(\mathcal{S}) \xrightarrow{j_!} \mathrm{Fun}(\Theta, \mathcal{S}) \xrightarrow{L} \mathrm{Seg}_{\Theta}(\mathcal{S})$$

abstract localization

Θ is <u>extendable</u> if $j_!$ lands in $\mathrm{Seg}_{\Theta}(\mathcal{S})$ - then $F_{\Theta} \simeq j_! i_*$

$$F_{\Theta} \Phi(E) \simeq \operatorname*{colim}_{O \rightsquigarrow E \in \mathrm{Act}_{\Theta}(E)} \ \lim_{\substack{O \\ \downarrow \\ E'}\in \Theta^{el}_{O/}} \Phi(E')$$

∞-gpd of active mor-s to E

$T_{\Theta} = U_{\Theta} F_{\Theta}$ Free Segal Θ-space monad.

Propn. (Um-H.): $\odot$ extendable $\Rightarrow$ $T_\odot$ is polynomial

Patterns from Polynomial Monads

(Weber)

Defn.: T polynomial monad on $\mathrm{Fun}(\mathcal{I}, \mathcal{S})$. Then T is a local right adjoint: $T : \mathrm{Fun}(\mathcal{I}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{S})_{/T(*)}$ has left adjoint L_*

$\mathcal{U}(T)^{op} \subseteq \mathrm{Fun}(\mathcal{I}, \mathcal{S})$ full subcat. on ob.s of the form $L_* I$ for some map $I \to T(*)$, $I \in \mathcal{I}^{op}$

$\mathcal{W}(T)^{op} \subseteq \mathrm{Alg}_T(\mathrm{Fun}(\mathcal{I}, \mathcal{S}))$ full subcat. of free alg.s on $\mathcal{U}(T)^{op}$

<u>Nerve Theorem</u> (For 1-cat.s Leinster, Weber, Berger–Melliès–Weber):

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{T}}(\mathrm{Fun}(\mathcal{I},\mathcal{S})) \arrow[r, hook, "\nu"] \arrow[d, "U_{\mathcal{T}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\mathcal{W}(\mathcal{T}),\mathcal{S}) \arrow[d, "j_{\mathcal{T}}^*"] \\
\mathrm{Fun}(\mathcal{I},\mathcal{S}) \arrow[r, hook, "i_{\mathcal{T},*}"'] & \mathrm{Fun}(\mathcal{U}(\mathcal{T}),\mathcal{S})
\end{tikzcd}$$

is cartesian and $j_{\mathcal{T},!}\, i_{\mathcal{T},*} \simeq \nu F_{\mathcal{T}}$.

$$\mathcal{I} \xhookrightarrow{i_{\mathcal{T}}} \mathcal{U}(\mathcal{T}) \xrightarrow{j_{\mathcal{T}}} \mathcal{W}(\mathcal{T})$$

$\Rightarrow$ "$\mathcal{T}$ almost comes from alg. pattern str. on $\mathcal{W}(\mathcal{T})$"

<u>But:</u> $\mathcal{U}(\mathcal{T})$ may not contain all eq.ces in $\mathcal{W}(\mathcal{T})$

$$F_T X \xrightarrow{\varphi} F_T Y \quad \rightsquigarrow \quad F_T X \to F_T L_* X \xrightarrow{F_T \psi} F_T Y$$

$$\leftrightarrow \quad \begin{tikzcd} X \arrow[r] \arrow[dr] & TY \arrow[d, "T(Y \to *)"] \\ & T* \end{tikzcd} \quad \leftrightarrow \quad L_* X \xrightarrow{\psi} Y$$

φ is <u>inert</u> if $F_T X \to F_T L_* X$ is eq.v

<u>active</u> if $F_T L_* X \to F_T Y$ is eq.v

<u>Thm</u> (Chu–H.): Inert & active maps give a fact. system on $W(T)$

$\rightsquigarrow$ alg. pattern w/ $W(T)^{el} =$ free on J

$W(T)$ is extendable

$$
\begin{tikzcd}
\mathrm{Alg}_T(\mathrm{Fun}(\mathcal{J}, \mathcal{S})) \arrow[r, "\sim"] \arrow[d] \arrow[dd, bend right=60, "U_T"'] & \mathrm{Seg}_{W(T)}(\mathcal{S}) \arrow[d] \\
\mathrm{Fun}(W(T)^{el}_1, \mathcal{S}) \arrow[r, "\sim"'] \arrow[d] & \mathrm{Seg}_{W(T)^{int}}(\mathcal{S}) \\
\mathrm{Fun}(\mathcal{J}_1, \mathcal{S}) &
\end{tikzcd}
$$

Saturated Patterns and Complete Monads

T is <u>complete</u> if $\mathcal{J} \longrightarrow W(T)^{el}$ is eq.ce

Complete polynomial monads on a localization of polynomial monads & are those monads that come from extendable patterns

Θ is <u>slim</u> if every ob. admits an active mor. to an elementary ob.

Θ is <u>saturated</u> if slim, extendable and

for $X \in \Theta$, $\quad X \xrightarrow{\sim} \lim\limits_{X \rightarrowtail E \in \Theta^{el}_{X/}} E$

Saturated patterns are exactly those that arise from polynomial monads & give a localization of slim patterns.

<u>Thm.</u> (Chu–H.): $\{$complete polynomial monads$\} \simeq \{$saturated patterns$\}$