Lifting of polynomial functors for logical reasoning

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Outline

Where we are, so far

Topic

- This talk is on polynomial functors, as specific interpretations of polynomial expressions
- This talk is a tutorial on "classic" work, from the late nineties, on the associated logic
  - so new research — actually very old work!
- It combines my own favourite work from two of my books:
  - *Categorical Logic and Type Theory* (North Holland, 1999)
  - *Introduction to Coalgebra* (CUP, 2016)

My own involvement with polynomial functors

(1) In coalgebra
- E.g. deterministic and non-deterministic automata are coalgebras of polynomial functors, in:

$$X \longrightarrow X^A \times 2 \qquad X \longrightarrow \mathcal{P}(X)^A \times 2$$

- The *Introduction to Coalgebra* book concentrates on polynomial functors — since they are most relevant in examples — not on functors in general.

(2) In a principled approach to logic for algebras/coalgebras, as datatypes
- Topic for today.
- “Classic stuff”, from: Hermida & Jacobs, *Structural induction and coinduction in a fibrational setting*, Inform. & Computation 1998

What are polynomial functors?

Informally: (endo)functors built-up inductively from primitives, via products & coproducts.

Definition may include:
- identity functor, and constant functors $X \mapsto C$;
- Powerset, list, distribution . . . (on Sets);
- Closure under products $X \mapsto F_1(X) \times F_2(X)$;
- Closure under coproducts $X \mapsto F_1(X) + F_2(X)$, possibly infinite;
- Possibly closure under “constant exponent” $X \mapsto F(X)^A$;
- Possibly closure under initial (or final) fixed point $X \mapsto \mu Y.F(X, Y)$.

We concentrate on:
- inductive build-up, not on preservation of structure;
- on finite products & coproducts — yielding “simple” poynomial functors

Main points

(1) Many (co)datatypes are initial/final coalgebras of a polynomial functor, defined on a category of types

(2) Logical principles for these (co)datatypes are obtained by initiality/finality, but for a lifting of the polynomial functor
- These principles are induction and coinduction
- This lifting happens from a category of types to a category of predicates or a category of relations
- Technically, this involves a fibration, of predicates over types

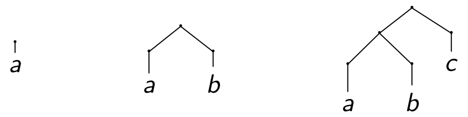
(3) Existence of initial/final objects for the lifted functor may result from comprehension or quotients in the logic.

Running example

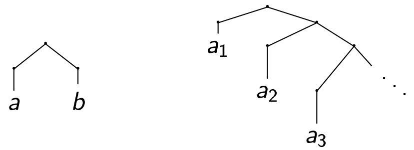
Fix a set of labels L and define a polynomial functor $T\colon \underline{\text{Sets}} \to \underline{\text{Sets}}$ as:

$$T(X) = L + (X \times X).$$

(1) Initial T-algebra $T(A) \xrightarrow{\cong} A$ Finite binary L-labeled trees, such as:

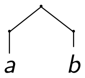


(2) Final T-coalgebra $Z \xrightarrow{\cong} T(Z)$ Finite & infinite binary L-labeled trees, like:



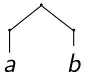
Explicit constructions

- If $\alpha = [\alpha_1, \alpha_2] \colon L + (A \times A) \xrightarrow{\cong} A$ is the initial algebra, then:



$$= \quad \alpha_2\big(\alpha_1(a), \alpha_1(b)\big) \;\in\; A.$$

- If $\zeta \colon Z \xrightarrow{\cong} L + (Z \times Z)$ is the final coalgebra, then:



$$= \quad \overline{f}(0)$$

where $\overline{f} \colon \{0,1,2\} \to Z$ is defined by finality in:

with:

$$f(0) = (1,2) \qquad f(1) = a \qquad f(2) = b$$

$$\begin{array}{ccc} L + (\{0,1,2\} \times \{0,1,2\}) & \xrightarrow{\mathrm{id} + (\overline{f} \times \overline{f})} & L + (Z \times Z) \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle \zeta}\,{\scriptstyle\cong} \\ \{0,1,2\} & \xrightarrow{\quad\overline{f}\quad} & Z \end{array}$$

Where we are, so far

Introduction

- The concept of fibration (or fibred category) arose in algebraic geometric, in the sixties, from work of Grothendieck and others
  - it's a categorical version of an indexed set $(X_i)_{i \in I}$, as function:

$$\begin{array}{c} \coprod_{i\in I} X_i \\ \downarrow \\ I \end{array} \qquad \text{or as} \qquad I \longrightarrow \underline{\mathrm{Sets}}$$

  - Categorically, this becomes:

$$\begin{array}{c} \mathbb{E} \\ \downarrow \\ \mathbb{B} \end{array} \qquad \text{or} \qquad \mathbb{B}^{\mathrm{op}} \longrightarrow \underline{\mathrm{Cat}}$$

- In logic and computer science, a fibration has become a standard categorical model for (typed) predicate logic
- See book *Categorical Logic and Type Theory* (North Holland, 1999).

The logical view on fibrations

$$\begin{array}{ll} \mathbb{E} \leftarrow\cdots & \text{category of predicates, over types} \\ {\scriptstyle p}\downarrow & \\ \mathbb{B} \leftarrow\cdots & \text{category of types} \end{array}$$

We will skip the formal definition, but only give the main idea, namely substitution

For each map $f \colon X \to Y$ in $\mathbb{B}$ and $Q \in \mathbb{E}$ "above" Y, that is, with $p(Q) = Y$, there is a suitably universal map $f^*(Q) \to Q$ above f.

$$\begin{array}{ccc} \mathbb{E} & \qquad & f^*(Q) \dashrightarrow Q \\ {\scriptstyle p}\downarrow & & \\ \mathbb{B} & & X \xrightarrow{\;f\;} Y \end{array}$$

In logical examples each fibre subcategory $\mathbb{E}_X \hookrightarrow \mathbb{E}$ of objects & maps above $X \in \mathbb{B}$ is a preorder.

A syntactic example (term/classifying model)

Definition

Let $\mathbb{T}$ have types σ as objects, in some type theory. A morphism $\sigma \to \tau$ is (an equivalence class of) a term $x\colon \sigma \vdash M(x)\colon \tau$.

Definition

Let $\mathbb{P}$ have type-proposition pairs (σ, φ) as objects, where $x\colon \sigma \vdash \varphi(x)\colon \mathrm{Prop}$. A map:

$$\big(x\colon \sigma \vdash \varphi\colon \mathrm{Prop}\big) \xrightarrow{M} \big(y\colon \tau \vdash \psi\colon \mathrm{Prop}\big) \quad \text{is} \quad \begin{cases} M\colon \sigma \to \tau \text{ with:} \\ x\colon \sigma \mid \varphi \vdash \psi[M/y] \end{cases}$$

Substitution is then substitution: for a term $M\colon \sigma \to \tau$ and a predicate $y\colon \tau \vdash \psi\colon \mathrm{Prop}$ on τ we get as predicate on σ,

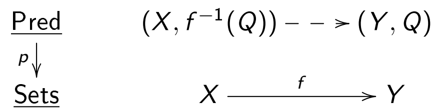
$$\big(x\colon \sigma \vdash \psi[M(x)/y]\colon \mathrm{Prop}\big) \xrightarrow{M} \big(y\colon \tau \vdash \psi\colon \mathrm{Prop}\big)$$

A set-theoretic example

Definition

Let category Pred have pairs (X, P) as objects, where $P \subseteq X$. A map $(X, P) \to (Y, Q)$ is a function $f\colon X \to Y$ with $x \in P \Rightarrow f(x) \in Q$, that is, if $P \subseteq f^{-1}(Q)$. It comes with Pred $\to$ Sets, given by $(X, P) \mapsto P$.

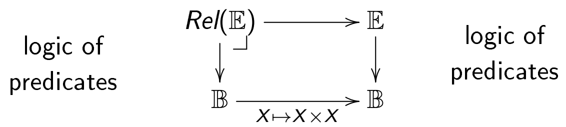
Substitution via inverse image:



There are many variations, like open/closed subsets of topological/metric/ordered spaces.

The logic of relations

If the base category $\mathbb{B}$ has products, we can form the fibration of relations via pullback:



Where we are, so far

(Co)products for types

We fix a fibration $\begin{array}{c}\mathbb{E}\\ \downarrow\\ \mathbb{B}\end{array}$ where the base category of types $\mathbb{B}$ has:

- finite products $(1, \times)$
- finite coproduct $(0, +)$
- distributivity of $\times$ over $+$.

Simple polynomial functors can then be interpreted as functors $F\colon \mathbb{B} \to \mathbb{B}$, once interpretations of constants are chosen.

For a set of labels L we thus have $T\colon \underline{\text{Sets}} \to \underline{\text{Sets}}$, via $T(X) = L + (X \times X)$.

(Co)products for predicates

The corresponding "logical" requirement is:

- each fibre $\mathbb{E}_X$ is a distributive lattice, with $\top, \wedge$ and $\bot, \vee$
- substitution f^* preserves this lattice structure.

Lemma

The total category $\mathbb{E}$ then has finite products: $\top \in \mathbb{E}_1$ is final, and the product of P, Q in $\mathbb{E}$ is given by:

$$\begin{array}{ccccc} P & \leftarrow\!- - & \pi_1^*(P) \wedge \pi_2^*(Q) & - -\!\rightarrow & Q \\ \\ X & \xleftarrow{\pi_1} & X \times Y & \xrightarrow{\pi_2} & Y \end{array}$$

Remark The final/top objects in the fibres give a right adjoint:

$$\begin{array}{c}\mathbb{E}\\ \downarrow \dashv \curvearrowright \top\\ \mathbb{B}\end{array}$$

Additional bifibration assumption

We also assume that our fibration is a bifibration. The easiest way formulation is: substitution functors f^* have left adjoints $\sum_f \dashv f^*$, as in:

$$\begin{array}{ccc} \mathbb{E}_X & \underset{\sum_f}{\overset{f^*}{\rightleftarrows}} \;(\top) & \mathbb{E}_Y \\ \\ X & \xrightarrow{f} & Y \end{array}$$

In presence of such sums, for $X \in \mathbb{B}$, consider the diagonal $\Delta = \langle \mathrm{id}, \mathrm{id} \rangle\colon X \to X \times X$ and define the equality relation as:

$$Eq(X) = \textstyle\sum_\Delta (\top_X). \qquad \text{giving} \qquad \begin{array}{c} Rel(\mathbb{E}) \\ Eq \curvearrowleft \downarrow \\ \mathbb{B} \end{array}$$

Coproduct in the global category

Lemma

In presence of sums $\sum$, the total category $\mathbb{E}$ has finite coproducts: $\bot \in \mathbb{E}_0$ is initial, and the coproduct of P, Q in $\mathbb{E}$ is given by:

$$\begin{array}{ccccc} P & - -\!\rightarrow & \sum_{\kappa_1}(P) \vee \sum_{\kappa_2}(Q) & \leftarrow\!- - & Q \\ \\ X & \xrightarrow{\kappa_1} & X + Y & \xleftarrow{\kappa_2} & Y \end{array}$$

Remark Basically the same constructions of products and coproducts work for relations — i.e. in $Rel(\mathbb{E})$

Predicate & relation lifting

- Under the previous assumptions, the total categories $\mathbb{E}$ and $Rel(\mathbb{E})$ have finite products & coproducts
- Hence, a polynomial functor F can not only be interpreted on the base category $\mathbb{B}$, but also on $\mathbb{E}$ and on $Rel(\mathbb{E})$
  - the only thing to decide is: what to do with constants?
  - an interpretation $C \in \mathbb{B}$ is changed to:
    - truth $\top \in \mathbb{E}_C$
    - equality $Eq(X) \in Rel(\mathbb{E})$
- This gives predicate lifting and relation lifting of F, by induction on the structure of F, in commuting rectangles:

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{Pred(F)} & \mathbb{E} \\ \downarrow & & \downarrow \\ \mathbb{B} & \xrightarrow{F} & \mathbb{B} \end{array} \qquad \begin{array}{ccc} Rel(\mathbb{E}) & \xrightarrow{Rel(F)} & Rel(\mathbb{E}) \\ \downarrow & & \downarrow \\ \mathbb{B} & \xrightarrow{F} & \mathbb{B} \end{array}$$

These lifted functors commute with truth $\top$ and equality Eq.

Where we are, so far

(Co)algebras of so many functors

- Computationally relevant are categories of (co)algebras $Alg(F)$ and $CoAlg(F)$ of $F: \mathbb{B} \to \mathbb{B}$
- But we can now also look at (co)algebras of lifted functors:

$Alg(Pred(F))$	$CoAlg(Pred(F))$	$Alg(Rel(F))$	$CoAlg(Rel(F))$
inductive predicates	invariants	congruences	bisimulations

- These are all predicate/relations which are suitably closed under the (co)algebraic operations.
- Nice illustrations of: *letting the formalism do the work for you*

Example: inductive predicate

Consider the set-theoretic fibration with a predicate $P \subseteq X$ carrying a $Pred(T)$-algebra, for $T(X) = L + (X \times X)$.

$$\begin{array}{ccc} Pred & & Pred(T)(P \subseteq X) \xrightarrow{\quad h \quad} (P \subseteq X) \\ \downarrow & & \\ \underline{\text{Sets}} & & L + (X \times X) \xrightarrow{h = [h_1, h_2]} X \end{array}$$

- where for $z \in T(X) = L + (X \times X)$,

$$Pred(T)(P \subseteq X)(a) \qquad \text{always holds, for } z = a \in L$$

$$Pred(T)(P \subseteq X)(x_1, x_2) \iff P(x_1) \wedge P(x_2), \quad \text{when } z = (x_1, x_2)$$

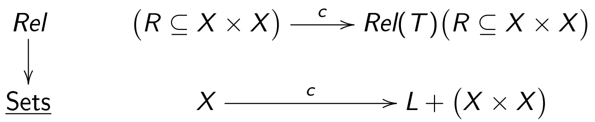
- The fact that $(P \subseteq X)$ carries an algebra thus means that it is closed under the algebra operations $h = [h_1, h_2]: L + (X \times X) \to X$, as in:

$$P(h_1(a)) \qquad \text{and} \qquad P(x_1) \wedge P(x_2) \implies P(h_2(x_1, x_2)).$$

  This is what we called an inductive predicate

Example: bisimulation

Consider a relation $R \subseteq X \times X$ carrying a $Rel(T)$-coalgebra



- where for $(z_1, z_2) \in (L + (X \times X)) \times (L + (X \times X))$,

$$Rel(T)(R \subseteq X \times X)(a_1, a_2) \iff a_1 = a_2$$
$$Rel(T)(R \subseteq X \times X)((x_1, x_2), (y_1, y_2)) \iff R(x_1, y_1) \wedge R(x_2, y_2)$$

- The relation $(R \subseteq X \times X)$ carries a coalgebra means that it is closed under the coalgebra operations $c\colon X \to L + (X \times X)$, as in:

$$R(x_1, x_2) \implies \begin{cases} c(x_1) = a_1 \in L \text{ iff } c(x_2) = a_2 \in L, \text{ and then } a_1 = a_2 \\ c(x_1) = (y_1, y_1') \text{ iff } c(x_2) = (y_2, y_2'), \text{and } R(y_1, y_2) \wedge R(y_1', y_2') \end{cases}$$

This is what's called a bisimulation

Where we are, so far

Back to diagrams

Lemma

Definition (The logic admits:)

(1) **induction** if $Alg(\top)\colon Alg(F) \to Alg(Pred(F))$ preserves initiality

(2) **coinduction** if $CoAlg(Eq)\colon CoAlg(F) \to CoAlg(Rel(F))$ preserves finality

Induction & coinduction

- induction means that each inductive predicate contains the image of the uniqe map from the initial algebra
- coinduction means that elements in a bisimulation are equal when mapped to the final coalgebra.

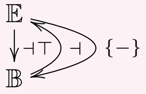
Aside: there is also a little-known relational version of induction: each congruence contains the image of the diagonal on the initial algebra.

- it's equivalent to the usual predicate version of induction

Induction from comprehension

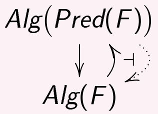
Definition (Lawvere)

A fibration admits comprehension if the truth functor has a right adjoint $\{-\}$, as in:



Lemma

Comprehension guarantees induction

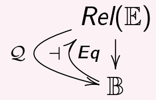


The functor $Alg(F) \to Alg(Pred(F))$ is then a left adjoint and thus preserves initiality

Coinduction from quotients

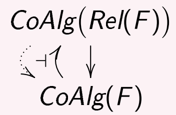
Definition (Jacobs)

A fibration admits qotients if the equality functor has a left adjoint $\mathcal{Q}$ in:



Lemma

Quotients guarantee coinduction



The functor $CoAlg(F) \to CoAlg(Rel(F))$ is then a right adjoint and thus preserves finality

Where we are, so far

Final remarks

- This structural approach to (co)induction has become mainstream in coalgebra
  - The paper from 1998 has 233 citations (Google Scholar)
  - sometimes called "Hermida-Jacobs" lifting
- Indeed, there are other / more general approaches to lifting functors, e.g.
  - via image-factorisation
  - codensity lifting
  - lifting via a parameter map, in presence of a generic object

  They typically coincide on simple polynomial functors.
- And many other variations & extensions, especially since the there are many variations of indistinguishability in coalgebra.

Thanks for your attention. Questions/remarks?