# Three kinds of polynomial functors

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This is a report on joint work with
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*Joachim Kock and David Spivak*

# (1)Introduction

The following set of notes was initially prepared for the talk I gave at the workshop on Polynomial Functors. I thank my collaborators for their precious help in improving the notes.

The goal of my talk was to show that the category of polynomial functors is *cartesian closed* for *three kinds* of polynomial functors. They are polynomial functors associated to the three kinds of categorical ring (or rig). Roughfly speaking, they are,

1. a symmetric monoidal closed category
2. a topos
3. a cartesian closed category

# (2)Introduction

The notions of categorical rig listed above are ambiguous, since we do not include the internal hom in the structure of these symmetric monoidal closed categories! We shall use the following alternative terminology for the three notions listed above:

1. a cosmos
2. a logos
3. a cartesian cosmos

A *cosmos* is a symmetric monoidal category $\mathcal{E}$ which is a (locally) presentable and in which the tensor product is distributive over colimits. These conditions imply that the internal hom exists, althought it is not generally preserved by homomorphisms of cosmoi.

# (3)Introduction

A cosmos is a ring-like structure in which the colimit operations are taking the role of the addition, and the tensor product is taking the role of the multiplication.

| *Sets* | *Categories* |
|---|---|
| *rigs* | *cosmos* |
| *addition* | *colimits* |
| *multiplication* | *tensor product* |

# (4)Introduction

Grothendieck topos can be viewed as a ring like structure, with the operations of colimits (resp. of finite limits) taking the role of the addition (resp. of the multiplication). We will call such a structure a *logos*.

A *cartesian cosmos* $\mathcal{E}$ is a cosmos in which the tensor product $\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is the cartesian product $\times : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$.

To each notion of categorical rig corresponds a notion of free rig and a notion of polynomial. If $\mathcal{V}$ is a cosmos, then every small $\mathcal{V}$-category $\mathbb{C}$ generates freely a $\mathcal{V}$-cosmos denoted $\mathcal{V}[\mathbb{C}]$. An element of $p \in \mathcal{V}[\mathbb{C}]$ is a *polynomial functor* with objective variables in $\mathbb{C}$. For example, if $\mathbb{C}$ is the unit $\mathcal{V}$-category with one object $X$, then a polynomial $p \in \mathcal{V}[X]$ is an exponential power series,

$$p(X) = \sum_{n \geq 0} p[n] \otimes_{\Sigma_n} X^{\otimes n}$$

# (5)Introduction

The category of $\mathcal{V}$-polynomial functors $Poly(\mathcal{V})$ is defined to be the opposite of the category of free $\mathcal{V}$-algebras. The category $Poly(\mathcal{V})$ is cartesian closed by [GJ][FGHW]. There is a similar result for the category of $L$-polynomial functors in the case of logoi, and the category of $C$-polynomial functors in the case of cartesian cosmoi [FJ]. The paper of Cattani and Winskel [CW] is an important source of categorification ideas. The cartesian closeness of the category of polynomial functors is true for a wider range of categorical rings ([Ol] Def. 2.3, Thm 3.1).

The characterisation of co-exponentiable $\mathcal{V}$-cosmoi is presently an open problem. When $\mathcal{V}$ is a quantale, a complete solution was obtained by Susan Niefield [Ni]. Our main conjecture is a categorification of her theory.

# (6)Introduction

In the first section of the notes, we recall the notions of cartesian closed category and of function space. In the second section, we recall a few basic aspects of the theory of commutative algebras and polynomials. In the third and fourth sections, we recall the basic aspects of theory of commutative quantales and we sketch Niefield's theory of co-exponentiable quantales. In the fifth section, we categorify the theory of quantales to obtain the theory of cosmoi and $\mathcal{V}$-cosmoi. In the sixth section, we describe the topos-theoretic analog of the theory of cosmoi. The seventh section is devoted to cartesian cosmoi.

We added an epilogue for some remarks and references.

# Content

Sections:

# Section 1: Cartesian closed categories and exponentials

Sub-sections:

- Cartesian closed categories
- Function spaces
- Co-exponentiable objects
- Co-exponentiable algebras
- Co-exponentiable boolean rings

# Cartesian closed categories

We will say that a category $\mathcal{C}$ is *cartesian* if it has finite cartesian products. Recall that a cartesian category $\mathcal{C}$ is said to be *closed* (to be a *CCC*) if the functor $A \times (-) : \mathcal{C} \to \mathcal{C}$ has a right adjoint $[A, -] : \mathcal{C} \to \mathcal{C}$ for every object $A \in \mathcal{C}$.

The object $[A, B] \in \mathcal{C}$ is often denoted $B^A$ and called the *space of maps* $A \to B$, or the *function space* $B^A$.

For examples, the category of sets *Set*, the category of posets *Poset*, and the category of small categories *Cat* are cartesian closed.

# Function spaces

Let $\mathcal{C}$ be a cartesian category.

The function space $B^A$ may exists for a pair of objects $A, B \in \mathcal{C}$, even if the category $\mathcal{C}$ is not closed.

By definition [Hyl], a *function space* $B^A$ is equipped with a map $\epsilon : B^A \times A \to B$ such that the map

$$\beta : Hom(Z, B^A) \to Hom(Z \times A, B)$$

defined by putting $\beta(f) = \epsilon \circ (f \times A)$ is bijective for every object $Z \in \mathcal{C}$.

An object $A \in \mathcal{C}$ is said to be *exponentiable* if the function space $B^A$ exists for every object $B \in \mathcal{C}$. This means that the functor $A \times (-) : \mathcal{C} \to \mathcal{C}$ has a right adjoint $(-)^A$.

A cartesian category $\mathcal{C}$ is closed if and only if every object of $\mathcal{C}$ is exponentiable.

# Co-exponentiable objects

We shall denote by $\mathcal{C}^{op}$ the opposite of a category $\mathcal{C}$.

To every morphism $f : A \to B$ in $\mathcal{C}$ corresponds a morphism $f^o : B^o \to A^o$ in $\mathcal{C}^{op}$.

We shall say that a category $\mathcal{C}$ is *co-cartesian* if the opposite category $\mathcal{C}^{op}$ is cartesian. We shall say that $\mathcal{C}$ is *co-cartesian closed* if $\mathcal{C}^{op}$ is cartesian closed.

Let $\mathcal{C}$ be a co-cartesian category, with the coproduct denoted $A \sqcup B$. We shall say that an object $A \triangleright B$ is the *co-exponential* of $B$ by $A$ if the object $(A \triangleright B)^o$ is the exponential of $B^o$ by $A^o$ in $\mathcal{C}^{op}$.

The object $A \triangleright B$ is equipped with a morphism $\eta : B \to A \sqcup (A \triangleright B)$ such the map

$$\beta : Hom(A \triangleright B, Z) \to Hom(B, A \sqcup Z)$$

defined by putting $\beta(f) = (A \sqcup f) \circ \eta$ is bijective for every object $Z \in \mathcal{C}$.

# (1) Co-exponentiable algebras

If $R$ is a commutative ring, we shall denote the category of commutative $R$-algebras by $CAlg(R)$.

Recall that the coproduct of two commutative $R$-algebras $A$ and $B$ is their tensor product $A \otimes_R B$. The initial $R$-algebra is the algebra $R$ itself. Hence the category $CAlg(R)$ is cocartesian.

If it exists, the $R$-algebra $A \triangleright B$ is equipped with an algebra homomorphism $\eta : B \to A \otimes_R (A \triangleright B)$ such that the map

$$\beta : Hom_R(A \triangleright B, Z) \to Hom_R(B, A \otimes_R Z)$$

defined by putting $\beta(f) = (A \otimes_R f) \circ \eta$ is bijective for every commutative $R$-algebra $Z$.

# (2) Co-exponentiable algebras

Let $R$ be a commutative ring. Not every $R$-algebra is co-exponentiable, but they have a simple characterisation.

Let $Mod(R)$ be the category of $R$-modules.

Recall that an $R$-module $M$ is said to be *dualisable* if the functor $M \otimes_R (-) : Mod(R) \to Mod(R)$ has a left adjoint. If $M$ is dualisable and $M^\star := Hom_R(M, R)$, then the functor $M^\star \otimes_R (-)$ is left adjoint to the functor $M \otimes_R (-)$. An $R$-module $M$ is dualisable if and only if it is finitely generated and projective.

**Theorem**

[Ni2] *An $R$-algebra $A$ is co-exponentiable if and only if the $R$-module $A$ is dualisable.*

# Co-exponentiable boolean rings

Recall that a commutative ring $R$ is *boolean* if $x^2 = x$ for every $x \in R$. A boolean ring is an $\mathbb{F}_2$-algebra.

Let $\mathbb{F}_2\langle S\rangle$ be the boolean ring freely generated by a set $S$.

For example, $\mathbb{F}_2\langle x\rangle := \mathbb{F}_2\langle\{x\}\rangle = \{0, 1, x, 1 + x\}$.

By Stone duality, a finite boolean ring is co-exponentiable.

Recall that the algebraic theory of boolean rings $\mathbb{B}$ has for objects the sets $2^n$ for $n \geq 0$ and for morphisms the maps $2^m \to 2^n$.

The category $\mathbb{B}$ is cartesian closed, since $(2^m)^{2^n} = 2^{m\times 2^n}$. It follows that

$$\mathbb{F}_2\langle S\rangle \triangleright \mathbb{F}_2\langle T\rangle = \mathbb{F}_2\langle 2^S \times T\rangle \tag{1}$$

for any finite sets $S$ and $T$. This formula is typical!

# Section 2: Rings and Polynomials

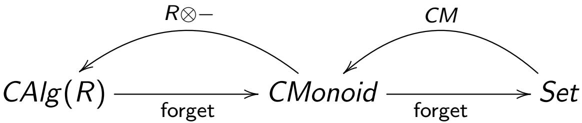
Sub-sections:

- Free commutative $R$-algebras
- $R$-polynomials
- The category $Poly(R)$

# Free commutative $R$-algebras

Let $R$ be a commutative ring.

Thanks to **Beck's distributive law**, the commutative $R$-algebra freely generated by a set $S$ can be constructed in two steps: $R[S] := R \otimes CM(S)$,




- $CM$ is the free commutative monoid functor
- $R \otimes (-)$ is the free $R$-module functor

If $M$ is a (commutative) monoid, then the $R$-module $R \otimes M$ has the structure of a (commutative) $R$-algebra, called the *envelopping $R$-algebra* of $M$.

# R-polynomials

An element $p \in R[S]$ is a *polynomial* with coefficients in $R$ and variables in $S$.

If $A$ is a commutative $R$-algebra, then $Hom(R[S], A) = A^S$.

If $\phi : R[S] \to R[T]$ is a homomorphism of $R$-algebras, then

$$Hom(\phi, A) : A^T \to A^S$$

is a *polynomial map*.

# The category $Poly(R)$

Let $R$ be a commutative ring.

We shall denote the category of free commutative $R$-algebras by $FreeCAlg(R)$.

The category of *$R$-polynomials* is defined to be the opposite category:

$$Poly(R) := FreeCAlg(R)^{op}$$

The category $Poly(R)$ is the algebraic theory of commutative $R$-algebras (in the Lawvere-Linton sense).

# Section 3: Rigs and Quantales

Sub-sections:

- Rigs
- Quantales
- Free sup-lattices
- Tensor product of sup-lattices
- Free commutative quantales
- Q-polynomials
- Co-exponentiable quantales
- The category *QPoly*
- What about frames?

# Rigs

Recall that a *rig* $R = (R, +, 0, \star, 1)$ is a ring-like structure except that the additive structure $(R, +, 0)$ is a commutative monoid, not a group in general. For example, the set of natural numbers $\mathbb{N} = (\mathbb{N}, +, 0, \times, 1)$ has the structure of a rig.

The notion of rig is vastly more general than that of ring, since the notion of commutative monoid is vastly more general than that of abelian group. For examples,

A distributive lattice $(L, \vee, 0, \wedge, 1)$ is a rig. In particular, a Boolean algebra $(B, \vee, 0, \wedge, 1, \neg)$ is a rig. Of course, every Boolean algebra has also the structure of a Boolean ring $(B, +, 0, \cdot, 1)$ if we let $x + y := (\neg x \wedge y) \vee (x \wedge \neg y)$ and $x \cdot y := x \wedge y$.

# Quantales

Recall that a poset $(R, \leq)$ is a *sup-lattice* if every subset $A \subseteq R$ as a supremum $sup(A) \in R$.

## Definition

A *quantale* is a sup-lattice $(R, \leq)$ equipped with a monoid structure $(R, \star, 1)$ such that

$$sup(A \star B) = sup(A) \star sup(B) \tag{2}$$

for subsets $A, B \subseteq R$, where $A \star B := \{a \star b \mid a \in A \text{ and } b \in B\}$.

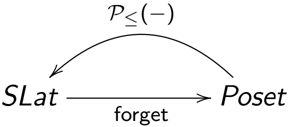
A quantale $(R, \leq, \star, 1)$ is *commutative* if the monoid $(R, \star, 1)$ is commutative.

For example, the interval of real numbers $[0, 1]$ equipped with the product of real numbers, is a commutative quantale.

# Free sup-lattices

Let us denote by *SLat* the category of sup-lattices and maps preserving suprema.

The forgetful functor $SLat \to Poset$ has a left adjoint:




If $S$ is a poset, then $\mathcal{P}_{\leq}(S)$ is the poset of downward closed subsets of $S$.

The map $y : S \to \mathcal{P}_{\leq}(S)$ defined by putting $y(s) = \{x \in S |\ x \leq s\}$ exihibits the sup-lattice freely generated by $S$.

# Tensor product of sup-lattices

The category *SLat* is symmetric monoidal closed.

The tensor product of two sup-lattices $X$ and $Y$ is the target of a map

$$X \times Y \to X \hat{\otimes} Y$$

which preserves suprema in each variables, and which is universal.

The unit object of the tensor product of sup-lattices is the poset $\mathbf{2}$.

The internal hom $SLat(X, Y)$ is the poset of sup-lattice maps $X \to Y$.

If $S$ and $T$ are posets, then the canonical map

$$P_{\leq}(S) \hat{\otimes} P_{\leq}(T) \to P_{\leq}(S \times T)$$

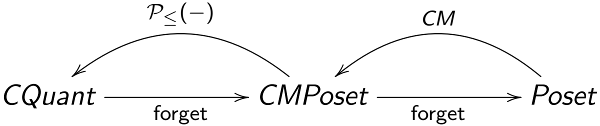
is an isomorphism.

# Free commutative quantales

Let us denote the category of commutative quantales by *CQuant*.

The commutative quantale freely generated by a poset $S$ can be constructed in two steps:

$$Q[S] = \mathcal{P}_{\leq}(CM(S)).$$




- $CM$ is the free commutative monoid functor
- If $M$ is a partially ordered monoid, then $\mathcal{P}_{\leq}(M)$ has the structure of a quantale: it is the *envelopping quantale* of $M$.

# Q-polynomials

An element $p \in Q[S]$ is a *quantalic polynomial* with variables in $S$.

For example, every $Q$-polynomial $p \in Q[x]$ is a supremum

$$p(x) = \bigvee_{n \in E} x^n$$

for some subset $E \subseteq \mathbb{N}$.

If $A$ is a commutative quantale, then $CQuant(Q[S], A) = A^S$.

If $\phi : Q[S] \to Q[T]$ is a homomorphism of quantales, then

$$Hom(\phi, A) : A^T \to A^S$$

is a *polynomial map*.

# Co-exponentiable quantales

A sup-lattice $L$ is said to be *dualisable* if the functor $L \otimes_S (-) : SLat \to SLat$ has a left adjoint.

A sup-lattice $L$ is dualisable if and only if it is a retract (by sup-lattice maps) of a free sup-lattice $\mathcal{P}_{\leq}(S)$.

### Theorem

[Ni] *A commutative quantale $A$ is co-exponentiable if and only if its underlying sup-lattice is dualisable.*

### Corollary

*The envelopping quantale of a commutative monoid $M \in Poset$ is co-exponentiable.*

In particular, a free commutative quantale is co-exponentiable.

# The category *QPoly*

### Theorem

[Lam] *If S and T are posets, then*

$$Q[S] \triangleright Q[T] = Q[CM(S)^{op} \times T]$$

Let us denote the category of free commutative quantales by *FreeCQuant*.

The opposite category $QPoly := FreeQant^{op}$ is the category of *Q-polynomials*.

### Corollary

[Lam] *The category QPoly is cartesian closed.*

# What about frames ?

A quantale $R = (R, \leq, \star, 1)$ is called a *frame* if $x \star y = x \wedge y := inf\{x, y\}$ for every $x, y \in R$.

The frame $Fr[S]$ freely generated by a poset $S$ is constructed in two steps: $Fr[S] = \mathcal{P}_{\leq}(Infs(S))$, where $Infs(S)$ is the inf-semi-lattice freely generated by $S$.

The frame $Fr[x] = \{0, x, 1\}$ is the lattice of open subsets of the *Sierpinski locale* $\mathbb{S}$.

A frame $R$ is co-exponentiable if and only if it is a continuous lattice [Sc][Hyl]. In particular, free frames are co-exponentiable.

We have

$$Fr[S] \triangleright Fr[T] = Fr[Infs(S)^{op} \times T]$$

for any pair of posets $S$ and $T$.

# Section 4: $R$-quantales

Let $R = (R, \leq, \star, 1)$ be a commutative quantale.

A commutative *$R$-quantale* is defined to be a commutative quantale $A$ equipped with a morphism of quantales $R \to A$. Niefield's theory [Ni] is devoted to the study of commutative $R$-quantales, for an arbitrary quantale $R$.

The relative theory is more powerful than the "absolute" theory. We shall briefly sketch the main lines.

# Section 4: Content

- R-posets
- Tensor product of R-posets
- R-monoids
- R-modules
- Free R-modules
- Tensor product of R-modules
- R-quantales
- Free commutative $R$-quantales
- $R$-polynomials
- Co-exponentiable $R$-quantales
- The category $QPoly(R)$

# $R$-posets

Let $R = (R, \leq, \star, 1)$ be a commutative quantale.

An *$R$-preorder* is a set $S$ equipped with a map $[-,-] : S \times S \to R$ satisfying

1. $[x, y] \star [y, z] \leq [x, z]$
2. $1 \leq [x, x]$

The relation $1 \leq [x, y]$ is a preorder relation $x \leq y$ on $S$.

An $R$-preorder $S$ is an *$R$-poset* if the preorder $x \leq y$ is a partial order.

The quantale $R$ is itself an $R$-poset.

A *morphism* of $R$-posets $S \to T$ is a map $f : S \to T$ such that $[x, y] \leq [fx, fy]$ for all $x, y \in S$.

We shall denote the category of $R$-posets by $Poset(R)$.

# Tensor product of $R$-posets

Let $R = (R, \leq, \star, 1)$ be a commutative quantale.

The category of $R$-posets $Poset(R)$ is symmetric monoidal closed.

The *tensor product* $S \otimes_R T$ of two $R$-posets $S$ and $T$ is their cartesian product $S \times T$, with

$$[(x, x'), (y, y')] := [x, y] \star [x', y']$$

The internal hom $Map_R(S, T)$ is the set of $R$-poset maps $S \to T$ with

$$[f, g] = \bigwedge_{x \in S} [f(x), g(x)]$$

for $f, g : S \to T$.

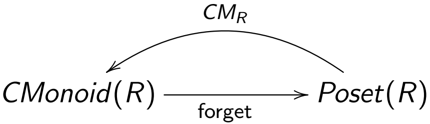
# $R$-monoids

Let $R = (R, \leq, \star, 1)$ be a commutative quantale.

An $R$-*monoid* is defined to be a monoid in the monoidal category of $R$-posets $Poset(R)$.

We shall denote the category of commutative $R$-monoids by $CMonoid(R)$.

Every $R$-poset $S$ generates freely a commutative $R$-monoid $CM_R(S)$.



# $R$-modules

Let $R = (R, \leq, \star, 1)$ be a commutative quantale.

The multiplication $\star : R \times R \to R$ preserves suprema in each variable.

A (commutative) quantale $R$ is the same thing as a (commutative) monoid $\star : R \hat{\otimes} R \to R$ in the symmetric monoidal category $SLat$.

An *$R$-module* is defined to be a sup-lattice $E \in SLat$ equipped with an associative action $R \hat{\otimes} E \to E$ of the monoid $R$.

A morphism of $R$-modules $f : E \to F$ is a morphism of sup-lattices respecting the actions by $R$; we shall say that $f$ is *$R$-linear*.

Every $R$-module has the structure of an $R$-poset.

We shall denote the category of $R$-modules and $R$-linear maps by $Mod(R)$.

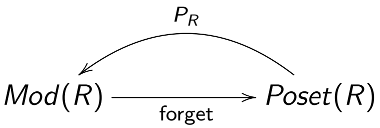
# Free $R$-modules

Let $R = (R, \leq, \star, 1)$ be a commutative quantale.

If $S$ is an $R$-poset, then the $R$-poset

$$P_R(S) := Map_R(S^{op}, R)$$

has the structure of an $R$-module. Moreover, the Yoneda map $y : S \to P_R(S)$ exhibits the $R$-module freely generated by $S$.



# Tensor product of $R$-modules

Let $R = (R, \leq, \star, 1)$ be a commutative quantale.

The category of $R$-modules $Mod(R)$ is symmetric monoidal closed.

The tensor product of two $R$-modules $E$ and $F$ is the target of a map

$$E \otimes_R F \to E \hat{\otimes}_R F$$

which is $R$-linear in each variable and universal.

The unit object of the tensor product of $R$-modules is the $R$-module $R$.

If $S$ and $T$ are $R$-posets, then the canonical map

$$P_R(S) \hat{\otimes}_R P_R(T) \to P_R(S \otimes_R T)$$

is an isomorphism.

# $R$-quantales

An $R$-*quantale* can be defined to be a monoid object in the category of $R$-modules.

A commutative $R$-*quantale* is the same thing as a commutative quantale $A$ equipped with a morphism of quantales $R \to A$.

We shall denote the category of commutative $R$-quantales by $CAlg(R)$.

If $M$ is an $R$-monoid, then the $R$-module $P_R(M)$ has the structure of an $R$-quantale: it is the *envelopping* $R$-*quantale* of $M$.

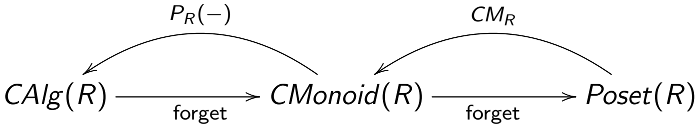
The quantale $P_R(M)$ is commutative if $M$ is commutative.

# Free commutative $R$-quantales

Let $R = (R, \leq, \star, 1)$ be a commutative quantale.

The commutative $R$-quantale freely generated by an $R$-poset $S$ can be constructed in two steps:

$$R[S] = P_R(CM_R(S)).$$



# $R$-polynomials

If $R$ is a commutative quantale and $S$ is an $R$-poset, then an element $p \in R[S]$ is a *polynomial* with coefficients in $R$ and (commutative) variables in $S$.

For example, a polynomial $p \in R[x]$ is a supremum

$$p(x) = \bigvee_{n=0}^{\infty} r[n] \cdot x^n$$

for a sequence of coefficients $r[-] : \mathbb{N} \to R$

If $A$ is a commutative $R$-quantale, then $Hom_R(R[S], A) = A^S$.

If $\phi : R[S] \to R[T]$ is a homomorphism of $R$-quantales, then

$$Hom_R(\phi, A) : A^T \to A^S$$

is a *polynomial map*.

# Co-exponentiable quantales

Let $R$ be a commutative quantale.

An $R$-module $E$ is said to be *dualisable* if the functor $E \hat{\otimes}_R (-) : Mod(R) \to Mod(R)$ has a left adjoint.

An $R$-module $E$ is dualisable if and only if it is a retract (by $R$-module maps) of a free $R$-module $P_R(S)$.

## Theorem

[Ni2] *A commutative $R$-quantale $A$ is co-exponentiable if and only if the underlying $R$-module is dualisable.*

## Corollary

*The envelopping $R$-quantale of a commutative $R$-monoid $M \in Poset(R)$ is co-exponentiable. A free commutative $R$-quantale is co-exponentiable.*

# The category $Poly(R)$

**Theorem**

[Lam] *If $S$ and $T$ are $R$-posets, then*

$$R[S] \triangleright R[T] = R[CM_R(S)^{op} \otimes_R T]$$

Let us denote the category of free commutative $R$-quantales by $CAlg(R)$.

The opposite category $Poly(R) := FreeCAlg(R)^{op}$ is the category of (quantalic) *$R$-polynomials*.

**Corollary**

[Lam] *The category $Poly(R)$ is cartesian closed.*

# Categorification

| | |
|---|---|
| *Sets, Posets* | *Categories* |
| *functions* | *functors* |
| *commutative monoids* | *symmetric monoidal categories* |
| *sup-lattices* | *presentable categories* |
| *commutative quantales* | *cosmoi, logoi and CCC* |

# Section 4: Cosmoi, $\mathcal{V}$-cosmoi and $\mathcal{V}$-polynomials

Sub-sections:

- Cosmoi
- $\mathcal{V}$-category theory
- Free symmetric monoidal $\mathcal{V}$-categories
- Day convolution product
- Presentable $\mathcal{V}$-categories
- $\mathcal{V}$-cosmos
- Free $\mathcal{V}$-cosmos
- $\mathcal{V}$-polynomials
- Co-exponentiable $\mathcal{V}$-cosmoi
- The category $Poly(\mathcal{V})$

# Cosmoi

## Definition

[Be][St] We will say that a symmetric monoidal category $\mathcal{V} = (\mathcal{V}, \otimes, I)$ is a *cosmos* if the following conditions hold:

1. the category $\mathcal{V}$ is (locally) presentable;
2. the functor $Z \otimes (-) : \mathcal{V} \to \mathcal{V}$ is cocontinuous for every object $Z \in \mathcal{V}$.

A cosmos is symmetric monoidal closed.

A *homomorphism* of cosmoi $\phi : \mathcal{R} \to \mathcal{A}$ is a cocontinuous symmetric monoidal functor.

Every commutative quantale is a cosmos.

# $\mathcal{V}$-category theory

Let $\mathcal{V}$ be a cosmos.

Category theory can be fully extended to $\mathcal{V}$-categories (Benabou[Ben], Borceux[Bor], Dubuc[Dub], Eilenberg-Kelly[EK], Kelly[Kel], Lack[KL], Day[Day], Street[Str]).

The category of $\mathcal{V}$-functors between a small $\mathcal{V}$-category $\mathbb{C}$ and a $\mathcal{V}$-category $\mathcal{E}$ (possibly large) is itself a $\mathcal{V}$-category $Fun_{\mathcal{V}}(\mathbb{C}, \mathcal{E})$.

The category of $\mathcal{V}$-categories (small or large) is symmetric monoidal.

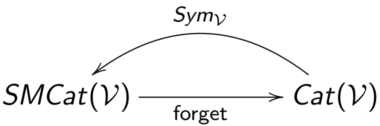
We shall denote the tensor product of two $\mathcal{V}$-categories $\mathcal{E}$ and $\mathcal{F}$ by $\mathcal{E} \otimes_{\mathcal{V}} \mathcal{F}$. The unit object is the $\mathcal{V}$-category $\mathbf{1}_{\mathcal{V}}$ with one object $\star$ and with $hom(\star, \star) := I$, where $I$ is the unit object of $\mathcal{V}$.

# Free symmetric monoidal $\mathcal{V}$-categories

The symmetric monoidal $\mathcal{V}$-category freely generated by a $\mathcal{V}$-category $\mathbb{C}$ has the following construction:

$$Sym_{\mathcal{V}}(\mathbb{C}) = \coprod_{n \geq 0} \mathbb{C}^{\otimes n}/\Sigma_n$$

where $\mathbb{C}^{\otimes n}/\Sigma_n$ denotes the wreath product $\Sigma_n \int \mathbb{C}^{\otimes n}$ for the natural action of the symmetric group $\Sigma_n$ on $\mathbb{C}^{\otimes n}$.

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# Presentable $\mathcal{V}$-categories

We say that a cocomplete $\mathcal{V}$-category $\mathcal{E}$ is *presentable* if it is presentable as an ordinary category.

Presentable $\mathcal{V}$-category $=$ $\mathcal{V}$-module

We shall denote by $Mod(\mathcal{V})$ the category of presentable $\mathcal{V}$-categories and cocontinuous $\mathcal{V}$-functors.

The category $Mod(\mathcal{V})$ is symmetric monoidal closed [Ke].

The *tensor product* of two presentable $\mathcal{V}$-categories $\mathcal{E}$ and $\mathcal{F}$ is the target of a $\mathcal{V}$-functor

$$\mathcal{E} \otimes_{\mathcal{V}} \mathcal{F} \to \mathcal{E} \hat{\otimes}_{\mathcal{V}} \mathcal{F}$$

cocontinuous in each variable and universal.

# Day convolution product

Recall that if $\mathbb{A}$ is a small monoidal $\mathcal{V}$-category, then the *convolution product* of two $\mathcal{V}$-presheaves $F, G : \mathbb{A}^{op} \to \mathcal{V}$ is the $\mathcal{V}$-presheaf $F \star G : \mathbb{A}^{op} \to \mathcal{V}$ defined by putting

$$(F \star G)(a) = \int^{a_1 \in \mathbb{A}} \int^{a_2 \in \mathbb{A}} F(a_1) \otimes F(a_2) \otimes Hom(a, a_1 \otimes a_2)$$

for every object $a \in \mathbb{A}$.

By a theorem of Day, the convolution product defines a monoidal structure on the $\mathcal{V}$-category

$$P_{\mathcal{V}}(\mathbb{A}) = Fun_{\mathcal{V}}(\mathbb{A}^{op}, \mathcal{V})$$

The monoidal structure is symmetric if the monoidal $\mathcal{V}$-category $\mathbb{A}$ is symmetric.

# $\mathcal{V}$-cosmos

A $\mathcal{V}$-*cosmos* is defined to be a cosmos $\mathcal{A}$ equipped with a homomorphism of cosmoi $\phi : \mathcal{V} \to \mathcal{A}$.

A $\mathcal{V}$-cosmos $\mathcal{A}$ is a symmetric monoidal object in the symmetric monoidal category of $\mathcal{V}$-modules $Mod(\mathcal{V})$.

We shall denote the category of $\mathcal{V}$-cosmoi by $CAlg(\mathcal{V})$.

If $\mathbb{A}$ is a small symmetric monoidal $\mathcal{V}$-category, then Day's convolution product gives the $\mathcal{V}$-category $P_{\mathcal{V}}(\mathbb{A})$ the structure of a $\mathcal{V}$-cosmos. Moreover, the Yoneda functor $Y : \mathbb{A} \to P_{\mathcal{V}}(\mathbb{A})$ exhibits the $\mathcal{V}$-cosmos freely generated by $\mathbb{A}$. We shall say that $P_{\mathcal{V}}(\mathbb{A})$ is the *envelopping* $\mathcal{V}$-*cosmos* of $\mathbb{A}$.

# (1) Free $\mathcal{V}$-cosmoi

The $\mathcal{V}$-cosmos freely generated by a small $\mathcal{V}$-category $\mathbb{C}$ can be constructed in two steps [GJ]:

$$\mathcal{V}[\mathbb{C}] = P_{\mathcal{V}}(Sym_{\mathcal{V}}(\mathbb{C})).$$

$$CAlg(\mathcal{V}) \underset{\text{forget}}{\dashrightarrow} SMCat(\mathcal{V}) \xrightarrow[\text{forget}]{} Cat(\mathcal{V}) \qquad P_{\mathcal{V}} : SMCat(\mathcal{V}) \to CAlg(\mathcal{V}), \quad Sym_{\mathcal{V}} : Cat(\mathcal{V}) \to SMCat(\mathcal{V}) \tag{4}$$

Remark: the first forgetful functor in the diagram above does not really exist, since a $\mathcal{V}$-cosmos can be large.

# (2) Free $\mathcal{V}$-cosmoi

The $\mathcal{V}$-cosmos freely generated by an *ordinary category* $\mathbb{C}$ can also be constructed in two steps:

$$\mathcal{V}[\mathbb{C}] = \mathit{Fun}(\mathit{Sym}(\mathbb{C})^{op}, \mathcal{V})$$

where $\mathit{Sym}(\mathbb{C})$ denotes the *ordinary* symmetric monoidal category freely generated by $\mathbb{C}$, and where $\mathit{Fun}(\mathit{Sym}(\mathbb{C})^{op}, \mathcal{V})$ is the category of *ordinary* functors.

For example, the $\mathcal{V}$-cosmos $\mathcal{V}[X]$ freely generated by *one object* $X$ is the category of *ordinary* functors $\mathit{Bij} \to \mathcal{V}$, where $\mathit{Bij}$ is the *ordinary* groupoid of finite sets and bijections.

# $\mathcal{V}$-polynomials

An object of the category $\mathcal{V}[\mathbb{C}]$ is a $\mathcal{V}$-*polynomial* with *objective variables* in the $\mathcal{V}$-category $\mathbb{C}$.

For example, a $\mathcal{V}$-polynomial in one variable $p(X) \in \mathcal{V}[X]$ is actually an *exponential power series*

$$p(X) = \sum_{n=0}^{\infty} p[n] \otimes_{\Sigma_n} X^{\otimes n}$$

where $p[-]$ is a functor $Bij \to \mathcal{V}$.

# On co-exponentiable $\mathcal{V}$-cosmoi

We say that a presentable $\mathcal{V}$-category $\mathcal{E}$ is *dualisable* if the functor $\mathcal{E} \,\hat{\otimes}_{\mathcal{V}}\, (-) : \mathit{Mod}(\mathcal{V}) \to \mathit{Mod}(\mathcal{V})$ has a left adjoint.

A presentable $\mathcal{V}$-category $\mathcal{E}$ is dualisable if and only if it is a retract (by $\mathcal{V}$-cocontinuous functors) of a free $\mathcal{V}$-category $\mathcal{P}_{\mathcal{V}}(\mathbb{C})$.

**Conjecture**: *A $\mathcal{V}$-cosmos $\mathcal{A}$ is co-exponentiable if and only if it is dualisable in the category $\mathit{Mod}(\mathcal{V})$.*

### Corollary

*The envelopping $\mathcal{V}$-cosmos $P_{\mathcal{V}}(\mathbb{A})$ of a small symmetric monoidal $\mathcal{V}$-category $\mathbb{A}$ is co-exponentiable.*

In particular, a free $\mathcal{V}$-cosmos $\mathcal{V}[\mathbb{C}]$ is co-exponentiable.

# The category $Poly(\mathcal{V})$

**Theorem**

[GJ][FGHW] *If $\mathbb{C}$ and $\mathbb{D}$ are small $\mathcal{V}$-categories, then*

$$\mathcal{V}[\mathbb{C}] \triangleright \mathcal{V}[\mathbb{D}] = \mathcal{V}[Sym_{\mathcal{V}}(\mathbb{C})^{op} \otimes_{\mathcal{V}} \mathbb{D}]$$

We shall denote the category of free $\mathcal{V}$-cosmoi by $FreeCAlg(\mathcal{V})$.

The category of $\mathcal{V}$-*polynomials* (or of $\mathcal{V}$-*analytic functors*) is defined to be the opposite category:

$$Poly(\mathcal{V}) := FreeCAlg(\mathcal{V})^{op}$$

**Corollary**

*The category of $\mathcal{V}$-polynomials $Poly(\mathcal{V})$ is cartesian closed.*

# Section 5: Topoi, logoi and $L$-polynomials

Sub-sections:

- Topoi and logoi
- Lex categories
- Free logoi
- $L$-polynomials
- The category $LPoly$

# Topoi and logoi

Recall that a *morphism* of (Gothendieck) topoi $\mathcal{E} \to \mathcal{F}$ is a pair of adjoint functor $\phi^\star \dashv \phi_\star$, where the left adjoint $\phi^\star : \mathcal{F} \to \mathcal{E}$ preserves finite limits.

We shall denote by *Logos* the opposite of the category of (Gothendieck) topoi. By definition, an object of the category *Logos* is a topos, but a morphism $\mathcal{E} \to \mathcal{F}$ is a cocontinuous functor $\phi : \mathcal{E} \to \mathcal{F}$ preserving finite limits.

We may say that an object of the category *Logos* is a *logos*.

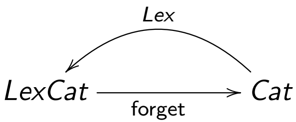
The notion of logos is to the notion of frame what the notion of topos is to the notion of locale.

The category of topoi is the opposite of the category of logoi.

# Lex categories

We shall say that a category $\mathbb{C}$ is *lex* if it has finite limits, and say that a functor between lex categories $\phi : \mathcal{E} \to \mathcal{F}$ is *lex* (= left exact) if it preserves finite limits.

We shall denote by *LexCat* the category of small lex categories and lex functors. The forgetful functor *LexCat* $\to$ *Cat* has a left adjoint which associates to a small category $\mathbb{C}$ the lex category *Lex*$(\mathbb{C})$ freely generated by $\mathbb{C}$.

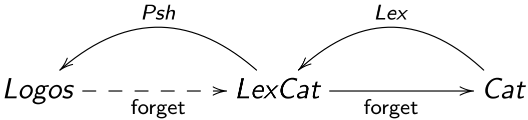


# Free logoi

If $\mathbb{A}$ is a lex category, then the Yoneda functor $Y : \mathbb{A} \to Psh(\mathbb{A})$ exihibits the logos freely generated by the lex category $\mathbb{A}$. We may say that $Psh(\mathbb{A})$ is the *envelopping logos* of the lex category $\mathbb{A}$.

The logos freely generated by a small category $\mathbb{C}$ is constructed in two steps, [SGA4] [Joh]:

$$LSet[\mathbb{C}] := Psh(Lex(\mathbb{C})) = [Lex(\mathbb{C})^{op}, Set]$$



# The category *LPoly*

A topos is exponentiable if and only if it is a continuous category [JJ].

## Theorem

[JJ][FJ] *Free logoi are co-exponentiable. If* $\mathbb{C}$ *and* $\mathbb{D}$ *are small categories, then*

$$LSet[\mathbb{C}] \triangleright LSet[\mathbb{D}] = LSet[Lex(\mathbb{C})^{op} \times \mathbb{D}]$$

We shall denote the category of free logoi by *FreeLogos*. The category of *L-polynomials* is defined to be the opposite category,

$$LPoly := (FreeLogos)^{op}$$

## Corollary

*The category LPoly is cartesian closed.*

# Section 6: Cartesian cosmoi and $C$-polynomials

Sub-sections:

- Cartesian cosmoi
- Free cartesian categories
- Free cartesian cosmoi
- $C$-polynomials
- The category *CPoly*

# Cartesian cosmoi

We say that a cosmos $(\mathcal{R}, \otimes, I)$ is *cartesian* if its tensor product is the cartesian product: $A \otimes B := A \times B$.

- A cartesian cosmos is cartesian closed.
- Every topos is a cartesian cosmos.
- Every frame is a cartesian cosmos.

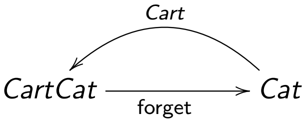
A *morphism* of cartesian cosmoi $\phi : \mathcal{E} \to \mathcal{F}$ is a cocontinuous functor preserving finite cartesian products.

We shall denote the category of cartesian cosmoi by $CCosmos$.

# Free cartesian categories

We shall say that a category $\mathbb{C}$ is *cartesian* if it has finite products. And we shall say that a functor between cartesian categories is *cartesian* if it preserves finite products.

We shall denote by *CartCat* the category of small cartesian categories and cartesian functors. The forgetful functor *CartCat* $\rightarrow$ *Cat* has a left adjoint which associates to a small category $\mathbb{C}$ the cartesian category *Cart*$(\mathbb{C})$ freely generated by $\mathbb{C}$.

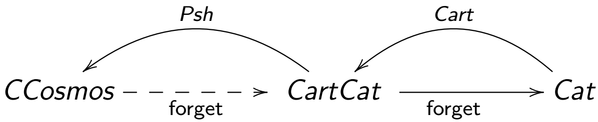


# Free cartesian cosmoi

If $\mathbb{A}$ is a cartesian category, then the Yoneda functor $Y : \mathbb{A} \to Psh(\mathbb{A})$ exihibits the cartesian cosmos freely generated by the cartesian category $\mathbb{A}$. We may say that $Psh(\mathbb{A})$ is the *envelopping cartesian cosmos* of the cartesian category $\mathbb{A}$.

The cartesian cosmos freely generated by a small category $\mathbb{K}$ can be constructed in two steps:

$$CSet[\mathbb{K}] := Psh(Cart(\mathbb{K})) = [Cart(\mathbb{K})^{op}, Set]$$



# The category *CPoly*

### Theorem

[FJ][Gal][Ol] *Free cartesian cosmoi are co-exponentiable. If* $\mathbb{K}$ *and* $\mathbb{D}$ *are small categories, then*

$$CSet[\mathbb{K}] \triangleright CSet[\mathbb{D}] = CSet[Cart(\mathbb{K})^{op} \times \mathbb{D}]$$

Let us denote the category of free cartesian cosmoi by *FreeCCosmos*. The category of *C-polynomials* is defined to be the opposite category,

$$CPoly := (FreeCCosmos)^{op}.$$

### Corollary

[FJ][Gal][Ol] *The category CPoly is cartesian closed.*

# Epilogue

The construction of a cartesian closed category from continuous lattices is due to Scott [Sc]. Exponentiable locales where studied by Hyland [Hyl] and by Niefield [Ni1].

The fact that the category of free commutative quantales is co-cartesian closed was first observed by Lamarche [Lam]. Co-exponentiable quantales were characterized by Niefield [Ni2].

The tensor product of categories that are cocomplete for a class of colimits was introduced by Kelly [Kel]. The tensor product of Grothendieck abelian categories was introduced by Deligne [Del] and studied by Di Liberti & Gonzales [DLG]. See also López Franco [LF].

The theory of exponentiable topoi was extended to $\infty$-topoi by Anel & Lejay [AL] and also by Lurie [HA]. The theory of $\mathcal{V}$-cosmoi was extended to $\infty$-categories by Lurie [HA].

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