Polynomials as spans

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Workshop on Polynomial Functors
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Initial idea

- A polynomial from X to Y in a category $\mathscr{C}$ is a diagram of the shape $X \xleftarrow{m_2} E \xrightarrow{m_1} S \xrightarrow{p} Y$ with m_1 a powerful (= exponentiable) morphism in $\mathscr{C}$.
- Such diagrams can be thought of as generalizing spans: a span $X \xrightarrow{(m_2,S,p)} Y$ amounts to the case where $E = S$ and m_1 is the identity.
- Our simple idea was to make the diagram more complicated by including an identity thus:

$$X \xleftarrow{m_2} E \xrightarrow{m_1} S \xleftarrow{1_S} S \xrightarrow{p} Y \ ,$$

resulting in a span

$$X \xleftarrow{(m_1,E,m_2)} S \xrightarrow{(1_S,S,p)} Y$$

of spans from X to Y.

Initial idea, continued

- Of course, the bicategory $\mathrm{Spn}\mathscr{C}$ of spans in $\mathscr{C}$ does not have all bicategorical pullbacks.
- Fortunately, polynomials are not general spans and sufficient pullbacks can be constructed.
- Indeed, that is what Weber's distributivity pullbacks around a pair of composable morphisms in $\mathscr{C}$ construct.
- That construction requires the use of powerful morphisms in $\mathscr{C}$.
- So what is it about the bicategory $\mathrm{Spn}\mathscr{C}$ that allows these restricted spans to form the bicategory of polynomials in $\mathscr{C}$.

Eine kleine Kategorientheorie: Cartesian morphisms

- Let $p : E \to B$ be a functor. A morphism $\chi : e' \to e$ in E is called *cartesian*[1] for p when the square (1) is a pullback for all $k \in E$.

$$\begin{array}{ccc} E(k,e') & \xrightarrow{E(k,\chi)} & E(k,e) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B(pk,pe') & \xrightarrow[B(pk,p\chi)]{} & B(pk,pe) \end{array} \qquad (1)$$

- Note that all invertible morphisms in E are cartesian.

- If p is fully faithful then all morphisms of E are cartesian.

[1] Classically called "strongly cartesian"

Eine kleine Kategorientheorie: Groupoid fibrations

We call the functor $p : E \to B$ a *groupoid fibration* when

(i) for all objects $e \in E$ and morphisms $\beta : b \to pe$ in B, there exist a morphism $\chi : e' \to e$ in E and isomorphism $b \cong pe'$ whose composite with $p\chi$ is β, and

(ii) every morphism of E is cartesian for p.

From the pullback (1), it follows that groupoid fibrations are conservative (that is, reflect invertibility).

Proposition

The Grothendieck fibration construction (wreath-product-like) 2-functor

$$\wr : \mathrm{Hom}(B^{\mathrm{op}}, \mathrm{Gpd}) \longrightarrow \mathrm{GFib}B \tag{2}$$

is a biequivalence.

Eine kleine Kategorientheorie: Fundamental groupoid

- The 2-adjunction

$$\mathrm{Cat} \underset{\mathrm{incl}}{\overset{\pi_1}{\rightleftarrows}} \mathrm{Gpd}$$

induces a biadjunction

$$\mathrm{Fib}B \underset{\mathrm{incl}}{\overset{\pi_1}{\rightleftarrows}} \mathrm{GFib}B$$

Eine kleine Kategorientheorie: Ultimate functors

- A functor $j : A \to E$ is called *ultimate* when, for all objects $e \in E$, the fundamental groupoid $\pi_1(e/j)$ of the comma category e/j (called $e \downarrow j$ by Mac Lane) is equivalent to the terminal groupoid:

$$\pi_1(e/j) \simeq \mathbf{1} \ .$$

- Every right adjoint functor is ultimate.
- Every coinverter (localization) is ultimate.

Proposition

The ultimate functors and groupoid fibrations form a bicategorical factorization system on Cat*. In particular, every functor* $f : A \to B$ *factors uniquely up to equivalence as* $f \cong (A \xrightarrow{j} E \xrightarrow{p} B)$ *where* j *is ultimate and* p *is a groupoid fibration.*

Eine kleine Kategorientheorie: Abstract polynomial functors

- A functor $f : A \to B$ is an *abstract polynomial functor* when, in its factorization
$$f \cong (A \xrightarrow{j} E \xrightarrow{p} B)$$
as per the last Proposition, the functor j is a right adjoint.
- I define the abstract polynomial inducing f simply to be the span
$$A \xleftarrow{j_*} E \xrightarrow{p} B$$
where $j_* \dashv j$.

Proposition

Polynomial functors compose up to isomorphism.

Proof.

Take $A \xrightarrow{j} E \xrightarrow{p} B \xrightarrow{k} F \xrightarrow{q} C$ with $j_* \dashv j$, $k_* \dashv k$ and with p, q groupoid fibrations. Form the pseudopullback

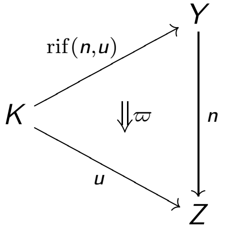
$$\begin{array}{ccc} P & \xrightarrow{\quad p' \quad} & F \\ {\scriptstyle k'_*}\downarrow & \overset{\theta}{\underset{\cong}{\Longleftarrow}} & \downarrow{\scriptstyle k_*} \\ E & \xrightarrow[\quad p \quad]{} & B \end{array} \tag{3}$$

to obtain the required "distributive law". One easily verifies there exists $k'_* \dashv k'$, p' is a groupoid fibration and the Chevalley-Beck condition

$$p' \circ k' \cong k \circ p$$

holds. So $q \circ k \circ p \circ j \cong q \circ p' \circ k' \circ j$ where $q \circ p'$ is a groupoid fibration and $k' \circ j$ is a right adjoint. □

Groupoid fibrations and lifters in bicategories

- Groupoid fibrations in a bicategory $\mathscr{M}$ are defined representably: a morphism $p : E \to B$ is a *groupoid fibration* when, for all $K \in \mathscr{M}$, the functor $\mathscr{M}(K, p) : \mathscr{M}(K, E) \to \mathscr{M}(K, B)$ is a groupoid fibration.
- 

  The defining property of a right lifting $\mathrm{rif}(n, u)$ of u through n is that pasting a 2-cell $v \Longrightarrow \mathrm{rif}(n, u)$ onto the triangle to give a 2-cell $nv \Longrightarrow u$ defines a bijection.
- A morphism $n : Y \to Z$ is called a *right lifter* when $\mathrm{rif}(n, u)$ exists for all $u : K \to Z$.

Examples of lifters

Example

Left adjoint morphisms in any $\mathscr{M}$ are right lifters (since the lifting is the composite with the right adjoint). In Cat all lifters are left adjoints.

Example

Composites of right lifters are right lifters.

Example

Suppose $\mathscr{M} = \mathrm{Spn}\mathscr{C}$ with $\mathscr{C}$ a finitely complete category. If $f : A \to B$ is powerful (= exponentiable, meaning that the functor $\mathscr{C}/B \to \mathscr{C}/A$, which pulls back along f, has a right adjoint Π_f) in $\mathscr{C}$ then $f^* : B \to A$ is a right lifter. The formula is $\mathrm{rif}(f^*, (v, T, q)) = (w, U, r)$ where

$$(U \xrightarrow{(w,r)} K \times B) = \Pi_{1_K \times f}(T \xrightarrow{(v,q)} K \times A) \ .$$

More examples

Example

Suppose $m = (m_1, E, m_2)$ is a morphism in $\mathscr{M} = \mathrm{Spn}\mathscr{C}$ with $\mathscr{C}$ a finitely complete category. Then m is a right lifter if and only if m_1 is powerful. The previous Examples imply "if". Conversely, apply Dubuc's Adjoint Triangle Theorem.

Example

Let $\mathscr{E}$ be a regular category and let $\mathrm{Rel}\mathscr{E}$ be the locally ordered bicategory of relations in $\mathscr{E}$. The objects are those of $\mathscr{E}$ and the morphisms $(r_1, R, r_2) : X \to Y$ are jointly monomorphic spans $X \xleftarrow{r_1} R \xrightarrow{r_2} Y$ in $\mathscr{E}$. Put $\mathrm{Sub}X = \mathrm{Rel}\mathscr{E}(1, X)$. For $f : Y \to X$, pulling back subobjects of X along f defines an order-preserving function $f^{-1} : \mathrm{Sub}X \to \mathrm{Sub}Y$ whose right adjoint, if it exists, is denoted by $\forall_f : \mathrm{Sub}Y \to \mathrm{Sub}X$. We can see that $(r_1, R, r_2) : X \to Y$ is a right lifter in $\mathrm{Rel}\mathscr{E}$ if and only if $\forall_{r_1}$ exists.

Bipullbacks of spans and Weber's distributivity pullbacks

Proposition

Suppose $\mathscr{C}$ is a category with pullbacks. Then the pseudofunctor $(-)_* : \mathscr{C} \to \mathrm{Spn}\mathscr{C}$ takes pullbacks to bipullbacks.

$$\begin{array}{ccccc} X & \xrightarrow{\;p\;} & Z & \xrightarrow{\;g\;} & A \\ {\scriptstyle q}\big\downarrow & & & & \big\downarrow{\scriptstyle f} \\ Y & & \xrightarrow[\;r\;]{} & & B \end{array} \qquad\qquad \begin{array}{ccc} Y & \xrightarrow{\;r_*\;} & B \\ {\scriptstyle p_* q^*}\big\downarrow & \overset{\Longleftarrow}{\cong} & \big\downarrow{\scriptstyle f^*} \\ Z & \xrightarrow[\;g_*\;]{} & A \end{array} \tag{4}$$

Proposition

Take $Z \xrightarrow{g} A \xrightarrow{f} B$ in a category $\mathscr{C}$ with pullbacks. The left diagram in (4) *is a pullback around (f, g) in the category $\mathscr{C}$ iff a square as on the right of* (4) *exists in the bicategory $\mathrm{Spn}\mathscr{C}$. The left diagram is a distributivity pullback around (f, g) in $\mathscr{C}$ iff the right diagram is a bipullback in $\mathrm{Spn}\mathscr{C}$.*

Calibrations of bicategories

Definition (Modelled on Jean Bénabou's notion for categories)

A class $\mathscr{P}$ of "neat" morphisms is a *calibration of the bicategory* $\mathscr{M}$ when:

- P0. all equivalences are neat and, if p is neat and there exists an invertible 2-cell $p \cong q$, then q is neat;
- P1. for all neat p, the composite $p \circ q$ is neat if and only if q is neat;
- P2. every neat morphism is a groupoid fibration;
- P3. every cospan of the form $S \xrightarrow{p} Y \xleftarrow{n} T$, with n a right lifter and p neat, has a bipullback (5) in $\mathscr{M}$ with $\tilde{p}$ neat.

$$\begin{array}{ccc} P & \xrightarrow{\tilde{p}} & T \\ {\scriptstyle \tilde{n}}\downarrow & \overset{\theta}{\underset{\cong}{\Longleftarrow}} & \downarrow{\scriptstyle n} \\ S & \xrightarrow[p]{} & Y \end{array} \qquad (5)$$

Calibrated bicategories

- A bicategory equipped with a calibration is called *calibrated*.
- Notice that the class GF of all groupoid fibrations in any bicategory $\mathscr{M}$ satisfies all the conditions for a calibration except perhaps the bipullback existence part of P3 (automatically $\tilde{p}$ will be a groupoid fibration).
- A bicategory $\mathscr{M}$ is called *polynomic* when GF is a calibration of $\mathscr{M}$.
- Cat is polynomic.
- If $\mathscr{C}$ is a finitely complete category then the bicategory $\mathrm{Spn}\mathscr{C}$ is polynomic. The groupoid fibrations are those spans with left leg invertible.
- If $\mathscr{E}$ is a regular category then the bicategory $\mathrm{Rel}\mathscr{E}$ is calibrated where neat means those relations with left leg invertible and right leg a monomorphism.

Polynomials in calibrated bicategories

Definition

A *polynomial* (m, S, p) from X to Y in $\mathscr{M} = (\mathscr{M}, \mathscr{P})$ is a span

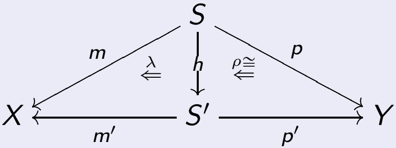
$$X \xleftarrow{m} S \xrightarrow{p} Y$$

in $\mathscr{M}$ with m a right lifter and p neat.

Morphisms of polynomials in a calibrated bicategory

Definition

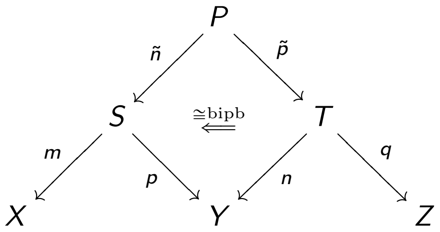
A *polynomial morphism* $(\lambda, h, \rho) : (m, S, p) \to (m', S', p')$ is a diagram



(6)

in which ρ is invertible. We call (λ, h, ρ) *strong* when λ is invertible. A 2-cell $\sigma : h \Rightarrow k : (m, S, p) \to (m', S', p')$ is a 2-cell $\sigma : h \Rightarrow k : S \to S'$ in $\mathscr{M}$ compatible with λ and ρ. Actually, σ must be invertible. Write $\mathrm{Poly}\mathscr{M}(X, Y)$ for the Poincaré category of the bicategory of polynomials from X to Y so obtained.

The bicategory of polynomials in a calibrated bicategory



(7)

This usual composition of spans is the effect on objects of functors

$$\circ : \mathrm{Poly}\mathscr{M}(Y, Z) \times \mathrm{Poly}\mathscr{M}(X, Y) \longrightarrow \mathrm{Poly}\mathscr{M}(X, Z)\ . \qquad (8)$$

Proposition

There is a bicategory $\mathrm{Poly}\mathscr{M}$ *of polynomials in a calibrated bicategory* $\mathscr{M}$. *The objects are those of* $\mathscr{M}$, *the homcategories are the* $\mathrm{Poly}\mathscr{M}(X, Y)$. *Composition is given by the functors* (8). *The vertical and horizontal stacking properties of bipullbacks provide the associativity isomorphisms.*

Some interpretations

Example

If $\mathscr{C}$ is a finitely complete category then the bicategory $\mathrm{PolySpn}\mathscr{C}$ is biequivalent to the bicategory denoted by $\mathrm{Poly}_{\mathscr{C}}$ by Gambino-Kock and by $\mathrm{Poly}(\mathscr{C})$ by Charles Walker. Moreover, $\mathrm{Poly}_{\mathrm{strong}}\mathrm{Spn}\mathscr{C}$ is biequivalent to Walker's bicategory $\mathrm{Poly}_{\mathrm{c}}(\mathscr{C})$.

Example

If $\mathscr{E}$ is a regular category then the bicategory $\mathrm{PolyRel}\mathscr{E}$ is biequivalent to the subbicategory of the usual $\mathrm{Poly}_{\mathscr{E}}$ consisting of those polynomials $X \xleftarrow{m_2} E \xrightarrow{m_1} S \xrightarrow{p} Y$ for which $(m_1, m_2) : E \to S \times X$ and $p : S \to Y$ are monomorphisms.

Profunctors = distributors = bimodules = directed modules

Objects of the bicategory Mod are categories.
The homcategories are the functor categories $\mathrm{Mod}(A, B) = [B^{\mathrm{op}} \times A, \mathrm{Set}]$ whose objects $m : B^{\mathrm{op}} \times A \to \mathrm{Set}$ are called modules directed from A to B.
Composition is defined by the coends $(n \circ m)(c, a) = \int^b m(b, a) \times n(c, b)$.
Each functor $f : A \to B$ gives a module $f_* : A \to B$ defined by $f_*(b, a) = B(b, fa)$.

Example

The bicategory Mod is calibrated by taking as neat modules those equivalent to p_* for p a discrete fibration. The bicategory $\mathrm{PolyMod}$ is biequivalent to the subbicategory of the Weber polynomial bicategory of the category Cat consisting of those polynomials $X \xleftarrow{m_2} E \xrightarrow{m_1} S \xrightarrow{p} Y$ for which $S \xleftarrow{m_1} E \xrightarrow{m_2} X$ is a two-sided discrete fibration from S to X and p is a discrete fibrations. Such polynomials are equivalent to parametric right adjoint functors $[X^{\mathrm{op}}, \mathrm{Set}] \longrightarrow [Y^{\mathrm{op}}, \mathrm{Set}]$.

Opposites of Kleisli categories of composite monads

There is another viewpoint on $\mathrm{PolyRel}\mathscr{E}$ and $\mathrm{PolyMod}$ described in my Cahiers paper with the same title as this talk. It seems in the spirit of André Joyal's talk of yesterday. The Kleisli category of a monad is the Linton theory corresponding to the monad.

Example

An elementary topos $\mathscr{E}$ admits two basic constructions, the power object $\mathcal{P}X$ and the partial map classifier $\widetilde{X}$. Both define object assignments for monads on $\mathscr{E}$. There is a distributive law $d_X : \mathcal{P}\widetilde{X} \to \widetilde{\mathcal{P}X}$ between the two monads. The classifying category of $\mathrm{PolyRel}\mathscr{E}$ is equivalent to the opposite of the Kleisli category $\mathscr{E}_{\widetilde{\mathcal{P}(-)}}$ for the composite monad $X \mapsto \widetilde{\mathcal{P}X}$.

Opposites of Kleisli categories of composite monads, continued

Example

The bicategory $\mathrm{PolyMod}$ is biequivalent to the opposite of the Kleisli bicategory for the composite $X \mapsto \mathrm{Fam}^{\mathrm{op}}[X^{\mathrm{op}}, \mathrm{Set}]$ of the colimit-completion pseudomonad and the product-completion pseudomonad (modulo obvious size issues).

Some pseudofunctors

Remark

i. *If the bicategory* $\mathscr{M}$ *is calibrated then each* $\mathscr{M}(K,-) : \mathscr{M} \to \mathrm{Cat}$ *is a calibrated bicategory pseudofunctor.*

ii. *Recall from an earlier Proposition that polynomial functors compose. That provides a pseudofunctor*

$$\mathrm{PolyCat} \longrightarrow \mathrm{Cat}, \qquad (X \xleftarrow{m} S \xrightarrow{p} Y) \mapsto (X \xrightarrow{pm^*} Y) \; .$$

From polynomials in bicategories to polynomial functors

Proposition

If the bicategory $\mathscr{M}$ *is calibrated then, for each* $K \in \mathscr{M}$*, there is a pseudofunctor* $\mathbb{H}_K : \mathrm{Poly}\mathscr{M} \longrightarrow \mathrm{Cat}$ *taking the polynomial* $X \xleftarrow{m} S \xrightarrow{p} Y$ *to the abstract polynomial functor which is the composite*

$$\mathscr{M}(K,X) \xrightarrow{\mathrm{rif}(m,-)} \mathscr{M}(K,S) \xrightarrow{\mathscr{M}(K,p)} \mathscr{M}(K,Y)$$

in Cat*.*

For $\mathscr{M} = \mathrm{SpnSet}$ and $K = \mathbf{1}$, the displayed composite is the usual polynomial functor $\mathrm{Set}/X \longrightarrow \mathrm{Set}/Y$ associated to a polynomial from X to Y.

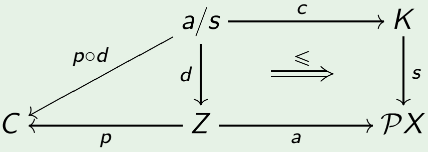
From polynomials in $\mathrm{Rel}\mathscr{E}$ to polynomial functors

Example

For topos $\mathscr{E}$ and $\mathscr{M} = \mathrm{Rel}\mathscr{E}$, the pseudofunctor $\mathbb{H}_K : \mathrm{PolyRel}\mathscr{E} \longrightarrow \mathrm{Ord}$ takes $C \xleftarrow{p} Z \xrightarrow{a} \mathcal{P}X$ to the order-preserving function

$$\mathrm{Rel}\mathscr{E}(K, X) \xrightarrow{\mathrm{rif}(a,-)} \mathrm{Rel}\mathscr{E}(K, Z) \xrightarrow{p\circ -} \mathrm{Rel}\mathscr{E}(K, C)$$

whose value at a relation $(s_1, S, s_2) : K \to X$ is the relation $(c, a/s, p \circ d) : K \to C$ as in the diagram below in which the square has the comma property and s classifies the relation (s_1, S, s_2).



From polynomials in Mod to polynomial functors

Example

For $\mathscr{M} = \mathrm{Mod}$, the pseudofunctor

$$\mathbb{H}_K : \mathrm{PolyMod} \longrightarrow \mathrm{Cat}$$

takes the morphism $Y \xleftarrow{p} S \xrightarrow{m} \mathrm{Psh}$ to the functor

$$[K, \mathrm{Psh}X] \longrightarrow [K, \mathrm{Psh}Y] \;,\; \ell \mapsto \bar{\ell}$$

where

$$(\bar{\ell}k)y = \sum_{s \in S_y} \mathrm{Psh}X(ms, \ell k)$$

for $k \in K$, for $y \in Y$ and for S_y the fibre of $p : S \to Y$ over y.

Thank You

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