Tutorial on *Polynomial Functors and Type Theory* Part II

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Outline

4. Universes in presheaves

Recall the notion of a **Hofmann-Streicher universe**

$$\dot{\mathsf{V}} \to \mathsf{V}$$

in a category of presheaves $\widehat{\mathbb{C}} = \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}}$.

1. Let set $\hookrightarrow$ Set be the full subcategory of *small* sets $s < \kappa$.
2. Let $\dot{\mathsf{set}} = 1/\mathsf{set}$ be the category of small *pointed* sets.
3. Then for $c \in \mathbb{C}$ let:

$$\mathsf{V}(c) = \mathsf{Cat}\big(\mathbb{C}/_c{}^{\mathsf{op}}, \mathsf{set}\big) \text{ the } \textit{set} \text{ of small presheaves on } \mathbb{C}/_c,$$
$$\dot{\mathsf{V}}(c) = \mathsf{Cat}\big(\mathbb{C}/_c{}^{\mathsf{op}}, \dot{\mathsf{set}}\big) \text{ ... small } \textit{pointed} \text{ presheaves on } \mathbb{C}/_c.$$

4. The action on $d \to c$ is given by *pre*composition with *post*composition $\mathbb{C}/_d \to \mathbb{C}/_c$.
5. There is a natural transformation $\dot{\mathsf{V}} \to \mathsf{V}$ determined by composing with the forgetful functor $\dot{\mathsf{set}} \to \mathsf{set}$

4. Universes in presheaves

Definition

In a category $\widehat{\mathbb{C}} = \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ of presheaves,

- an object A is *small* if its values $A(c)$ are small, for all $c \in \mathbb{C}$,
- a map $A \to X$ is *small* if its fibers $A_x = x^*A$ are small, for all $x : \mathrm{y}c \to X$,

$$\begin{array}{ccc} A_x & \longrightarrow & A \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathrm{y}c & \xrightarrow[x]{} & X. \end{array}$$

Note that small maps are stable under pullback.
And that the map $\dot{\mathsf{V}} \to \mathsf{V}$ is small, since the fiber $\dot{\mathsf{V}}_S$ over $S : \mathrm{y}c \to \mathsf{V}$ has as elements pointed presheaves $\dot{S} : \mathbb{C}/_c \to \dot{\mathsf{set}}$.

4. Universes in presheaves

Proposition

For every small map $A \to X$ there is a canonical classifying map $\alpha : X \to \mathsf{V}$ fitting into a pullback diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & \dot{\mathsf{V}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ X & \xrightarrow[\alpha]{} & \mathsf{V}. \end{array}$$

Proof.

Do it first for the small maps $A_x \to \mathsf{y}c$, for all $x : \mathsf{y}c \to X$, for which there is a canonical choice of $\alpha_x : \mathsf{y}c \to \mathsf{V}$. Then use the presentation of X as a colimit over its category of elements $(c, x) \in \int_{\mathbb{C}} X$ to get $\alpha : X \to \mathsf{U}$. □

4. Universes in presheaves

Remark

For large enough κ the small maps are closed under the adjoints $\Sigma_A \dashv A^* \dashv \Pi_A$ to pullback along small maps $A \to X$.

This fact gives rise to natural operations on the universe $\dot{\mathsf{V}} \to \mathsf{V}$ that can be used to (coherently!) model the corresponding type-forming operations, as follows:

- a universe $\dot{\mathsf{V}} \to \mathsf{V}$ is a natural model on the category of contexts $\widehat{\mathbb{C}}$,
- a universe $\dot{\mathsf{V}} \to \mathsf{V}$ *generates* a polynomial endofunctor

$$P : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}.$$

- The type forming operations in the natural model will be seen to correspond to an algebraic structure on the polynomial endofunctor.

5. Polynomial monad and type formers

Let $p : \dot{\mathsf{U}} \to \mathsf{U}$ be a natural model on an arbitrary category $\mathbb{C}$, and consider the associated *polynomial endofunctor*,

$$P = \mathsf{U}_! \circ p_* \circ \dot{\mathsf{U}}^* : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}\,,$$

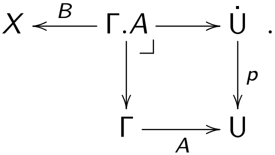
which we can write as,

$$P(X) \;=\; \sum_{A\,:\,\mathsf{U}} X^{[A]},$$

where $[A] = p^{-1}(A)$ is the fiber of $p : \dot{\mathsf{U}} \to \mathsf{U}$ at $A : \mathsf{U}$.

Lemma

Maps $\Gamma \to P(X)$ *correspond naturally to pairs* (A, B) *where*



□

5. Polynomial monad and type formers

Applying P to U itself therefore gives the object

$$P\mathsf{U} = \sum_{A:\mathsf{U}} \mathsf{U}^{[A]}$$

for which maps $\Gamma \to P\mathsf{U}$ correspond naturally to pairs (A, B) of the form,

$$\begin{array}{ccccc}
\mathsf{U} & \xleftarrow{B} & \Gamma.A & \longrightarrow & \dot{\mathsf{U}} \\
 & & \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\
 & & \Gamma & \xrightarrow[A]{} & \mathsf{U}
\end{array}$$

Since maps $\Gamma \to \mathsf{U}$ correspond naturally to types in context $\Gamma \vdash A$, we see that maps $\Gamma \to P\mathsf{U}$ correspond naturally to types in the extended context $\Gamma.A \vdash B$.

5. Polynomial monad and type formers

Proposition

For a natural model $\dot{\mathsf{U}} \to \mathsf{U}$, the polynomial object

$$P\mathsf{U} = \sum_{A:\mathsf{U}} \mathsf{U}^{[A]}$$

classifies types in context. Specifically, there is a natural isomorphism between maps $\Gamma \to P\mathsf{U}$ and pairs (A, B) where

$$\Gamma.A \vdash B.$$

Similarly, the object

$$P\dot{\mathsf{U}} = \sum_{A:\mathsf{U}} \dot{\mathsf{U}}^{[A]}$$

models *terms* in context: pairs $(A, b : B)$ where $\Gamma.A \vdash b : B$, for (A, B) the composite with $P\dot{\mathsf{U}} \to P\mathsf{U}$.

5. Polynomial monad and type formers: Π

Proposition

The natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for products just if there are maps λ, Π making the following a pullback.

$$\begin{array}{ccc} P\dot{\mathsf{U}} & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\ {\scriptstyle Pp}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ P\mathsf{U} & \xrightarrow[\Pi]{} & \mathsf{U} \end{array}$$

5. Polynomial monad and type formers: Π

Proposition

The map $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for products just if there are maps λ, Π making the following a pullback.

Proof:

$$\sum_{A:\mathsf{U}} \mathsf{U}^{[A]}$$

$$\begin{array}{ccc} P(\dot{\mathsf{U}}) & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\ \downarrow & & \downarrow \\ P(\mathsf{U}) & \xrightarrow[\Pi]{} & \mathsf{U} \end{array}$$

5. Polynomial monad and type formers: Π

Proposition

The map $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for products just if there are maps λ, Π making the following a pullback.

Proof:

$$\begin{array}{ccc} P(\dot{\mathsf{U}}) & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\ \downarrow & & \downarrow \\ \sum_{A:\mathsf{U}} \mathsf{U}^{[A]} \qquad P(\mathsf{U}) & \xrightarrow[\Pi]{} & \mathsf{U} \end{array}$$

$A \vdash B$ $\qquad\qquad$ $\Pi_A B$

5. Polynomial monad and type formers: Π

Proposition

The map $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for products just if there are maps λ, Π making the following a pullback.

Proof:

$A \vdash b : B$ $\qquad\qquad$ $\lambda_A b$

$$
\begin{array}{ccccc}
\sum_{A:\mathsf{U}} \dot{\mathsf{U}}^{[A]} & & P(\dot{\mathsf{U}}) & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\
 & & \downarrow & & \downarrow \\
\sum_{A:\mathsf{U}} \mathsf{U}^{[A]} & & P(\mathsf{U}) & \xrightarrow[\Pi]{} & \mathsf{U}
\end{array}
$$

$A \vdash B$ $\qquad\qquad$ $\Pi_A B$

5. Polynomial monad and type formers: Π

Proposition

The map $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for products just if there are maps λ, Π making the following a pullback.

Proof:

$$\begin{array}{ccccc} & & & & f \\ \sum_{A:\mathsf{U}} \dot{\mathsf{U}}^{[A]} & & P(\dot{\mathsf{U}}) & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\ & & \downarrow & & \downarrow \\ \sum_{A:\mathsf{U}} \mathsf{U}^{[A]} & & P(\mathsf{U}) & \xrightarrow[\Pi]{} & \mathsf{U} \\ & & & & \\ A \vdash B & & & & \Pi_A B \end{array}$$

5. Polynomial monad and type formers: Π

Proposition

The map $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for products just if there are maps λ, Π making the following a pullback.

Proof:

$A \vdash fx : B$ $\qquad\qquad$ $\lambda_A fx = f$

$$\begin{array}{ccccc}
\sum_{A:\mathsf{U}} \dot{\mathsf{U}}^{[A]} & & P(\dot{\mathsf{U}}) & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\
 & & \downarrow & & \downarrow \\
\sum_{A:\mathsf{U}} \mathsf{U}^{[A]} & & P(\mathsf{U}) & \xrightarrow[\Pi]{} & \mathsf{U}
\end{array}$$

$A \vdash B$ $\qquad\qquad$ $\Pi_A B$

5. Polynomial monad and type formers: Σ

Proposition

The map $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for sums just if there are maps (pair, Σ) making the following a pullback

$$\begin{array}{ccc} Q & \xrightarrow{\;\mathsf{pair}\;} & \dot{\mathsf{U}} \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ P(\mathsf{U}) & \xrightarrow[\;\Sigma\;]{} & \mathsf{U} \end{array}$$

where $q = p \triangleleft p : Q \to P(\mathsf{U})$ is the generating map of the composite $P_q = P_{p \triangleleft p} = P_p \circ P_p$.

Explicitly:

$$Q \;=\; \sum_{A:\mathsf{U}} \sum_{B:\mathsf{U}^A} \sum_{x:A} B(x)$$

5. Polynomial monad and type formers: T

Rules for a terminal type T

$$\frac{}{\vdash \mathsf{T}} \qquad \frac{}{\vdash * : \mathsf{T}} \qquad \frac{}{x : \mathsf{T} \vdash x = * : \mathsf{T}}$$

Proposition

The map $p : \dot{\mathsf{U}} \to \mathsf{U}$ *models the rules for a terminal type just if there are maps* $(*, \mathsf{T})$ *making the following a pullback.*

$$\begin{array}{ccc} 1 & \xrightarrow{\;*\;} & \dot{\mathsf{U}} \\ \downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow[\;\mathsf{T}\;]{} & \mathsf{U} \end{array}$$

5. Polynomial monad

Consider the pullback squares for T and Σ.

$$
\begin{array}{ccc}
1 & \xrightarrow{\;*\;} & \dot{\mathsf{U}} \\
\big\downarrow & & \big\downarrow{\scriptstyle p} \\
1 & \xrightarrow[\;\mathsf{T}\;]{} & \mathsf{U}
\end{array}
\qquad\qquad
\begin{array}{ccc}
Q & \xrightarrow{\;\mathsf{pair}\;} & \dot{\mathsf{U}} \\
{\scriptstyle p \triangleleft p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
P(\mathsf{U}) & \xrightarrow[\;\Sigma\;]{} & \mathsf{U}
\end{array}
$$

These determine cartesian natural transformations between the corresponding polynomial endofunctors.

$$\tau : 1 \Rightarrow P \qquad\qquad \sigma : P \circ P \Rightarrow P$$

5. Polynomial monad

Theorem (A-Newstead)

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the $\top$ and Σ type formers just if the associated polynomial endofunctor P has the structure maps of a cartesian monad.

$$\tau : 1 \Rightarrow P \qquad\qquad \sigma : P \circ P \Rightarrow P$$

What about the monad laws?

5. Polynomial monad

The monad laws correspond to the following type isomorphisms.

$\sigma \circ P\sigma = \sigma \circ \sigma_P$	$\sum_{a:A} \sum_{b:B(a)} C(a,b) \;\cong\; \sum_{(a,b):\sum_{a:A} B(a)} C(a,b)$
$\sigma \circ P\tau = 1$	$\sum_{a:A} 1 \;\cong\; A$
$\sigma \circ \tau_P = 1$	$\sum_{x:1} A \;\cong\; A$

5. Polynomial monad

The pullback square for Π

$$\begin{array}{ccc} P\dot{\mathsf{U}} & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\ {\scriptstyle Pp}\downarrow & & \downarrow{\scriptstyle p} \\ P\mathsf{U} & \xrightarrow[\Pi]{} & \mathsf{U} \end{array}$$

determines a cartesian natural transformation

$$\pi : P^{\mathbf{2}}p \Rightarrow p$$

where $P^{\mathbf{2}} : \hat{\mathbb{C}}^{\mathbf{2}} \to \hat{\mathbb{C}}^{\mathbf{2}}$ is the extension of P to the arrow category.

5. Polynomial monad

Theorem (A-Newstead)

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the Π type former just if it has an algebra structure for the extended endofunctor $P^{\mathbf{2}}$,

$$\pi : P^{\mathbf{2}} p \Rightarrow p.$$

5. Polynomial monad

The algebra laws correspond to the following type isomorphisms.

$\pi \circ P\pi \; = \; \pi \circ \sigma$	$\prod_{a:A} \prod_{b:B(a)} C(a,b) \; \cong \; \prod_{(a,b):\sum_{a:A} B(a)} C(a,b)$
$\pi \circ \tau \; = \; 1$	$\prod_{x:1} A \; \cong \; A$

6. Propositions and types

We can compare these operations on types

$$\Sigma, \Pi : P\mathsf{U} \longrightarrow \mathsf{U}$$

with those on subobjects of objects A in the topos $\widehat{\mathbb{C}}$,

$$\exists_A, \forall_A : \Omega^A \longrightarrow \Omega.$$

Consider

$$P\Omega \;=\; \sum_{A:\mathsf{U}} \Omega^A$$

for the polynomial endofunctor of $\dot{\mathsf{U}} \to \mathsf{U}$.
We then have the comparable maps

$$\exists, \forall : P\Omega \longrightarrow \Omega.$$

6. Propositions and types

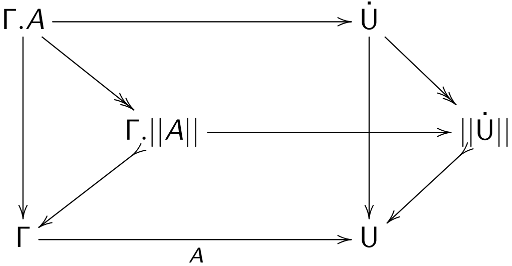
Proposition

There is a retraction $i : \Omega \rightarrowtail \mathsf{U}$, $s : \mathsf{U} \twoheadrightarrow \Omega$ *such that the following squares commute.*

$$
\begin{array}{ccc}
P\Omega & \xrightarrow{\exists} & \Omega \\
{\scriptstyle Pi}\downarrow & & \uparrow{\scriptstyle s} \\
P\mathsf{U} & \xrightarrow[\Sigma]{} & \mathsf{U}
\end{array}
\qquad\qquad
\begin{array}{ccc}
P\Omega & \xrightarrow{\forall} & \Omega \\
{\scriptstyle Pi}\downarrow & & \uparrow{\scriptstyle s} \\
P\mathsf{U} & \xrightarrow[\Pi]{} & \mathsf{U}
\end{array}
$$

6. Propositions and types

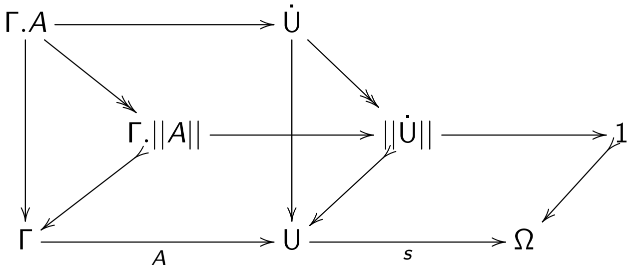
For the proof, factor the natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ as on the right below.



So $||\dot{\mathsf{U}}|| \rightarrowtail \mathsf{U}$ is a universal family of small propositions.

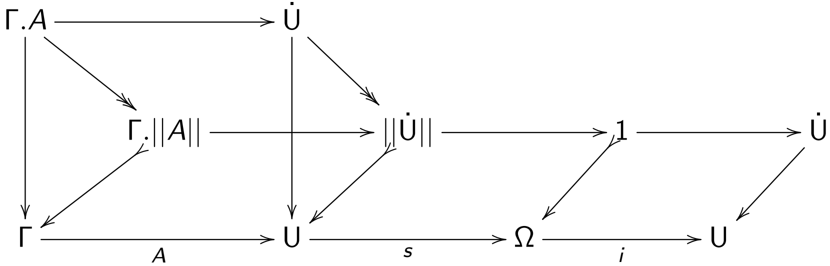
6. Propositions and types

Let $s : \mathsf{U} \to \Omega$ classify the mono $||\dot{\mathsf{U}}|| \rightarrowtail \mathsf{U}$.



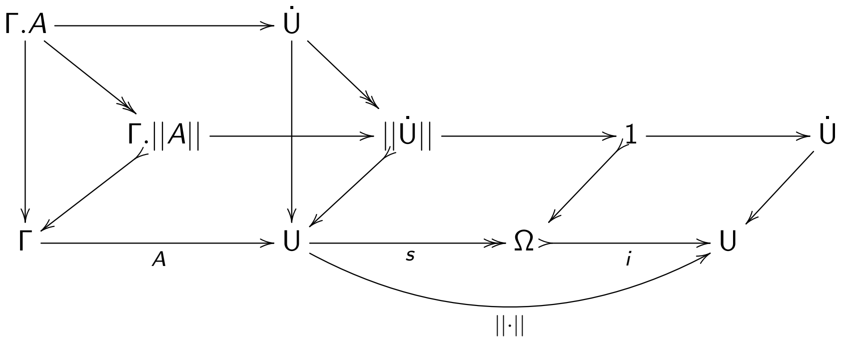
6. Propositions and types

Let $s : \mathsf{U} \to \Omega$ classify the mono $||\dot{\mathsf{U}}|| \rightarrowtail \mathsf{U}$.



Let $i : \Omega \to \mathsf{U}$ classify the family of small propositions $1 \rightarrowtail \Omega$.

6. Propositions and types



Let

$$||\cdot|| := i \circ s : \mathsf{U} \to \mathsf{U}.$$

We have

$$s \circ i = 1 : \Omega \to \Omega.$$

So

$$\Omega = \mathrm{im}(||\cdot||).$$

6. Propositions and types

The following diagrams then commute, as required.

$$
\begin{array}{ccc}
\sum_{A:\mathsf{U}} \Omega^A & \xrightarrow{\exists} & \Omega \\
{\scriptstyle Pi}\downarrow & & \uparrow{\scriptstyle s} \\
\sum_{A:\mathsf{U}} \mathsf{U}^A & \xrightarrow[\Sigma]{} & \mathsf{U}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\sum_{A:\mathsf{U}} \Omega^A & \xrightarrow{\forall} & \Omega \\
{\scriptstyle Pi}\downarrow & & \uparrow{\scriptstyle s} \\
\sum_{A:\mathsf{U}} \mathsf{U}^A & \xrightarrow[\Pi]{} & \mathsf{U}
\end{array}
$$

$\square$

References

1. Awodey, S. (2017) Natural models of homotopy type theory, MSCS 28(2). arXiv:1406.3219
2. Awodey, S. and N. Gambino and S. Hazratpour (2022) Kripke-Joyal semantics for homotopy type theory. arXiv:2110.14576
3. Awodey, S. and C. Newstead (2018) Polynomial pseudomonads and dependent type theory. arXiv:1802.00997
4. Spivak, D. (2022) A summary of categorical structures in Poly. arXiv:2202.00534
5. Newstead, C. (2018) Algebraic Models of Dependent Type Theory, CMU PhD thesis. arXiv:2103.06155

Appendix: Natural models of HoTT

Theorem

A category $\mathbb{C}$ with a terminal object 1 admits a natural model of Homotopy Type Theory if it has a class of maps $\mathcal{D}$ satisfying the following conditions:

- **total:** *every $C \to 1$ is in $\mathcal{D}$,*
- **stable:** *$\mathcal{D}$ is closed under pullbacks along all maps in $\mathbb{C}$,*
- **closed:** *$\mathcal{D}$ is closed under composition and under dependent products along all maps in $\mathcal{D}$,*
- **factorizing:** *every map $f : A \to B$ in $\mathbb{C}$ factors as $f = d \circ a$ with $a \in {}^{\pitchfork}\mathcal{D}$ and $d \in \mathcal{D}$.*

Proof.

Uses the main idea of the Lumsdaine-Warren coherence theorem: a left-adjoint splitting of the fibration of $\mathcal{D}$-maps. □

Appendix: Natural models of HoTT

Examples of categories satisfying the conditions of the theorem:

- Kan complexes with the right wfs on sSets.
- Any right-proper Cisinski model category (restricted to the fibrant objects).
- Groupoids, n-Groupoids, ∞-Groupoids.
- Joyal's πh-tribes.
- The syntactic category of contexts of type theory itself.