From polynomial functors to functor calculi

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Plan

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   2.2 Abelian functor calculus
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Perspective – Polynomial Functors via Calculus

Taylor polynomials

Given a nice function $f : \mathbb{R} \to \mathbb{R}$, its nth Taylor polynomial

- is a polynomial of degree $\leq n$, and, hence, may be easier to work with than f,
- provides an approximation of f in a prescribed way (at least within the radius of convergence), and
- becomes a better approximation to f as n increases.

Perspective – Polynomial Functors via Calculus

Goal:
Define new ways to associate a sequence of functors $\{P_n F\}_{n\geq 0}$ to a functor F so that

- $P_n F$ is "nice" in the sense of some property indexed by n ($P_n F$ is degree n),
- there is a natural transformation $\eta_n : F \Rightarrow P_n F$ such that $P_n F$ is universal among degree n functors with natural transformations from F ($P_n F$ is related to F),
- the induced natural transformations $P_m \eta_n : P_m F \Rightarrow P_m P_n F$ and $\eta_n : P_m F \to P_n P_m F$ are equivalences when $m \leq n$ ($P_n F$ preserves degree m part of F).

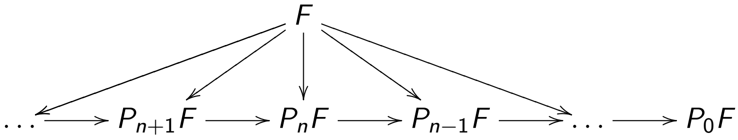
Perspective – Equivalence

Coming from a topological point of view, we work in settings with a weaker notion of equivalence, which we'll denote $\simeq$:

- Topological spaces: $f : X \xrightarrow{\simeq} Y$ iff f is a weak homotopy equivalence, that is, $f_j : \pi_j(X) \to \pi_j(Y)$ is an isomorphism for all $j \geq 0$.
- Chain complexes: $f : A_* \xrightarrow{\simeq} B_*$ iff f is a chain homotopy equivalence or, alternatively, a f is a quasi-isomorphism.
- (Simplicial) model categories: $f : C \xrightarrow{\simeq} D$ iff f is a weak equivalence.

Theorem [Goodwillie, 2003]

For a functor F of spaces or spectra that preserves weak homotopy equivalences, there is a Taylor tower of functors and natural transformations



such that

- for all $n \geq 0$, P_nF is an n-excisive functor,
- if F is "nice," the tower converges to F on sufficiently "nice" objects ($F(x) \to P_nF(x)$ is roughly $(n+1)k$-connected when x is k-connected), and
- P_nF is universal (up to a zig-zag of weak equivalences) among n-excisive functors with natural transformations from F.

n-excisive functors – n-cubical diagrams

$\mathcal{P}(n)$ is the poset of subsets of $\mathrm{n} = \{1, 2, \ldots, n\}$.
E.g., $\mathcal{P}(2)$ is

$$\begin{array}{ccc} \emptyset & \longrightarrow & \{1\} \\ \downarrow & & \downarrow \\ \{2\} & \longrightarrow & \{1,2\} \end{array}$$

An n-cube in a category $\mathcal{C}$ is a functor from $\mathcal{P}(n)$ to $\mathcal{C}$. E.g., a 2-cube is a commuting square

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_{12} \end{array},$$

a 3-cube diagram is a commuting cube in $\mathcal{C}$, etc.

n-excisive functors

- A n-cubical diagram of spaces (spectra) is strongly cocartesian if every 2-face is a homotopy pushout square.
- A functor of spaces (spectra) is n-excisive if it takes strongly cocartesian $(n+1)$-cubical diagrams to homotopy cartesian (homotopy pullback) diagrams.

Examples

- The identity functor of spaces Id is not n-excisive for any n. (Homotopy pullback n-cubes are not the same as homotopy pushout n-cubes in spaces.)
- (Snaith splitting) For the functor from spaces to spectra, $F : X \mapsto \Sigma^\infty \Omega \Sigma X$,

$$P_m F(X) \simeq \prod_{1 \leq n \leq m} \Sigma^\infty (X^{\wedge n}).$$

- For the identity functor Id of spaces, $P_1 \mathrm{Id} \simeq \Omega^\infty \Sigma^\infty = Q$, the stable homotopy functor.

Applications

- Homotopy Theory: The Taylor tower of the identity functor of spaces interpolates between stable homotopy theory and unstable homotopy theory, and has contributed to new perspectives on homotopy theory.
- Algebraic K-theory: Under mild hypotheses, two functors F and G agree "up to a constant" if there is a natural transformation $F \Rightarrow G$ such that the induced map

$$\text{hofiber}\left(P_1 F(X) \to F(*)\right) \to \text{hofiber}\left(P_1 G(X) \to G(*)\right)$$

is a weak equivalence for "nice" X. Used to compare algebraic K-theory to topological Hochschild and cyclic homology.

Abelian Functor Calculus – Context

- $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F : \mathcal{B} \to \mathcal{A}$ is a functor.
- Eilenberg and Mac Lane (1954) defined "polynomial degree n" functors in this context in terms of *cross effects*.
- Eilenberg and Mac Lane (1951); and Dold and Puppe (1961) constructed new functors QF (for stable homology of R-modules with coefficients in S) and DF (for derived functors of non-additive functors) that are degree 1 polynomial approximations to F.

Cross Effects

Definition:

For $F : \mathcal{B} \to \mathcal{A}$ where $\mathcal{B}$ and $\mathcal{A}$ are abelian categories, the *nth cross effect functor* $cr_n F : \mathcal{B}^n \to \mathcal{A}$ is defined recursively by

$$cr_0 F = F(0)$$

$$F(X) \cong F(0) \oplus cr_1 F(X),$$

$$cr_1 F(X_1 \oplus X_2) \cong cr_1 F(X_1) \oplus cr_1 F(X_2) \oplus cr_2 F(X_1, X_2),$$

and, in general,

$$\begin{aligned} cr_{n-1} F(X_1, \ldots, X_{n-2}, X_{n-1} \oplus X_n) \cong\ & cr_{n-1} F(X_1, \ldots, X_{n-2}, X_{n-1}) \\ & \oplus cr_{n-1} F(X_1, \ldots, X_{n-2}, X_n) \\ & \oplus cr_n F(X_1, \ldots, X_{n-1}, X_n). \end{aligned}$$

Degree n functors

Definition:

$F : \mathcal{B} \to \mathcal{A}$ is *degree n* if and only if $cr_{n+1} F \simeq 0$.

Example

A is an object in an abelian category $\mathcal{A}$, $F : \mathcal{A} \to \mathcal{A}$ with $F(X) = A \oplus X$. Then

$$A \oplus X = F(X) \cong F(0) \oplus cr_1 F(X).$$

Thus,

$$cr_1 F(X) \cong X,$$
$$cr_1 F \cong \mathrm{id}.$$

And,

$$X \oplus Y \cong cr_1 F(X \oplus Y) \cong cr_1 F(X) \oplus cr_1 F(Y) \oplus cr_2 F(X, Y),$$

$$cr_2 F \cong 0.$$

Abelian Functor Calculus

Theorem (J-McCarthy, 2004)

Given a functor $F : \mathcal{B} \to \mathcal{A}$ between abelian categories $\mathcal{B}$ and $\mathcal{A}$, there exists a Taylor tower of functors and natural transformations

$$\begin{array}{ccccccccccc} & & & & & F & & & & & \\ & \swarrow & & \swarrow & & \downarrow & & \searrow & & \searrow & \\ \cdots & \longrightarrow & P_{n+1}F & \longrightarrow & & P_nF & \longrightarrow & P_{n-1}F & \longrightarrow & \cdots & \longrightarrow P_0F \end{array}$$

such that

- for all $n \geq 0$, P_nF is a degree n functor,
- if F is "nice," the tower converges to F on "nice" objects, and
- P_nF is universal (in an appropriate homotopy category) among degree n functors with natural transformations from F.

Abelian functor calculus and cartesian differential categories

Bauer, J, Osborne, Riehl, Tebbe, 2018

There is a notion of directional derivative coming out of abelian functor calculus that endows a category $\mathsf{HoAbCat}_{\mathsf{Ch}}$ with the structure of a Cartesian differential category.

Constructing P_nF in the abelian functor calculus

Lemma:

There is an adjunction

$$\mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A}) \underset{cr_n}{\overset{\Delta^*}{\rightleftarrows}} \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \quad (\perp)$$

where $\mathrm{Fun}(\mathcal{B}, \mathcal{A})$ is the category of functors from $\mathcal{B}$ to $\mathcal{A}$ and $\mathrm{Fun}_*(\mathcal{B}^n, \mathcal{A})$ is the category of functors of n variables from $\mathcal{B}$ to $\mathcal{A}$ that are reduced in each variable, and Δ is the diagonal functor.

Consequence:

$C_n := \Delta^* cr_n$ is a comonad on $\mathrm{Fun}(\mathcal{B}, \mathcal{A})$. For $F : \mathcal{B} \to \mathcal{A}$, $X \in \mathcal{A}$,

$$C_nF(X) := cr_nF(X, X, \ldots, X).$$

Constructing P_nF in the abelian functor calculus

Definition:
For $F : \mathcal{B} \to \mathcal{A}$, $P_nF : \mathcal{B} \to Ch_{\geq 0}\mathcal{A}$ is the chain complex

$$\dots \longrightarrow C_{n+1}^{\times 3}F \xrightarrow{\epsilon - C_{n+1}\epsilon + C_{n+1}^{\times 2}\epsilon} C_{n+1}^{\times 2}F \xrightarrow{\epsilon - C_{n+1}\epsilon} C_{n+1}F \xrightarrow{\epsilon} F$$

where $\epsilon : C_{n+1} = \Delta^* cr_{n+1} \Rightarrow \mathrm{id}$ is the counit of the adjunction (Δ^*, cr_{n+1}).

Constructing P_nF in the abelian functor calculus

Proposition:

For $F : \mathcal{B} \to \mathcal{A}$, $P_nF : \mathcal{B} \to Ch_{\geq 0}\mathcal{A}$ is a degree n functor.

Proof:

There is a natural contracting homotopy on $cr_{n+1}P_nF$:

$$\cdots \rightleftarrows cr_{n+1}C_{n+1}^{\times 2}F \underset{s_1}{\overset{cr_{n+1}(\epsilon - C_{n+1}\epsilon)}{\rightleftarrows}} cr_{n+1}C_{n+1}F \underset{s_0}{\overset{cr_{n+1}\epsilon}{\rightleftarrows}} cr_{n+1}F$$

given by $s_k = \eta cr_{n+1}(C_{n+1})^{\times k}$ where $\eta : \mathrm{id} \Rightarrow cr_{n+1}\Delta^*$ is the unit of the adjunction (Δ^*, cr_{n+1}).

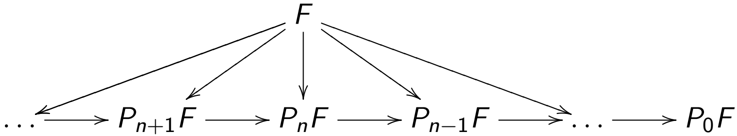
Questions

- Can we do something like this in a more topological (or homotopy-theoretical) context?
- How would it compare with Goodwillie's calculus?
- Can we make other calculi this way?

Discrete Functor Calculus

Theorem (Bauer, J, McCarthy, 2015; Mauer-Oats, 2006)

Given a functor $F : \mathcal{C} \to \mathcal{D}$ between a simplicial model category $\mathcal{C}$ and a pointed stable simplicial model category $\mathcal{D}$, there exists a Taylor tower of functors and natural transformations



such that

- for all $n \geq 0$, P_nF is a degree n functor,
- P_nF is universal (in an appropriate homotopy category) among degree n functors with natural transformations from F.

Degree n functors

Definition:
F is degree n iff $cr_{n+1}F \simeq *$.

Definition:
For an n-tuple $\mathsf{X} = (X_1, \ldots, X_n)$ of objects in $\mathcal{C}$,

$$cr_n F(\mathsf{X}) := \text{ihofiber}\,(U \in \mathcal{P}(n) \mapsto F(\mathsf{X}_1(U) \sqcup \cdots \sqcup \mathsf{X}_n(U)))$$

where

$$\mathsf{X}_i(U) = \begin{cases} X_i & i \notin U, \\ * & i \in U. \end{cases}$$

Example

$cr_2 F(X_1, X_2)$ is the iterated homotopy fiber of the diagram

$$\begin{array}{ccc} F(X_1 \sqcup X_2) & \longrightarrow & F(* \sqcup X_2) \\ \downarrow & & \downarrow \\ F(X_1 \sqcup *) & \longrightarrow & F(* \sqcup *). \end{array}$$

Construction of $P_n F$ for the discrete calculus

Lemma

$\perp_{n+1} = \Delta^* cr_{n+1}$ defines a comonad on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Definition

For a functor $F : \mathcal{C} \to \mathcal{D}$,

- $k \mapsto \mathrm{Bar}_k^{n+1} F := \perp_{n+1}^{k+1} F$ defines a simplicial object in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.
- The counit $\epsilon : \perp_{n+1} \Rightarrow \mathrm{id}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}$ makes this an augmented simplicial object:

$$\mathrm{Bar}_\bullet^{n+1} F \xrightarrow{\epsilon} F.$$

- $P_n F := \text{hocofiber}\left(\|\mathrm{Bar}_\bullet^{n+1} F\| \to F\right).$

Construction of P_nF for the discrete calculus

Proposition

P_nF is degree n.

Proof:

The comonad $\perp_{n+1}$ arises from a composite of adjunctions

$$\mathrm{Fun}(\mathcal{C}^{n+1}, \mathcal{D})_t \underset{t^+}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{Fun}(\mathcal{C}^{n+1}, \mathcal{D}) \underset{\sqcup^*}{\overset{\Delta^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

with $cr_{n+1} = U \circ t^+ \circ \sqcup^*$.

P_nF is degree n, cont.

The unit for the adjunction provides a contracting homotopy for $cr_{n+1}\mathrm{Bar}_\bullet^{n+1}F$ via an extra degeneracy, so that

$$\begin{aligned} cr_{n+1}P_nF &= \text{hocofiber}\left(\|cr_{n+1}\mathrm{Bar}_\bullet^{n+1}F\| \to cr_{n+1}F\right) \\ &\simeq \text{hocofiber}\left(cr_{n+1}F \to cr_{n+1}F\right) \\ &\simeq *. \end{aligned}$$

Theorem

- For a functor F that commutes with realization ($|F(-)| \xrightarrow{\simeq} F \circ |-|$), the discrete P_nF is weakly equivalent to the n-excisive P_nF.
- In general, the discrete P_nF and the n-excisive P_nF agree on the initial object of $\mathcal{C}$.

Calculus from Comonads (Hess-J, 20??)

Questions

For a comonad K acting on a (simplicial) model category $\mathcal{C}$ and an object x in $\mathcal{C}$, we can always construct a new object

$$\Gamma_K(x) := \text{hocofiber}\left(\|\text{Bar}_\bullet^K(x)\| \to x\right).$$

- If we define degree n in terms of a comonad K (e.g., x is degree n iff $Kx \simeq *$), what conditions on K guarantee that $\Gamma_K(x)$ will be degree n for all x?

Questions

- What conditions on a tower of comonads and comonad maps

$$\dots K_n \to K_{n-1} \to \cdots \to K_2 \to K_1$$

  will guarantee that

$$\dots \Gamma_{K_n}(x) \to \Gamma_{K_{n-1}}(x) \to \cdots \to \Gamma_{K_2}(x) \to \Gamma_{K_1}(x)$$

  is a Taylor tower for x?
- What are the essential properties of a Taylor tower (what makes a tower a calculus)?
- Can the process that produced the comonads for the discrete calculus tower be generalized?
- What kinds of new examples are produced?

What is a calculus?

Definition

Let $\mathcal{M}$ be a model category, and let $\mathcal{M}'$ be a subcategory of $\mathcal{M}$. Let Γ be a functor that assigns to an object x in $\mathcal{M}'$, a coaugmented tower of objects in $\mathcal{M}$:

$$\begin{array}{c} x \\ \swarrow \quad \swarrow \quad \downarrow \eta_n x \quad \searrow \quad \searrow \\ \cdots \longrightarrow \Gamma_{n+1}x \longrightarrow \Gamma_n x \longrightarrow \Gamma_{n-1}x \longrightarrow \cdots \longrightarrow \Gamma_0 x. \end{array}$$

If the following conditions hold, then Γ is a calculus on $\mathcal{M}'$ with values in $\mathcal{M}$.

What is a calculus?

1. For all $m \leq n$ and all objects x in $\mathcal{M}'$, the natural transformation

$$\Gamma_m \eta_n(x) : \Gamma_m x \to \Gamma_m \Gamma_n x$$

is a weak equivalence.

2. For all $m \geq n$ and all objects x in $\mathcal{M}'$, the natural transformation

$$\eta_m(\Gamma_n x) : \Gamma_n x \to \Gamma_m \Gamma_n x$$

is a weak equivalence.

3. If $f : x \to x'$ is a weak equivalence in $\mathcal{M}'$, then $\Gamma_n f : \Gamma_n x \to \Gamma_n x'$ is a weak equivalence in $\mathcal{M}$ for all n.

Example of a calculus

Let Ch be the category of unbounded chain complexes over a commutative ring R (with the projective model structure).
For $n \geq 0$, define $\Gamma_n : \mathsf{Ch} \to \mathsf{Ch}$ by

$$\Gamma_n(X,d)_k = \begin{cases} X_k & : k \leq n \\ X_{n+1}/\ker d_{n+1} & : k = n+1 \\ 0 & : k > n+1, \end{cases}$$

Then

$$H_k\big(\Gamma_n(X,d)\big) = \begin{cases} H_k(X,d) & : k \leq n \\ 0 & : k > n. \end{cases}$$

The natural transformations $\eta_n : \mathrm{Id}_{\mathsf{Ch}} \to \Gamma_n$ and $\gamma_n : \Gamma_n \to \Gamma_{n-1}$ are given by taking appropriate quotients.
The chain complexes that are of degree at most n with respect to Γ are those with homology concentrated in degree at most n

Conditions on comonads

Definition, current version

Let $\mathcal{M}$ be a pointed simplicial model category, and let $\mathcal{M}'$ be a subcategory of $\mathcal{M}$. A comonad $\mathbb{K} = (K, \Delta, \varepsilon)$ on $\mathcal{M}$ is compliant with respect to $\mathcal{M}'$ if

1. $\mathrm{Bar}^{\mathbb{K}}_\bullet(x)$ is levelwise cofibrant for all objects x in $\mathcal{M}'$,
2. K^s sends weak equivalences in $\mathcal{M}'$ to weak equivalences, and
3. K admits natural transformations $|K^s(-)| \to K^s \circ |-|$ that are componentwise weak equivalences (plus a little more).

Two comonads K and L on $\mathcal{M}$ that are compliant with respect to $\mathcal{M}'$ are jointly compliant if all of the simplicial objects

$$\mathrm{Bar}^{\mathbb{L}}_\bullet\big(|\mathrm{Bar}^{\mathbb{K}}_\bullet(x)|\big), \quad \mathrm{Bar}^{\mathbb{K}}_\bullet\big(|\mathrm{Bar}^{\mathbb{L}}_\bullet(x)|\big), \quad \big|\mathrm{Bar}^{\mathbb{L}}_\bullet\mathrm{Bar}^{\mathbb{K}}_\bullet(x)\big|_h, \quad \big|\mathrm{Bar}^{\mathbb{L}}_\bullet\mathrm{Bar}^{\mathbb{K}}_\bullet(x)\big|_v$$

are levelwise cofibrant for all objects x in $\mathcal{M}'$.

Calculi from comonads

Ingredients

- $\mathcal{M}$ is a pointed simplicial model category.
- $\mathcal{M}'$ is a subcategory of $\mathcal{M}$.
- $\mathcal{K} = (\mathbb{K}_{n+1} \xrightarrow{\sigma_n} \mathbb{K}_n)_{n \geq 1}$ is a tower of comonads.

Theorem

If each comonad $\mathbb{K}_n$ is compliant with respect to $\mathcal{M}'$, and each pair of comonads $(\mathbb{K}_m, \mathbb{K}_n)$ is jointly compliant with respect to $\mathcal{M}'$, then the coaugmented tower obtained from $\mathcal{K}$ is a calculus.

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Abelian Functor Calculus – Cross effects

An analogy:

For $f : \mathbb{R} \to \mathbb{R}$, f is degree 1 $\Rightarrow f(x) = ax + b$ for some a and b. Then

$$cr_1 f(x) := f(x) - f(0) = ax$$

is linear, and

$$cr_2 f(x, y) = cr_1 f(x + y) - cr_1 f(x) - cr_1 f(y) = 0.$$

For $f : \mathbb{R} \to \mathbb{R}$:
f is degree 2 $\Rightarrow f(x) = ax^2 + bx + c$ for some a, b, and c. Then

$$\begin{aligned} cr_2 f(x,y) &= cr_1 f(x+y) - cr_1 f(x) - cr_1 f(y) \\ &= a(x+y)^2 + b(x+y) - ax^2 - bx - ay^2 - by \\ &= 2axy \end{aligned}$$

is linear in both x and y and

$$\begin{aligned} cr_3 f(x,y,z) &= cr_2 f(x, y+z) - cr_2 f(x,y) - cr_2 f(x,z) \\ &= 2ax(y+z) - 2axy - 2axz = 0. \end{aligned}$$

In fact, f is degree n iff $cr_{n+1} f(x_1, x_2, \ldots, x_{n+1}) = 0$.