Dependent products of polynomials

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Dialectica categories, polynomials, models of type theory

- [von Glehn, 2014]:
  $(\mathbb{B}, \mathcal{F}) \mapsto (\mathbf{Poly}_{\mathcal{F}}, \mathcal{F}_{\mathbf{Poly}_{\mathcal{F}}})$ is an operation on display map categories.

$\Longrightarrow$ A dependently-typed version of Dialectica categories

[de Paiva, 1989], [Hyland, 2002],. . .

$\Longrightarrow$ A model of dependent types from freely adding sums and products

- Dependent types for other variants?

[Moss, 2018], [Moss, von Glehn, 2018]

Outline

1. $\Sigma\Pi(\mathbb{C})$: polynomials in $\mathbb{C}$
2. Tensors and (local) exponentials in $\Sigma\Pi(\mathbb{C})$
3. Dialectica categories
4. $\Sigma(\mathbb{A})$ where $\mathbb{A}$ has biproducts

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1. $\Sigma\Pi(\mathbb{C})$: polynomials in $\mathbb{C}$
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3. Dialectica categories
4. $\Sigma(\mathbb{A})$ where $\mathbb{A}$ has biproducts

The category of polynomials

$$y \coloneqq \mathrm{id} \cong \mathbf{Set}(1, -) \in [\mathbf{Set}, \mathbf{Set}]$$

$$\mathbf{Poly} \simeq \text{(closure of } \{y\} \text{ under sums and products)} \subseteq [\mathbf{Set}, \mathbf{Set}]$$

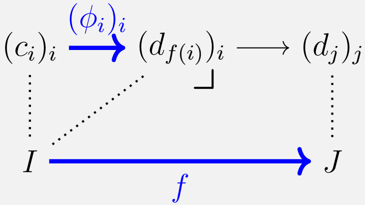
$\Sigma(\mathbb{C})$, the free sum completion of $\mathbb{C}$

objects

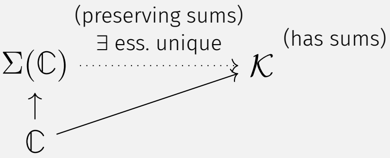
$$\begin{array}{c}(c_i)_{i\in I}\\ \vdots\\ I\end{array} = \text{“}\sum_{i\in I} c_i\text{”}$$

with $I \in \mathbf{Set}$ and $c : I \to \mathrm{ob}\,\mathbb{C}$.

morphisms



universal property



as presheaves

$\Sigma(\mathbb{C}) \simeq$ closure of

$$\{\mathbb{C}(-, c)\}_{c\in\mathbb{C}} \subseteq [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$$

under sums.

$\Pi(\mathbb{C})$: the free product completion of $\mathbb{C}$

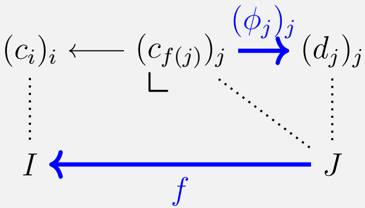
$$\Pi(\mathbb{C}) \coloneqq (\Sigma(\mathbb{C}^{\mathsf{op}}))^{\mathsf{op}} \simeq \text{(closure of } \{\mathbb{C}(c,-)\}_{c\in\mathbb{C}} \text{ under products)} \subseteq [\mathbb{C}, \mathbf{Set}]^{\mathsf{op}}$$

objects

$$\begin{matrix} (c_i)_{i\in I} \\ \vdots \\ I \end{matrix} \;=\; \text{“}\prod_{i\in I} c_i\text{”}$$

with $I \in \mathbf{Set}$ and $c : I \to \operatorname{ob}\mathbb{C}$.

morphisms



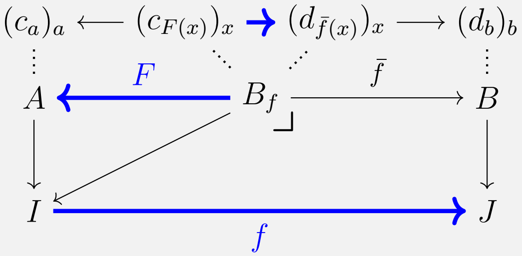
$\Sigma\Pi(\mathbb{C})$: adding products, then adding sums

objects

Bundles of bundles in $\mathrm{ob}\,\mathbb{C}$.

$$\begin{array}{c} (c_a)_{a\in A} \\ \vdots \\ A \\ p\big\downarrow \\ I \end{array} \quad = \text{“}\sum_{i\in I}\prod_{a\in A_i} c_a\text{”}$$

morphisms



Basic examples of $\Sigma(-)$, $\Pi(-)$, $\Sigma\Pi(-)$

$$\Sigma(\mathbb{1}) \simeq \mathbf{Set} \qquad \Pi(\mathbb{1}) \simeq \mathbf{Set}^{\mathsf{op}} \qquad \Sigma\Pi(\mathbb{1}) \simeq \mathbf{Poly}$$

$\Sigma\Pi(\mathbb{2}) =$ ‘dependently-typed Dialectica category’

$\Sigma(I) \simeq \mathbf{Set}^I$ (where $I \in \mathbf{Set}$)

$\Sigma\Pi(I) \simeq$ polynomial functors $\mathbf{Set}^I \to \mathbf{Set}$

Products in $\Sigma(\mathbb{C})$

If $\mathbb{C}$ has products then $\Sigma(\mathbb{C})$ has products given by

$$\prod_{j \in J} \sum_{i \in I_j} c_{i,j} \cong \sum_{f \in \Pi_{j \in J} I_j} \prod_{j \in J} c_{f(j),j}$$

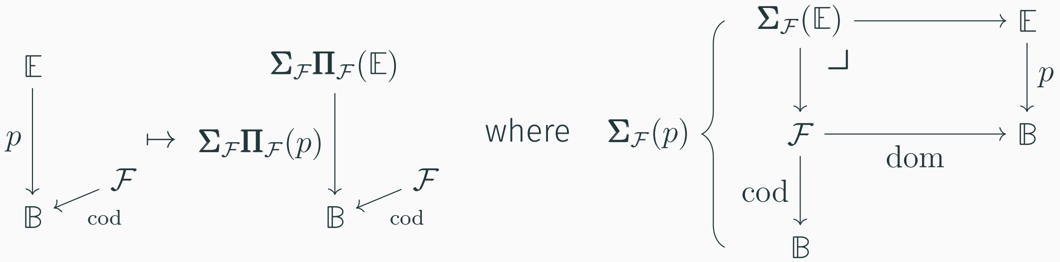
and $\mathbb{C} \to \Sigma(\mathbb{C})$ preserves products.

Corollary

$\Sigma\Pi(\mathbb{C})$ has products, and these distribute over sums.

(Actually a distributive law $\Pi\Sigma \xrightarrow{\delta} \Sigma\Pi$).

The fibration of $\mathcal{F}$-polynomials in a fibration $p : \mathbb{E} \to \mathbb{B}$



Recovering the basic setting

$$(\mathbb{E} \xrightarrow{p} \mathbb{B}) = (\Sigma(\mathbb{C}) \xrightarrow{\Sigma(!)} \Sigma(\mathbb{1}) \simeq \mathbf{Set}) \qquad \mathcal{F} = \operatorname{mor} \mathbf{Set}$$

$$\Sigma\Pi(\mathbb{C}) = (\mathbf{\Sigma}_{\mathcal{F}}\mathbf{\Pi}_{\mathcal{F}}(\mathbb{E}))_1$$

Summary of polynomials in $\mathbb{C}$

- $\Sigma\Pi(\mathbb{C})$, the 'polynomials in $\mathbb{C}$', is given by formally/freely adding sums and products to the category $\mathbb{C}$.
- $\mathbf{Poly} \simeq \Sigma\Pi(\mathbb{1})$.
- The fibrational setting gives lots of flexibility. e.g.
  (a) control the 'sizes' of sums/products are added,
  (b) non-standard notions of 'bundles' of polynomials.

Outline

The Day tensor on $\Sigma(\mathbb{C})$ (e.g. when $\mathbb{C}$ is small)

Let $(\mathbb{C}, \otimes, I)$ be symmetric monoidal.

- 'Convolution' extends $\otimes$ to the Day tensor on $[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$...

$$(F \otimes G)c := \int^{x,y \in \mathbb{C}} Fx \times Gy \times \mathbb{C}(c, x \otimes y).$$

- ... which restricts to $\Sigma(\mathbb{C})$:

$$\left(\sum_{i \in I} c_i\right) \otimes \left(\sum_{j \in J} d_j\right) := \sum_{(i,j) \in I \times J} c_i \otimes d_j.$$

- $\otimes$ has exponentials in $[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$

$$(G \multimap H)c := \int_{y,z \in \mathbb{C}} (\mathbb{C}(z, y \otimes c) \times Gy) \Rightarrow Hz$$

but not necessarily in $\Sigma(\mathbb{C})$.

Lifting tensors to $\Sigma\Pi(\mathbb{C})$

- Day convolution takes $\times$ on $\Pi(\mathbb{C})$ to $\times$ on $\Sigma\Pi(\mathbb{C})$.

$$\left(\sum_{i\in I}\prod_{a\in A_i} c_a\right) \times \left(\sum_{j\in J}\prod_{b\in B_j} d_b\right) = \sum_{(i,j)\in I\times J}\ \prod_{x\in A_i+B_j} (\ldots)$$

- If $(\mathbb{C}, \otimes, I)$ is monoidal, extend by Day convolution twice.

$$\left(\sum_{i\in I}\prod_{a\in A_i} c_a\right) \otimes \left(\sum_{j\in J}\prod_{b\in B_j} d_b\right) = \sum_{(i,j)\in I\times J}\left(\left(\prod_{a\in A_i} c_a\right) \otimes \left(\prod_{b\in B_j} d_b\right)\right)$$

$$= \sum_{(i,j)\in I\times J}\ \prod_{(a,b)\in A_i\times B_j} c_a \otimes d_b$$

Day exponentials in $\Sigma(\mathbb{C})$?

$$\mathbb{C} \subseteq \Sigma(\mathbb{C}),$$

is a *dense* subcategory closed under $\otimes$.

$\implies$ Sufficient to check exponentials on $c \in \mathbb{C}$:

$$\frac{c \otimes Y \to Z}{c \to Y \multimap Z}$$

We will see: when $\Sigma(\mathbb{C})$ has products, enough that $(c \multimap d) \in \Sigma(\mathbb{C})$.

Exponentiating by connected objects

- $X \in \mathcal{E}$ is connected (or coprime) iff $\mathcal{E}(X, -) : \mathcal{E} \to \mathbf{Set}$ preserves coproducts.
- The connected objects in $\Sigma(\mathbb{C})$ are precisely $\mathbb{C} \subseteq \Sigma(\mathbb{C})$.

$$\implies \qquad c \multimap \left(\sum_j d_j\right) \cong \sum_j (c \multimap d_j)$$

Exponentiating when $\Sigma(\mathbb{C})$ is closed under products

Suppose $\Sigma(\mathbb{C})$ has products, e.g. because $\mathbb{C}$ has products.

$$\left(\sum_i c_i\right) \multimap \left(\sum_j d_j\right) \cong \prod_i \sum_j (c_i \multimap d_j)$$

Then $(\Sigma(\mathbb{C}), \otimes, I)$ is closed under $\multimap$ iff $(c \multimap d) \in \Sigma(\mathbb{C})$ for all $c, d \in \mathbb{C}$.

$\Sigma\Pi(\mathbb{C})$ is cartesian closed (for $\mathbb{C}$ locally small)

[Altenkirch, Levy, Staton, 2010] for $\mathbf{Poly}$

- $\times$ in $\Pi(\mathbb{C})$ is *free* over $\mathbb{C}$:

$$\Pi(\mathbb{C})(X \times Y, c) \cong \Pi(\mathbb{C})(X, c) + \Pi(\mathbb{C})(Y, c)$$

$\Longrightarrow$ For $Y \in \Pi(\mathbb{C})$, $c \in \mathbb{C}$,

$$Y \Rightarrow c \quad \cong \quad c + \sum_{f \in \Pi(\mathbb{C})(Y,c)} 1.$$

- For general $\left(\prod_{k \in K} c_k\right) \in \Pi(\mathbb{C})$,

$$Y \Rightarrow \left(\prod_k c_k\right) \cong \prod_k \left(Y \Rightarrow c_k\right).$$

$\square$

Tensors and exponentials so far

- Easy to lift a tensor $\otimes$ from $\mathbb{C}$ to $\Sigma(\mathbb{C})$ or to $\Sigma\Pi(\mathbb{C})$.
- $\Sigma\Pi(\mathbb{C})$ admits $\multimap$ iff $c \multimap d$ exists in $\Sigma\Pi(\mathbb{C})$ for $c, d \in \mathbb{C}$.
- $\Sigma\Pi(\mathbb{C})$ cartesian closed.

Local exponentials?
Pullbacks in $\Sigma\Pi(\mathbb{C})$ are not guaranteed, so let's focus on $\mathbf{Poly}$ where having all finite limits simplifies things.

Dependent products in a category $\mathcal{E}$ with finite limits

Dependent product along $f : B \to A$ is a right adjoint $\Pi_f : \mathcal{E}/B \to \mathcal{E}/A$ to pullback $f^* : \mathcal{E}/A \to \mathcal{E}/B$.

- Gives *local* exponentials: $\Pi_f(f^*(-)) \cong f \Rightarrow (-)$ in $\mathcal{E}/A$.
- Conversely: can construct $\Pi_f(-)$ from $f \Rightarrow (-)$ (and pullbacks).

$\Longrightarrow$ Dependent products along f exist iff f is exponentiable in $\mathcal{E}/A$.

- Also: exponentiable maps are stable under pullback.

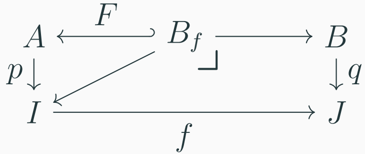
$\Longrightarrow$ If $\mathcal{E}$ is a CCC (with pullbacks), then product projections are exponentiable.

Exponentiable maps in $\mathbf{Poly}$

1. Product projections are exponentiable.

$$y^B \xrightarrow{y^f} y^A \text{ is exponentiable whenever } f : A \to B \text{ is injective}$$

2. In $\mathbf{Poly}/y^A$, exponentiable objects are closed under sums.
3. $\mathbf{Poly}/\sum_i y^{A_i} \simeq \prod_i \left(\mathbf{Poly}/y^{A_i}\right)$ therefore exponentiable maps are closed under sums.



No other maps in $\mathbf{Poly}$ are exponentiable [von Glehn], [Altenkirch, Levy, Staton].

Summary of (local) exponentials

- $\Sigma\Pi(\mathbb{C})$ is always cartesian closed.
- $\mathbf{Poly}$ is not locally cartesian closed, but we know exactly which maps are exponentiable.
- In general $\Sigma\Pi(\mathbb{C})$ might not have all pullbacks. Instead, find a class

$$\mathcal{F} = \mathcal{F}_{\Sigma\Pi(\mathbb{C})} \subseteq \mathrm{mor}(\Sigma\Pi(\mathbb{C}))$$

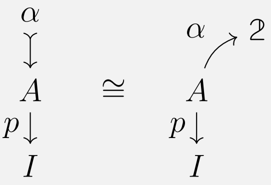
  of display maps modelling "Π-types": Π_f where $f \in \mathcal{F}$, defined on $\mathcal{F}$, valued in $\mathcal{F}$.
- The $\mathcal{F}$ that works for $\mathbf{Poly}$ generalizes to $\Sigma\Pi(\mathbb{C})$.

(cf. "Reedy fibrations" of [Shulman, 2014], [Uemura, 2017]).

Outline

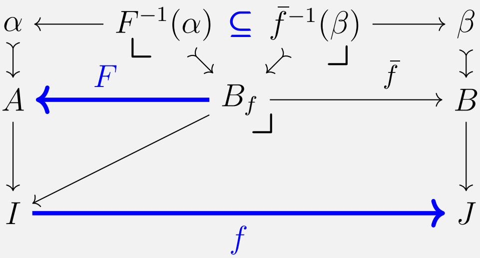
The ‘Dialectica polynomials’ $\Sigma\Pi(2)$ where $2 = \{\bot \leq \top\}$

objects



$= \text{“}\exists i \in I. \forall x \in A_i . \alpha(i, x)\text{”}$

morphisms



“The proof of $\beta(f(i_0), y)$ must consume precisely one predetermined instantiation of $\forall x \in A_{i_0} . \alpha(i_0, x)$”.

$\mathbf{Dial}$ ('classic Dialectica') [de Paiva, 1989]

$\mathbf{Dial} \subseteq \Sigma\Pi(2)$ is the full subcategory on objects

$$\alpha \rightarrowtail I \times A \to I$$

- $\mathbf{Dial}$ has sums of '$\forall$-homogeneous' summands.

$$(\exists i \in I.\, \forall x \in A.\, \alpha(i,x)) + (\exists j \in J.\, \forall x \in A.\, \beta(j,x))$$
$$\cong (\exists x \in I + J.\, \forall a \in A.\, \texttt{match}\, x\, \{\texttt{inl}(i) \to \alpha(i,a), \texttt{inr}(j) \to \beta(j,a)\}).$$

- [Hofstra, 2011]:
  Take $\mathcal{F}$ = product projections in $\mathbf{Set}$ and add $\mathcal{F}$-sums and $\mathcal{F}$-products to a fibration of propositions $\mathbb{P} \to \mathbf{Set}$.

Cf. $\mathbf{Lens} \subseteq \mathbf{Poly}$

Products and tensor in $\mathbf{Dial}$

$\mathbf{Dial}$ is closed under the products of $\Sigma\Pi(2)$:

$$\left(\sum_{i\in I}\prod_{x\in A}\alpha(i,x)\right)\times\left(\sum_{j\in J}\prod_{y\in B}\beta(j,y)\right) = \sum_{(i,j)\in I\times J}\underbrace{\left(\prod_{x\in A}\alpha(i,x)\right)\times\left(\prod_{y\in B}\beta(j,y)\right)}_{\text{product over } A+B}$$

and also the tensor product induced by $(2,\wedge,\top)$:

$$\left(\sum_{i\in I}\prod_{x\in A}\alpha(i,x)\right)\otimes\left(\sum_{j\in J}\prod_{y\in B}\beta(j,y)\right) = \sum_{(i,j)\in I\times J}\ \prod_{(x,y)\in A\times B}(\alpha(i,x)\wedge\beta(j,y))$$

$\Sigma\Pi(2)$ has exponentials for both, $\mathbf{Dial}$ only for the latter.

Dialectica categories so far

- The simplest version is $\Sigma\Pi(2)$ — objects are "$\exists i \in I.\forall x \in A_i.\alpha(i, x)$". The original $\mathbf{Dial}$ just requires $(A_i)_i$ be constant in $i \in I$.
- $\mathbf{Dial}$ is SMC with $\otimes$, cartesian, has only weak sums. $\Sigma\Pi(2)$ is SMC with $\otimes$, cartesian closed, has sums.
- 'Really' about adding quantifiers to a fibration $\mathbb{P} \to \mathbf{Set}$ of predicates.

Another way to 'make' $\mathbf{Dial}$ cartesian closed is to take the co-Kleisli category of a well-chosen comonad, e.g. such that $\otimes$ becomes the cartesian product (cf. ! modality in linear logic).

- Diller-Nahm variant [de Paiva, 1989] using $\mathcal{M}_{\text{fin}}$.
- 'Error' variant [Biering, 2008] using $(-) + 1$.

Outline

The ‘type theory part’ of $\mathbf{Dill}$

- The finite multisets monad $\mathcal{M}_{\text{fin}} : \mathbf{Set} \to \mathbf{Set}$ induces a comonad L on $\mathbf{Poly}$.

$$\sum_{i \in I} y^{A_i} \mapsto \sum_{i \in I} y^{\mathcal{M}_{\text{fin}} A_i}$$

- $(\mathbf{Poly})_L \simeq \Sigma((\mathbf{Set}_{\mathcal{M}_{\text{fin}}})^{\mathsf{op}}) \simeq \Sigma(\mathbf{FreeCMon}^{\mathsf{op}})$.

Recent work on automatic differentiation uses CC structure of $\Sigma(\mathbf{CMon})$, $\Sigma(\mathbf{CMon}^{\mathsf{op}})$,... [Vákár, Smeding, Lucatelli Nunes].

$\Sigma(\mathbb{A})$ is cartesian closed — when $\mathbb{A}$ has biproducts and products

- Earlier: if $\mathbb{A}$ has products, suffices to exponentiate $\mathbb{A}$'s by $\mathbb{A}$'s.
- $\times = \oplus = +$ in $\mathbb{A}$:

$$\mathbb{A}(a \oplus b, c) \cong \mathbb{A}(a, c) \times \mathbb{A}(b, c)$$

$\Longrightarrow$ Exponentiate $\mathbb{A}$'s by $\mathbb{A}$'s:

$$b \Rightarrow c \quad \cong \quad c \times \left(\sum_{f \in \mathbb{A}(b,c)} 1\right) \quad \cong \quad \sum_{f \in \mathbb{A}(b,c)} c$$

$\square$

Polynomials via an enriched sum completion?

[Kelly, Basic concepts of enriched category theory]

- Let $(\mathcal{V}, \otimes, I)$ be a bicomplete SMC (e.g. $\mathbf{CMon}$, $\mathbf{Set}_*$).
- Free $\mathcal{V}$-category $\mathbb{C}_\mathcal{V}$ on a $\mathbf{Set}$-category $\mathbb{C}$:

$$\mathrm{ob}\, \mathbb{C}_\mathcal{V} := \mathrm{ob}\, \mathbb{C} \qquad \mathbb{C}_\mathcal{V}(c, d) := \mathbb{C}(c, d) \cdot I$$

- Free $\mathcal{V}$-sum completion $\Sigma_\mathcal{V}(\mathcal{A})$: close $\mathcal{A}$ under sums in $[\mathcal{A}^{\mathsf{op}}, \mathcal{V}]_\mathcal{V}$.

- $\mathbb{1}_{\mathbf{CMon}} \simeq \mathbb{N}$ (as a 'semiring'/one-object $\mathbf{CMon}$-category).
- $\mathbf{FreeCMon} \simeq \Sigma_{\mathbf{CMon}}(\mathbb{1}_{\mathbf{CMon}})$ (cf. [Mac Lane, Ex. VIII.2.5–6]).
- Thus $(\mathbf{Poly})_L \simeq \Sigma\Pi_{\mathbf{CMon}}(\mathbb{1}_{\mathbf{CMon}})$.