On the differential structure of polynomial functors

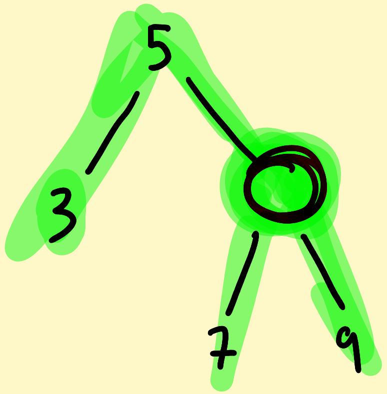
Fredrik Nordvall Forsberg
University of Strathclyde

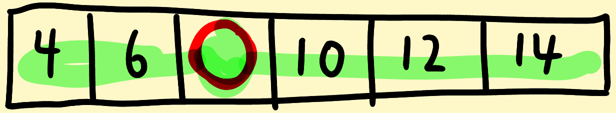
2nd Workshop on polynomial functors, 14 March 2022

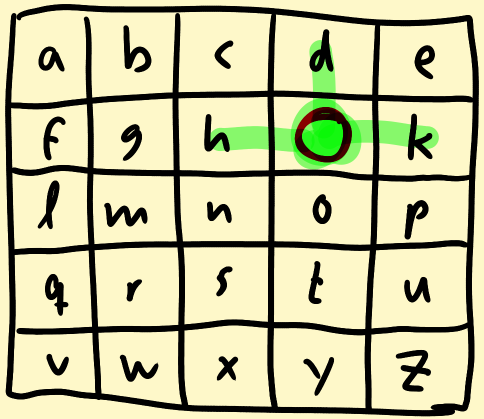
Joint work in progress with Neil Ghani and Conor McBride

One-hole contexts

Types for representing being in the middle of some operation.







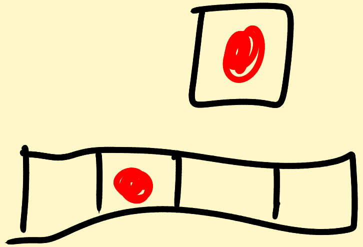
Text editors, proof state, file systems, window managers, ...

More generally: Zippers [Huet 1997].

Gérard Huet. The Zipper. Journal of Functional Programming, 7(5):549–554, 1997

Examples of one-hole contexts

What is an FX data structure with an hole in it?



$$\mathrm{Hole}\ X = 1$$

$$\mathrm{Hole}\ A = 0$$

$$\mathrm{Hole}\ (FX + GX) = (\mathrm{Hole}\ FX) + (\mathrm{Hole}\ GX)$$

$$\mathrm{Hole}\ (X^n) = n \times X^{n-1}$$

$$\mathrm{Hole}\ (FX \times GX) = \mathrm{Hole}\ FX \times GX + \mathrm{Hole}\ GX \times FX$$

$$\mathrm{Hole}\ (F(GX)) = (\mathrm{Hole}\ FX)_{GX} \times (\mathrm{Hole}\ GX)_{X}$$

$$\frac{\partial}{\partial x} x = 1$$

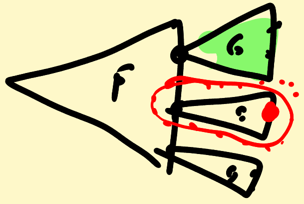
$$\frac{\partial}{\partial x} a = 0$$

$$\frac{\partial}{\partial x} (f+g) = \frac{\partial}{\partial x} f + \frac{\partial}{\partial x} g$$

$$\frac{\partial}{\partial x} x^n = n x^{n-1}$$

$$\frac{\partial}{\partial x} (f \cdot g) = \frac{\partial}{\partial x} f \cdot g + \frac{\partial}{\partial x} g \cdot f$$

$$\frac{\partial}{\partial x} \left(f(g(x))\right) = \left(\frac{\partial}{\partial x} f\right)\Big|_{g(x)} \cdot \frac{\partial}{\partial x} g \Big/ x$$



Conor McBride: The Derivative of a regular type is its type of one-hole contexts, 2001

Plugging a hole

Given a one-hole context and a thing, we should be able to reconstruct a whole structure again.

$$\text{plug}_F : (\text{Hole } F)(X) \times X \longrightarrow F(X)$$

$$\text{plug}_X \; * \; x = x$$

$$\text{plug}_A \; y \; x = \text{impossible! } y \in 0$$

$$\text{plug}_{F_1+F_2} \; (\text{in}_i \; y) \; x = \text{in}_i \; (\text{plug}_{F_i} \; y \; x)$$

$$\text{plug}_{F\times G} \; (\text{in}_1 \; (y, t)) \; x = (\text{plug}_F \; y \; x, t)$$

$$\text{plug}_{F\times G} \; (\text{in}_2 \; (y, t)) \; x = (\; t \;, \text{plug}_G \; y \; x)$$

$$\text{plug}_{F\circ G} \; (y, z) \; x = \text{plug}_F \; y \; (\overbrace{\text{plug}_G \; z \; x}^{Gx})$$

Note: syntactically linear.

$$\text{Hole } X = 1$$

$$\text{Hole } A = 0$$

$$\text{Hole } (F + G) = \text{Hole } F + \text{Hole } G$$

$$\text{Hole } (F \times G) = (\text{Hole } F) \times \textcircled{G} + (\text{Hole } G) \times F$$

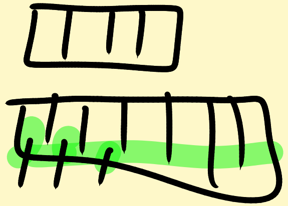
$$\text{Hole } (F \circ G) X = \underline{(\text{Hole } F)(GX)} \times (\text{Hole } G) X$$

What about more realistic data types?

What are the one-hole contexts of fixed points of functors?

$$\mathrm{List}\,X = \mu Y.\ 1 + X \times Y$$

Need multisorted holes!



For now, let's stay simpleminded and approach the problem differently.

We can represent such data types by polynomial functors, given by

a set S

a family $P : S \to \mathrm{Set}$

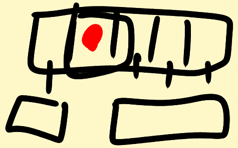
$$[\![S,P]\!] : \mathrm{Set} \to \mathrm{Set}$$

$$[\![S,P]\!](X) = \sum_{s \in S} (P(s) \to X)$$

$$\mathrm{List} = [\![\mathbb{N}, n \mapsto \mathrm{Fin}\ n]\!]$$

Theorem: Polynomial functors are closed under Id, K_A, $+$, $\times$, $\circ$, μ.

The derivative of a polynomial functor



$$\partial [\![S,P]\!] = [\![\sum_{s\in S} P(s), (s,h) \mapsto P(s)\setminus h$$

where $X \setminus y := \{x \in X \mid \neg(x=y)\}$

$\text{List } X = 1 + X + X^2 + \ldots$

$\Rightarrow \partial \text{List} = \text{List}^2$

Example: $[\![\partial \text{List}]\!] X = \sum_{n\in\mathbb{N}, h\in \text{Fin } n} ((\text{Fin } n \setminus h) \to X)$

$\cong \sum_{n\in\mathbb{N}, m\in\mathbb{N}} ((\text{Fin } n + \text{Fin } m) \to X) \cong (\text{List } X)^2$

We need $P(s)$ to have decidable equality for every $s : S$!

Can we define

$\text{plug} : \partial [\![S,P]\!](X) \times X \to [\![S,P]\!](X)$?

$$\text{plug } ((s,h), f : P(s)\setminus h \to X)\ x = (s, p \mapsto \begin{cases} f\,p & p \neq h \\ x & p = h \end{cases}$$

Polynomial functors with positions with decidable equality

Definition: A set X has decidable equality if there is

$$\mathrm{dec}_X : \prod_{x,y\in X} \overbrace{(x=y)}^{\{* \mid x=y\}} + \neg\underbrace{(x=y)}_{\cdots \to 0}$$

Lemma: Sets with decidable equality are closed under $0, 1, \times, +, \Sigma, =, \neq$.

Let us restrict ourselves to poly. functors with decidable equality on positions.

Theorem: Poly. func. with dec. eq. on positions are closed under Id, K_A, $+$, $\times$, $\circ$, μ.

That's nice, but what does it all mean?

With a little work, we can verify $\partial([\![S,P]\!] + [\![S',P']\!]) = \partial[\![S,P]\!] + \partial[\![S',P']\!]$ and Leibniz's other laws.

Is this a coincidence?

Abbott, Altenkirch, Ghani and McBride [2005] showed that ∂F has a universal property.

For this to work, we need to restrict to the subcategory of Cartesian morphisms between polynomial functors.

Michael Abbott, Thorsten Altenkirch, Neil Ghani, and Conor McBride. ∂ for Data: Differentiating Data Structures. Fundamenta Informatica XX, 1–29, 2005

Cartesian morphisms of polynomial functors

Recall that natural transformations $[\![S,P]\!] \xrightarrow{\cdot} [\![S',P']\!]$ are given by

$$f : S \to S'$$
$$g : \forall s.\ P'(f(s)) \to P(s)$$

Definition: A morphism $(f,g):(S,P)\to(S',P')$ is Cartesian if g_s is iso for every $s \in S$, i.e. $g : \forall s.\ P'(f(s)) \cong P(s)$.

[Terminology comes from families fibration
$$\begin{tikzcd} \mathrm{Fam}(\mathbb{C}) \arrow[d] \\ \mathrm{Set} \end{tikzcd}$$
: $\mathrm{Poly} = \mathrm{Fam}(\mathrm{Set}^{op})$.]

objects (X,P) where X set
$P : X \to \mathbb{C}$

The universal property of ∂F

Consider the Cartesian product $F \times G$ in Poly. In the subcategory $\mathrm{Poly}_{\mathrm{cart}}$, it is (confusingly) only monoidal $F \otimes G$.

Theorem: Let F be a polynomial functor with positions with dec. eq. We have the following natural iso:

$$\begin{array}{c} \boxed{\mathrm{Poly}_{\mathrm{cart}}}(G \otimes \mathrm{Id}, F) \\ \hline \hline \mathrm{Poly}_{\mathrm{cart}}(G, \underline{\partial F}) \end{array} \quad - \otimes \mathrm{Id} \dashv \partial F \qquad \partial F = \mathrm{Id} \multimap F$$

∂F

The counit $\varepsilon : \partial F \otimes \mathrm{Id} \overset{\cdot}{\longrightarrow} F$ is familiar — our old friend

$$\varepsilon_X = \mathrm{plug} : (\partial F)(X) \times X \longrightarrow F(X) \; !$$

That's nice, but what does it all mean (again)?

This result shows a connection between ∂ for Poly and "linear" approximation, but it still lives in the world of polynomial functors only.

Can we relate ∂ for Poly somehow with "ordinary" differentiation?

We construct a category where polynomial functors are the morphisms, and show that it satisfies the axioms of a Cartesian Differential Category [Blute, Cockett, and Seely 2009].

Rick Blute, Robin Cockett, and R.A.G. Seely. Cartesian differential categories. Theory and Applications of Categories, 22, 622–672, 2009

n-ary polynomial functors

CDCs talk about partial derivatives, so let us recall multisorted poly. functors:

Definition: An n-ary polynomial functor with positions with dec. eq. is given by a set S and $P : S \to \overset{\mathrm{Fam}\,(\mathrm{Fin}\ n)}{\cancel{\mathrm{Fin}\ n \to \mathrm{DecEqSet}}}$

$$[\![S,P]\!] : \mathrm{Set}^{n} \longrightarrow \mathrm{Set}$$

$$[\![S,P]\!](\bar{X}) = \sum_{s \in S} \prod_{p \in idx(P_s)} X_{sort(P_s)\,p}$$

$idx : \mathrm{Fam}\ A \to \mathrm{DecEqSet}$

$sort\ P : idx\ P \to A$

$(A \to \mathrm{Set}) \cong \mathrm{Fam}\ A$

Example: Projections $\pi_i = [\![1, P = \lambda\_.\,(1, \lambda\_.\,i)]\!]$

Cartesian Differential Categories

A CDC consists of:

- Left additive structure
- Cartesian structure
- Differential structure

(+ axioms relating the notions, of course)

Typical examples: Smooth functions $\mathbb{R}^n \to \mathbb{R}^m$, polynomials (ordinary kind!)

Rick Blute, Robin Cockett, and R.A.G. Seely. Cartesian differential categories. Theory and Applications of Categories, 22, 622–672, 2009

Polynomial functors as a Cartesian Differential Category

Define a category as follows:

Objects: $n \in \mathbb{N}$

Morphisms: $n \to m$ is m-tuple of n-ary pol. funs

Identities: tuples of projections

Composition: componentwise $\circ$ of poly funs

"Lawvere theory of polynomial functors"

Cartesian structure:

$$\underline{n} \times \underline{m} = \underline{n+m}$$

Related: Cockett's[2012] CDC where morphisms $X \to Y$ are indexed poly. funs.

$$\begin{tikzcd} X \arrow[r, bend left] & Y \end{tikzcd}$$

Robin Cockett. "Can you differentiate a polynomial?". Talk at FMCS 2012.

Left additive structure

Each homset should be a commutative monoid (cf. componentwise adding maps $\mathbb{R}^n \to \mathbb{R}^m$)

We define

$$(F_1, \dots, F_m) + (G_1, \dots, G_m) = (F_1 + G_1, \dots, F_m + G_m)$$

$$0 = (K_0, \dots, K_0)$$

Need to check

$$\begin{tikzcd} \underline{n} \arrow[r, "f"] & \underline{m} \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "+"] & \underline{k} \end{tikzcd} = \begin{tikzcd} & m \arrow[dr, "g"] & \\ n \arrow[ur, "f"] \arrow[dr, "f"'] & + & k \\ & m \arrow[ur, "h"'] & \end{tikzcd} \quad ✓$$

(Right additivity not true in general)

Also need to check that Cartesian structure is (right) additive. ✓

Differential operator

$$\frac{X \xrightarrow{f} Y}{X \times X \xrightarrow{D[f]} Y}$$

"original point"

"evaluation vector"

$$\frac{\partial f(x)}{\partial x}(a) \cdot b$$

Given n-ary polynomial functor (S,P), define $\overbrace{2 * n}^{n+n}$-ary $\partial(S,P) = (S',P')$ where

$$S' = \sum_{s \in S} idx\,(P(s))$$

$$P'(s,h) = \left(idx\,(P(s)),\ \lambda p.\,(p \overset{?}{=} h,\ sort^{(P(s))}\ p) \right)$$

Axioms

✓ [CD.1] $D[f+g] = D[f] + D[g]$, $D[0] = 0$

(✓) [CD.2] $D[f]$ additive in first argument

✓ [CD.3] $D[id] = \pi_0$, $D[\pi_0] = \pi_0 ; \pi_0$ $D[\pi_1] = \pi_0 ; \pi_1$

✓ [CD.4] $D[\langle f, g \rangle] = \langle D[f], D[g] \rangle$

(✓) [CD.5] $D[f ; g] = \langle D[f], (\pi_1 ; f) \rangle ; D[g]$

(✓) [CD.6] $D[f]$ linear in first variable

g linear if $D[g] = \pi_0 ; g$

(✓) [CD.7] Order of partial derivatives does not matter

put 0 in unwanted places

Summary

Rules for one-hole contexts corresponds to rules for differentiation.

∂ can be defined for polynomial functors with positions with decidable equality, and satisfies a universal property in a subcategory of Cartesian morphisms.

Ongoing work to construct a Cartesian Differential Category of polynomial functors.

Thanks!