Procontainers

a proposal from computational effects

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Workshop on polynomial functors

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Computational effects

```
let f (x : int) =
  if x % 2 = 0 then
    x / 2
  else
    3 * x + 1
```

$\rightsquigarrow \quad f : \mathbb{Z} \to \mathbb{Z}$

But generally, programs interact with env., such as:

- read input
- print
- non-determinism
- memory cell usage

Monads [Moggi '89]

One alternative is to assume a monad (T, μ, η) or equiv. an extension system $(T, -^*, \eta)$

```
      type t a = ...
-*    bind   : forall a b . t a -> (a -> t b) -> t b
η     return : forall a . a -> t a
```

```
read : () -> t int
print : string -> t ()
```
$1 \cong \{*\}$

```
let c () =
  bind (read ()) (fun n ->
  if (n > 10) then print "big!"
              else return ())
```

$\rightsquigarrow \quad c : 1 \longrightarrow T\,1$

Idioms (or applicative fu ors) [McBride, Paterson '08]

There are more "static" alternatives:

```
type f a = ...
app  : forall a b . f (a -> b) -> f a -> f b
pure : forall a . a -> f a
```

We still can write programs, but much more limited:

```
let c =
  app (app (pure (fun () () -> ()))
           (print "Hello!"))
           (print "Goodbye!")
```

CANNOT USE read's result to decide next computation

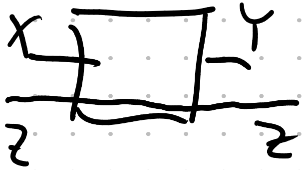
We can think of f as a (strong) lax monoidal functor:

(F, m, e) $\qquad m : FA \times FB \longrightarrow F(A \times B)$ $\qquad e : 1 \longrightarrow F1$

Arrows [Hughes '00]

$f : X \rightsquigarrow Y$

An intermediate point is obtained by using a type constructor in two arguments:




```
type a x y = ...
arr   : forall x y . (x -> y) -> a x y
(>>>) : forall x i y . a x i -> a i y -> a x y
first : forall x y z . a x y -> a (x * z) (y * z)
```

We can think of <u>a</u> as a "profunctor":

```
lmap : forall x y z . (y -> z) -> a x y -> a x z
rmap : forall x y z . (z -> x) -> a x y -> a z y
```

The operator <u>first</u> gives <u>a</u> a structure of strong profunctor.

Ex. parser which has static information about if it accepts empty word.

Transforming computational effects

Given $\underline{t}$ with monad structure, we can construct

$$a\ x\ y = x \to t\ y$$

$$\mathrm{arr} : (x \to y) \to a\ x\ y$$
$$f \mapsto \lambda x.\ \mathrm{return}\,(f\,x)$$

$$(\ggg) : a\ x\ i \to a\ i\ y \to a\ x\ y$$
$$f \qquad g \mapsto \lambda x.\ \mathrm{bind}\ (f\,x)\ (\lambda i.\ g\ i)$$

Given $\underline{a}$ with arrow structure,

$$f\ x = a\ 1\ x$$

$$\mathrm{pure} : x \to f\ x$$
$$x \mapsto \mathrm{arr}\ (\lambda *.\ x)$$

$$\mathrm{app} : f\,(x \to y) \to f\,x \to f\,y$$
$$f \qquad v \mapsto \ldots$$

Given $\underline{f}$ with idiom structure, we can construct

$$a\ x\ y = f\ (x \to y)$$

$$\mathrm{arr} : (x \to y) \to a\ x\ y$$
$$f \mapsto \mathrm{pure}\ f$$

$$(\ggg) : a\ x\ i \to a\ i\ y \to a\ x\ y$$
$$f \qquad g \mapsto \mathrm{app}\ (\mathrm{app}\ (\mathrm{pure}\ (\lambda f.\lambda g.\ g \cdot f))\ f)\ g$$

This talk: in case t or f are containers, can we propose a notion of procontainer that captures these transformations?

$$
\begin{tikzcd}
\overset{\text{Cont}}{t}\ \text{monad} \arrow[dr] & & f\ \overset{\text{cont}}{\text{idiom}} \arrow[dl] \\
& \underbrace{a\ x\ y}_{\text{ProCont}} = \ldots &
\end{tikzcd}
$$

A short review of containers

structure

$$S \lhd P \begin{cases} S : \mathrm{Set} \\ P : S \to \mathrm{Set} \end{cases} \rightsquigarrow \begin{array}{ccc} E & \longrightarrow & B \\ \sum_s P_s & \xrightarrow[\pi_1]{} & S \end{array}$$

morphisms

$$S \lhd P \xrightarrow{(f, f^\#)} T \lhd Q \begin{cases} f : S \to T \\ f^\# : \forall (s : S).\, Q(f\, s) \to P\, s \end{cases}$$

monoidal structures

$$\underset{\substack{\uparrow \text{monoids} \\ \text{monads}}}{(S \lhd P) \circ (T \lhd Q)} = \left(\sum_{s \in S} P(s) \Rightarrow T \right) \lhd \left(\lambda (s, h).\, \sum_{p \in P(s)} Q(h(p)) \right)$$

$$(S \lhd P) \ast (T \lhd Q) = (S \times T) \lhd \left(\lambda (s, t).\, P(s) \times Q(t) \right)$$

monoids ↓ ↳ Day's conv.

lax monoidal functors

Procontainers

$S : \mathrm{Set}$

$P^+ : S \to \mathrm{Set}$

$$\frac{P^- : \forall (s : S) \,.\, P^+(s) \to \mathrm{Set}}{\left[\sum_{s:S} P^+(s)\right] \to \mathrm{Set}}$$

$$\Big\downarrow \text{notation}$$

$S \triangleleft P^+ \triangleleft P^-$

$$F \longrightarrow E \longrightarrow B$$

$$\sum_{(s,p) : \sum_s P^+(s)} P^-(s,p) \longrightarrow \sum_s P^+(s) \longrightarrow S$$

Procontainers as profunctors

$$[\![S \triangleleft P^+ \triangleleft P^-]\!] : \mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}$$

$$[\![S \triangleleft P^+ \triangleleft P^-]\!](X, Y) = \sum_{s \in S} \left(X \Rightarrow \left(\sum_{p^+ \in P^+(s)} \left(P^-(s, p^+) \Rightarrow Y \right) \right) \right)$$

$$[\![S \triangleleft P^+ \triangleleft P^-]\!](f, g) : [\![S \triangleleft P^+ \triangleleft P^-]\!](A, B) \to [\![S \triangleleft P^+ \triangleleft P^-]\!](C, D)$$

$$\left(s, h : A \Rightarrow \sum_{p^+} P^-(p^+) \Rightarrow B\right) \longmapsto \left(s, \lambda c.\ \text{let } (p^+, k) = h\ (f\ c) \text{ in } (p^+, g \circ k)\right)$$

for $\quad f : C \to A$

$\qquad g : B \to D$

On strength

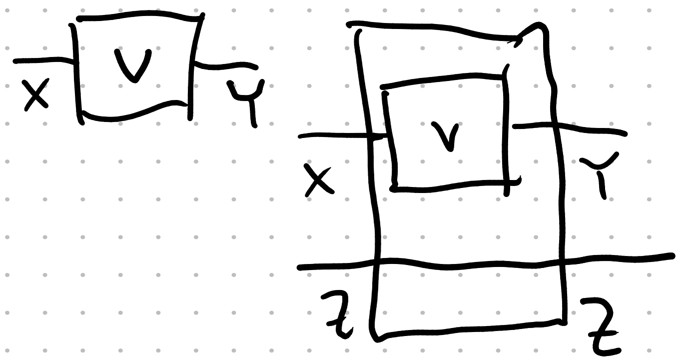
Q: What's a strength for a profunctor?

A: [Paré & Román '98] define a strength for P as

$$st_{X,Y,V} : P(X,Y) \longrightarrow P(X \otimes V, Y \otimes V)$$

nat. in X, Y
dinat. in V

$v : P(X,Y)$



subject to:

$$\begin{tikzcd}
P(X,Y) \arrow[rr, "st"] \arrow[dr, "P(\rho, id)"'] & & P(X \otimes I, Y \otimes I) \arrow[dl, "P(id, \rho)"] \\
& P(X \otimes I, Y) &
\end{tikzcd}$$

$$\begin{tikzcd}
P(X,Y) \arrow[r, "st"] \arrow[d, "st"'] & P(X \otimes (V \otimes W), Y \otimes (V \otimes W)) \arrow[d] \\
P(X \otimes V, Y \otimes V) \arrow[r, "st"'] & P((X \otimes V) \otimes W, (Y \otimes V) \otimes W)
\end{tikzcd}$$

In our case, we want $\otimes = \times$

⚠ Strength for $Set^{op} \times Set \to Set$ can fail to exist, or not be unique!

On strength for procontainers

Given a procontainer $S \triangleleft P^+ \triangleleft P^-$, define strength

$$\mathrm{st} : [\![S \triangleleft P^+ \triangleleft P^-]\!](X, Y) \longrightarrow [\![S \triangleleft P^+ \triangleleft P^-]\!](X \times V, Y \times V)$$

$$\Big(s, h : X \Rightarrow \sum_{p^+} (P^- p^+ \Rightarrow Y)\Big) \longmapsto \Big(s, \lambda(x, v).\ \mathrm{let}\ (p^+, k) = h\,x\ \mathrm{in}\ (p^+, \lambda p^-.(k(p^-), v))\Big)$$

- It satisfies previous axioms.
- Moreover, for $[\![\underline{S \triangleleft P^+ \triangleleft P^-}]\!]$ this strength is unique!
- All morphisms between procontainers are strong.

Procontainer morphisms

Look for a notion of morphism

$$S \lhd P^+ \lhd P^- \longrightarrow T \lhd Q^+ \lhd Q^-$$

$$\frac{\frac{[\![S \lhd P^+ \lhd P^-]\!] \to [\![T \lhd Q^+ \lhd Q^-]\!]}{[\![S \lhd P^+ \lhd P^-]\!](X,Y) \to [\![T \lhd Q^+ \lhd Q^-]\!](X,Y)}}{\sum_s X \Rightarrow \sum_{p^+} P^-(s,p^+) \Rightarrow Y \longrightarrow \sum_t X \Rightarrow \sum_{q^+} Q^-(t,q^+) \Rightarrow Y}$$

$$\rightsquigarrow$$

$$f : S \longrightarrow T$$

$$f^+ : \forall s : S .\ P^+ s \to Q^+ (f\, s)$$

$$f^- : \forall s : S .\ \forall p^+ : P^+(s) .\ Q^-(f(s), f^+\, s\, p^+) \to P^-(s, p^+)$$

$\boxed{\text{Defines a cat. ProCont}}$

Products and coproducts

$$(S \triangleleft P^+ \triangleleft P^-) \times (T \triangleleft Q^+ \triangleleft Q^-) =$$

$$S \times T \triangleleft \lambda(s,t).\ P^+(s) \times Q^+(t)$$

$$\triangleleft\ \lambda(s,t),(p^+,q^+).\ P^-(s,p^+) + Q^-(t,q^+)$$

$$(S \triangleleft P^+ \triangleleft P^-) + (T \triangleleft Q^+ \triangleleft Q^-) =$$

$$S + T \triangleleft \lambda\{\mathrm{inl}\ s \mapsto P^+(s)\ ;\ \mathrm{inr}\ t \mapsto Q^+(t)\}$$

$$\triangleleft\ \lambda\{\mathrm{inl}\ s, p^+ \mapsto P^-(s,p^+)\ ;$$
$$\mathrm{inr}\ t, q^+ \mapsto Q^-(t,q^+)\}$$

Convolutions

The product can be thought as conv. on $\mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}$,

where the monoidal structure on $\mathbf{Set}^{op} \times \mathbf{Set}$ is given by

$\times$ on $\mathbf{Set}^{op}$ and $+$ on $\mathbf{Set}$.

Another kind of conv. with $+$ in $\mathbf{Set}^{op}$ and $\mathbf{Set}$ is:

$$(S \triangleleft P^+ \triangleleft P^-) * (T \triangleleft Q^+ \triangleleft Q^-) =$$

$$S \times T \triangleleft \lambda(s,t).\ P^+(s) + Q^+(t)$$

$$\triangleleft\ \lambda \left\{ \begin{array}{l} (s,t), \mathrm{inl}\ p^+ \mapsto P^-(s, p^+) ; \\ (s,t), \mathrm{inr}\ q^+ \mapsto Q^-(t, q^+) \end{array} \right\}$$

Bénabou's tensor and arrows

$$P : \mathbb{C}^{op} \times \mathbb{D} \to \mathbf{Set} \qquad Q : \mathbb{D}^{op} \times \mathbb{E} \to \mathbf{Set}$$

$$(P \otimes Q)(X,Y) = \int^{I} P(X,I) \times Q(I,Y)$$

$$\dfrac{\dfrac{\dfrac{P \otimes P \to P}{(P\otimes P)(X,Y) \to P(X,Y)}}{\int^{I} P(X,I) \times P(I,Y) \longrightarrow P(X,Y)}}{P(X,I) \times P(I,Y) \longrightarrow P(X,Y)} \quad \rightsquigarrow \quad (\ggg) : a\ x\ i \to a\ i\ y \to a\ x\ y$$

Bénabou's tensor and arrows

$$(S \triangleleft P^+ \triangleleft P^-) \otimes (T \triangleleft Q^+ \triangleleft Q^-) =$$

$$S \times T \triangleleft \lambda(s,t). \sum_{p^+ \in P^+(s)} P^-(s,p^+) \Rightarrow Q^+(t)$$

$$\triangleleft \lambda(p^+,h). \sum_{p^- \in P^-(s,p^+)} Q^-(t, h(p^-))$$

Also, there's the unit:

$$\mathrm{Hom} \otimes P \simeq P \otimes \mathrm{Hom} \simeq P$$

$$\mathrm{Hom} = 1 \triangleleft \lambda *. 1 \triangleleft \lambda *. 1$$

Procontainers to containers

Given profunctor $P : Set^{op} \times Set \to Set$, we can obtain a functor by

$$P \mapsto P(1, -) \qquad \text{(this is the same as intro)}$$

In case we have a procont. $S \triangleleft P^+ \triangleleft P^-$, we do:

$$[\![S \triangleleft P^+ \triangleleft P^-]\!](1,-) = \sum_{s \in S} 1 \Rightarrow \sum_{p^+ \in P^+(s)} P^-(p^+) \Rightarrow - \simeq \sum_{s \in S} \sum_{p^+} P^-(p^+) \Rightarrow -$$

which we can think as cont. $\sum_{s \in S} P^+(s) \triangleleft P^-$

As bundles, $F \xrightarrow{q} E \xrightarrow{p} B$ gets to $F \xrightarrow{q} E$

Defines a functor $-^* : ProCont \to Cont$

Containers to procontainers: Kleisli

On the opposite direction, given $F : \mathrm{Set} \to \mathrm{Set}$, we form

$$F_* : \mathrm{Set} \to \mathrm{Set}$$

"direct image"

$$F_*(X, Y) = X \Rightarrow FY$$

Given $S \triangleleft P$, this time we construct

$$1 \triangleleft \lambda *.\, S \triangleleft P$$

As bundles, we have

$$E \xrightarrow{p} B \quad \longmapsto \quad E \xrightarrow{p} B \xrightarrow{!} 1$$

And this defines a functor $-_* : \mathrm{Cont} \to \mathrm{ProCont}$

Containers to procontainers: Cayley

There's another construction for a functor $F : \mathrm{Set} \to \mathrm{Set}$:

$$F_! (X, Y) = F(X \Rightarrow Y)$$ [Pastro, Street '07]

is also a profunctor

In the case of a container $S \triangleleft P$,

$$[\![S \triangleleft P]\!]_! (X, Y) = \sum_{s \in S} P(s) \Rightarrow (X \Rightarrow Y) = \sum_{s \in S} X \Rightarrow (P(s) \Rightarrow Y)$$

which is a procontainer $S \triangleleft \lambda s.\, 1 \triangleright \lambda(s, *).\, P(s)$

In terms of bundles, we have

$$E \xrightarrow{p} B \quad \longmapsto \quad E \xrightarrow{p} B \xrightarrow{id} B$$

This defines a functor $-_! : \mathrm{Cont} \to \mathrm{ProCont}$

Monoidality of Kleisli and Cayley

$-_*$ is monoidal w.r.t. $\circ$ in Cont and $\otimes$ in ProCont

$-_!$ is monoidal w.r.t. $*$ in Cont and $\otimes$ in ProCont

Monoidal functors send monoids to monoids:

$$\left.\begin{array}{l} \bullet\ (S \triangleleft P, \mu, \eta) \longmapsto (S \triangleleft P)_* \text{ monoid in } (\text{ProCont}, \otimes, \text{Hom}) \\[2em] \bullet\ (S \triangleleft P, m, e) \longmapsto (S \triangleleft P)_! \text{ monoid in } (\text{ProCont}, \otimes, \text{Hom}) \end{array}\right\} \text{arrows}$$

$-^*$ is monoidal w.r.t. $\otimes$ in ProCont and $*$ in Cont

- $(S \triangleleft P^+ \triangleleft P^-, \text{sss}, \text{arr}) \longmapsto (S \triangleleft P^+ \triangleleft P^-)^*$ monoid in $(\text{Cont}, *, \text{Id})$

Adjunctions of Kleisli and Cayley

There's a number of nice obs. for these functors

$$\frac{C_! \to P}{C \to P^*} \qquad\qquad \frac{P^* \to C}{P \to C_*}$$

$$\begin{tikzcd}
\mathrm{Cat} \arrow[r, bend right, "-_*"'] \arrow[r, phantom, "\perp"] & \mathrm{ProCat} \arrow[l, bend right, "-^*"'] \arrow[r, bend right, "-^*"'] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend right, "-_!"']
\end{tikzcd}$$

$\Downarrow$ $\qquad\qquad\qquad\qquad$ $\Downarrow$

$(-^*)_* = \Diamond$ monad on ProCat $\qquad$ $(-^*)_! = \Box$ comonad on ProCat

Combinatorial aspects: containers

Extra structure that give a container a monad or lax monoidal functor structure

[Uustalu '17]

We define an *mnd-container* to be a container (S, P) with operations

- $\mathsf{e} : S$
- $\bullet : \Pi s : S.\, (P\ s \to S) \to S$
- $q_0 : \Pi s : S.\, \Pi v : P\ s \to S.\, P\,(s \bullet v) \to P\,s$
- $q_1 : \Pi s : S.\, \Pi v : P\ s \to S.\, \Pi p : P\,(s \bullet v).\, P\,(v\,(v \nwarrow_s p))$

where we write $q_0\ s\ v\ p$ as $v \nwarrow_s p$ and $q_1\ s\ v\ p$ as $p \nearrow_v s$, satisfying

- $s = s \bullet (\lambda \_.\, \mathsf{e})$
- $\mathsf{e} \bullet (\lambda \_.\, s) = s$
- $(s \bullet v) \bullet (\lambda p''.\, w\,(v \nwarrow_s p'')\,(p'' \nearrow_v s)) = s \bullet (\lambda p'.\, v\,p' \bullet w\,p')$
- $p = (\lambda \_.\, \mathsf{e}) \nwarrow_s p$
- $p \nearrow_{\lambda \_.\, s} \mathsf{e} = p$
- $v \nwarrow_s ((\lambda p''.\, w\,(v \nwarrow_s p'')\,(p'' \nearrow_v s)) \nwarrow_{s \bullet v} p) = (\lambda p'.\, v\,p' \bullet w\,p') \nwarrow_s p$
- $((\lambda p''.\, w\,(v \nwarrow_s p'')\,(p'' \nearrow_v s)) \nwarrow_{s \bullet v} p) \nearrow_v s =$ $\mathsf{let}\ u\,p' \leftarrow v\,p' \bullet w\,p'\ \mathsf{in}\ w\,(u \nwarrow_s p) \nwarrow_{v\,(u \nwarrow_s p)} (p \nearrow_u s)$
- $p \nearrow_{\lambda p''.\, w\,(v \nwarrow_s p'')\,(p'' \nearrow_v s)} (s \bullet v) =$ $\mathsf{let}\ u\,p' \leftarrow v\,p' \bullet w\,p'\ \mathsf{in}\ (p \nearrow_u s) \nearrow_{w\,(u \nwarrow_s p)} v\,(u \nwarrow_s p)$

We define an *lmf-container* as a container (S, P) with operations

- $\mathsf{e} : S$
- $\bullet : S \to S \to S$
- $q_0 : \Pi s : S.\, \Pi s' : S.\, P\,(s \bullet s') \to P\,s$
- $q_1 : \Pi s : S.\, \Pi s' : S.\, P\,(s \bullet s') \to P\,s'$

where we write $q_0\ s\ s'\ p$ as $s' \nwarrow_s p$ and $q_1\ s\ s'\ p$ as $p \nearrow_{s'} s$, satisfying

- $\mathsf{e} \bullet s = s$
- $s = s \bullet e$
- $(s \bullet s') \bullet s'' = s \bullet (s' \bullet s'')$
- $\mathsf{e} \nwarrow_s p = p$
- $p \nearrow_s \mathsf{e} = p$
- $s' \nwarrow_s (s'' \nwarrow_{s \bullet s'} p) = (s' \bullet s'') \nwarrow_s p$
- $(s'' \nwarrow_{s \bullet s'} p) \nearrow_{s'} s = s'' \nwarrow_{s'} (p \nearrow_{s' \bullet s''} s)$
- $p \nearrow_{s''} (s \bullet s') = (p \nearrow_{s' \bullet s''} s) \nearrow_{s''} s'$

Combinatorial aspects: procontainers

We define an *arrow-procontainer* as a procontainer $(S \lhd P^+ \lhd P^-)$ with operations

- $e : S$
- $\bullet : S \to S \to S$
- $r : P^+ e$
- $q_0 : \Pi\, s_l : S.\ \Pi\, s_r : S.\ \Pi\, p_l^+ : P^+ s_l.\ (P^-(s_l, p_l^+) \to P^+ s_r) \to P^+(s_l \bullet s_r)$
- $q_1 : \Pi\, s_l : S.\ \Pi\, s_r : S.\ \Pi\, p_l^+ : P^+ s_l.\ \Pi\, h : P^-(s_l, p_l^+) \to P^+ s_r.$
  $P^-(s_l \bullet s_r, q_0(s_l, s_r, p_l^+, h)) \to P^-(s_l, p_l^+)$
- $q_2 : \Pi\, s_l : S.\ \Pi\, s_r : S.\ \Pi\, p_l^+ : P^+ s_l.\ \Pi\, h : P^-(s_l, p_l^+) \to P^+ s_r.$
  $\Pi\, p^- : P^-(s_l \bullet s_r, q_0(s_l, s_r, p_l^+, h)).\ P^-(s_r, h(q_0(s_l, s_r, p_l^+, h, p^-)))$

where we write $q_0(s_l, s_r, p_l^+, h)$ as $p_l^+ \nwarrow_{s_l,s_r} h$, $q_1(s_l, s_r, p_l^+, h, p^-)$ as $(p_l^+, p^-) \nearrow_{s_l,s_r} h$ and $q_2(s_l, s_r, p_l^+, h, p^-)$ as $(p_l^+, p^-) \searrow_{s_l,s_r} h$, satisfying

- $s \bullet e = s$
- $e \bullet s = s$
- $s \bullet (s' \bullet s'') = (s \bullet s') \bullet s''$
- $p^+ \nwarrow_{s,e} (\lambda\_.\ r) = p^+$
- $r \nwarrow_{e,s} (\lambda\_.p^+) = p^+$
- $(p^+, p^-) \nearrow_{s,e} (\lambda\_.r) = p^-$
- $(r, p^-) \searrow_{e,s} (\lambda\_.p^+) = p^-$
- $p^+ \nwarrow_{s,s' \bullet s''} (\lambda p^-.v(p^-) \nwarrow_{s',s''} w(p^-)) =$
  $p^+ \nwarrow_{s,s'} v \nwarrow_{s \bullet s',s''} (\lambda p^-.w\,((p^+, p^-) \nearrow_{s,s'} v)\,((p^+, p^-) \searrow_{s,s'} v))$
- $(p^+, p^-) \nearrow_{s,s' \bullet s''} \left(\lambda p_2^-.v(p_2^-) \nwarrow_{s',s''} w(p_2^-)\right) =$
  $\left(p^+, (p^+ \nwarrow_{s,s'} v, p^-) \nearrow_{s \bullet s',s''} \left(\lambda p_2^-.w(\left(p^+, p_2^+\right) \nearrow_{s,s'} v)(\left(p^+, p_2^-\right) \searrow_{s,s'} v)\right)\right) \nearrow_{s,s'} v$
- $\left(v(h), (p^+, p^-) \searrow_{s,s' \bullet s''} \left(\lambda p_2^-.v(p_2^-) \nwarrow_{s',s''} w(p_2^-)\right)\right) \nearrow_{s',s''} w(h) =$
  $\left(p^+, (p^+ \nwarrow_{s,s'} v, p^-) \nearrow_{s \bullet s',s''} \left(\lambda p_2^-.w(\left(p^+, p_2^-\right) \nearrow_{s,s'} v)(\left(p^+, p_2^-\right) \searrow_{s,s'} v)\right)\right) \searrow_{s,s'} v$
  w. $h = \left(p^+, (p^+ \nwarrow_{s,s'} v, p^-) \nearrow_{s \bullet s',s''} \left(\lambda p_2^-.w(\left(p^+, p_2^-\right) \nearrow_{s,s'} v)(\left(p^+, p_2^-\right) \searrow_{s,s'} v)\right)\right) \nearrow_{s,s'} v$
- $\left(v(h), (p^+, p^-) \searrow_{s,s' \bullet s''} \left(\lambda p_2^-.v(p_2^-) \nwarrow_{s',s''} w(p_2^-)\right)\right) \searrow_{s',s''} w(h) =$
  $(p^+ \nwarrow_{s,s'} v, p^-) \searrow_{s \bullet s',s''} (p^+ \nwarrow_{s,s'} v, p^-) \nearrow_{s \bullet s',s''} \left(\lambda p_2^-.w(\left(p^+, p_2^-\right) \nearrow_{s,s'} v)(\left(p^+, p_2^-\right) \searrow_{s,s'} v)\right)$
  w. $h = \left(p^+, (p^+ \nwarrow_{s,s'} v, p^-) \nearrow_{s \bullet s',s''} \left(\lambda p_2^-.w(\left(p^+, p_2^-\right) \nearrow_{s,s'} v)(\left(p^+, p_2^-\right) \searrow_{s,s'} v)\right)\right) \nearrow_{s,s'} v$

Discussion

An interesting class of arrows come from a comonad D, and considering:

$$A(X,Y) = DX \Rightarrow Y$$

Comonads which are containers are a nice structure, but given a comonad container, we don't get a procontainer:

$$D = S \triangleleft P \longmapsto \Big[\sum_{s \in S} (P(s) \Rightarrow X)\Big] \Rightarrow Y \cong \forall (s:S).(P(s) \Rightarrow X) \Rightarrow Y$$

$$(DX \Rightarrow Y) \Rightarrow D(X \times Z) \Rightarrow (Y \times Z)$$

$$h \;,\; d \longmapsto \begin{array}{l} \text{let } (x,z) = \varepsilon(d) \text{ in} \\ \text{let } \ell = D(\pi_1)(d) \text{ in} \\ (h(\ell), z) \end{array}$$

The strength already uses the comonadic structure!

Conclusion

Departing from some relationship between monads, idioms and arrows, we proposed a notion of procontainer which is closed under interesting operations and can mimic the situation for comp. effects.

- Bénabou's tensor
- Products & coproducts
- Other convolutions

Are there more general notions of procontainers?

Thanks!