The Berry Order
(Ideas from 1980s Stable Domain Theory)

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Preface

This talk is about things that I did over 30 years ago.

I have struggled to remember my own work,
let alone that of others, for which I apologise.

François Lamarche (and maybe others) also worked on these ideas at the time, but with different points of view, in particular different notions of "stability".

This presentation benefits from emotional distance and maturity.
Otherwise it is the mathematical point of view that I had at the time.

The goal is to present ideas that students in category theory
may wish to develop in the future.
(So details of the ordered version are left out.)

The focus is on cartesian (natural) transformations
since these seemed to be missing
from last year's Polynomial Functors Workshop.

Stable categories

In case you are expecting polynomial functors on **Set**, toposes or locally cartesian closed categories, ...

Stable categories have and stable functors preserve

- pullbacks

$$\begin{array}{ccc} P & \longrightarrow & X_2 \\ \downarrow & \lrcorner & \downarrow \\ X_1 & \longrightarrow & X_0 \end{array}$$

- and cofiltered limits,

$$\lim X_i \longrightarrow \ldots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0$$

- which together generate wide pullbacks, plus
- filtered colimits for domain theory, but
- not necessarily a terminal object or equalisers.

Laminated would have been a better word for this structure.

Yves Diers and multiadjoints

Diers worked in commutative algebra, for which he generalised notions from categorical algebra to disjunctive theories.

For example, a field K satisfies

$$x : K \quad \vdash \quad x = 0 \quad \vee \quad \exists !\, y{:}K.\ x \cdot y = 1,$$

where both $\vee$ and $\exists$ are uniquely satisfied.

The category of fields and homomorphisms
is the typical example of a stable category.

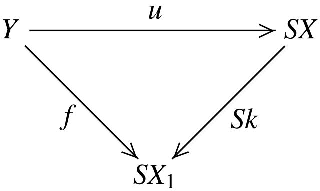
Constructions (functors) in this topic are *like* adjunctions,
except they can be multi-valued.
This applies in particular to forgetful functors and (multi)colimits.

They can also have automorphisms.

Diers wrote a lot of papers, containing a lot of examples.
`www.researchgate.net/scientific-contributions/Yves-Diers-20`
`ncatlab.org/nlab/show/Yves+Diers`

Reminder: universal maps

Let $S : \mathcal{X} \equiv \mathbf{Vect} \rightarrow \mathcal{Y} \equiv \mathbf{Set}$ be the forgetful functor
from the category of vector spaces to that of sets.
For any set (basis) Y
there is a vector space X
with a function $u : Y \rightarrow SX$ that is a universal map

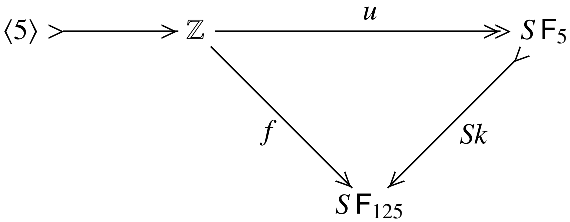


i.e. for any function ($\mathcal{Y}$-morphism) $f : Y \rightarrow SX_1$
to the underlying set (S) of a vector space $X_1 \in \mathcal{X}$
there is a unique linear map $k : X \rightarrow X_1$
($\mathcal{X}$-morphism, vector space homormorphism)
making the triangle of functions ($\mathcal{Y}$-maps) commute.

Fields and integral domains

Now let $S : \mathcal{X} \equiv \mathbf{Field} \to \mathcal{Y} \equiv \mathbf{IntDom}$ be the forgetful functor from fields to integral domains.

Any ring homomorphism f from a field to an integral domain,
eg $\mathbb{Z} \to \mathsf{F}_{125}$
has a prime kernel and factorises:

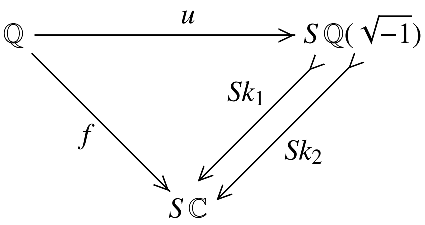


Therefore there is a universal map for each characteristic p.
The set of them is called the Zariski spectrum.

Splitting fields for polynomials

Now let $S : \mathcal{X} \equiv \textbf{Field}[x^2 + 1] \to \mathcal{Y} \equiv \textbf{Field}$
be the forgetful functor from fields that split a particular polynomial to all fields.

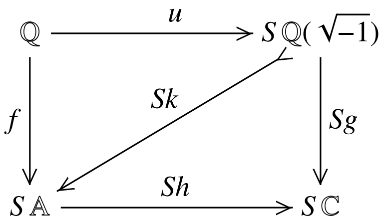
Then any field homomorphism $f : Y \to SX_1$ factorises:



However, the inclusion $k : \mathbb{Q}(\sqrt{-1}) \rightarrowtail \mathbb{C}$ isn't unique
because it could take $\sqrt{-1}$ to either $+i$ or $-i$.
Also, the "universal" object has automorphisms.

Uniqueness in a slice

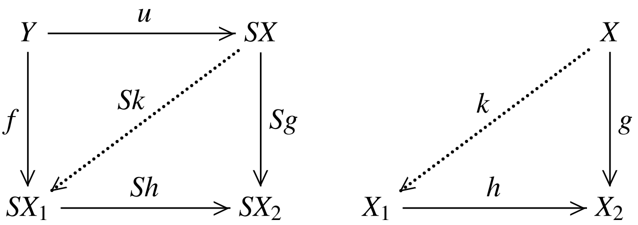
We restore uniqueness of $k : X \to X_1$ by working in a slice category:



Let $\mathbb{A}$ be a field in which the polynomial splits
accompanied by a homomorphism $h : \mathbb{A} \to \mathbb{C}$.
Now k is unique such that both triangles commute.

Candidates (my formulation)

Let $S : \mathcal{X} \to \mathcal{Y}$ be a functor.
A candidate is a $\mathcal{Y}$-morphism $u : Y \to SX$ such that
in every commutative square ($u\,;Sg = f\,;Sh$)

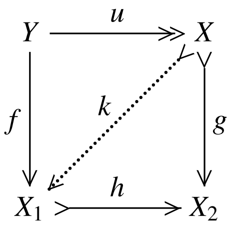


there is a unique $\mathcal{X}$-map $k : X \to X_1$
such that both $u\,;Sk = f$ and $k\,;h = g$.

Definition: The functor S is stable if
every $f : Y \to SX_1$ is $f = u\,;Sk$ where u is a candidate.

Example: Factorisation systems (a “dangerous bend”)

For any factorisation system $(\mathcal{E}, \mathcal{M})$ on a category $\mathcal{X}$,
the orthogonality property says that
the inclusion $S : \mathcal{M} \hookrightarrow \mathcal{X}$ is a stable functor.



(We will need this definition later.)

Such $S : \mathcal{M} \hookrightarrow \mathcal{X}$ is also bijective on objects
and injective on morphisms.

But this is not enough to make $\mathcal{M}$ part of a factorisation system:
The composite of two “epis” ($\mathcal{E}$-maps) is “epi”, but candidates need not compose. (This is a correction of my 1988 work.)

Polynomial functors

Any polynomial functor $S : \mathbf{Set} \to \mathbf{Set}$ of the form

$$SX \equiv \coprod_{i \in I} A_i \times X^{B_i}$$

is stable.

Its candidates $u : \mathbf{1} \to SX$ select

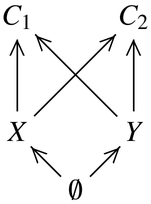
- some $i \in I$,
- some $a \in A_i$,
- $\mathsf{id} : X^{B_i}$ for $X \equiv B_i$.

Then the factorisation of $f : \mathbf{1} \to SX_1$ is $f = u \,;\, g$, where $g : B_i \to X$ is the element of X^{B_i} given by f.

This extends to $\coprod_{i \in I} A_i \times X^{B_i} / G_i$.

Multi-colimits

In particular, stable categories (those with wide pullbacks) have multi-colimits.



These could have automorphisms, in which case they're called polycolimits.

There are examples in the category of fields. (But I can't remember my Galois Theory.)

Slices

Another way to see stable functors is that
they have left adjoints on each slice.

Stable categories and functors should be thought of as
made up of their slices,
which is why laminated would be a better name.

So the extreme cases of stable functors are:

- homomorphisms have ordinary single left adjoints
- isotomies are equivalences on each slice,
  so they are equivalent to
  fibrations whose fibres are groupoids.

We will show that these form a factorisation system.

Project for a categorical algebra PhD thesis

Recall Gabriel–Ulmer duality between

- locally finitely presentable categories and
- lex categories (*i.e.* with finite limits).

This generalises to

- locally finitely poly-presentable categories and
- categories with finite poly-limits.

(We say poly- instead of multi- when
the candidates can have groups of autmorphisms.)

Also

These things should be linked to the classifying toposes for disjunctive theories.

Gérard Berry and sequential algorithms

Turning from algebra to computer science,

If a sequential algorithm
has produced a certain part of its output
then there is a unique minimal part of its input that was required.

So if $y \subset sx_2$ then
there is $x \subset x_2$ with $y \subset sx$
and whenever $y \subset sx_1$ and $x_1 \subset x_2$ then $x \subset x_1$.

Hence s satisfies the order analogue of our definition.

Reference: Bottom-up computation of recursive programs
Revue française d'automatique informatique recherche opérationnelle.
Informatique théorique.
tome 10, no R1 (1976), p. 47–82
`www.numdam.org/item?id=ITA_1976__10_1_47_0`
This cites the 1973 PhD and 1974 Doctorat d'Etat theses of J. Vuillemin.
This is where the word *stable* comes from, I believe.

The Berry order and CCCs of stable domains

Berry later gave a syntactical analysis of Plotkin's PCF and also an interpretation in the cartesian closed category of dI-domains.

For $f \sqsubseteq g$ and $x \sqsubseteq y$ we must have $fx = gx \sqcap fy$.

This is known as the Berry order.

In order to interpret the fixed points in PCF, we also require the domains to have and functions to preserve directed joins, as in Scott-style domain theory.

The categorical version of the Berry order is easier to see ...

Reference: Stable Models of Typed Lambda-calculi
Proc. 5th Coll. on Automata, Languages and Programming,
Lectures Notes in Computer Science 62,
Springer-Verlag, pp. 72-89, 1978

Stable binary functors and cartesian transformations

For any two morphisms $f : X_1 \to X_2$ in $\mathcal{X}$ and $g : Y_1 \to Y_2$ in $\mathcal{Y}$,

$$\begin{array}{ccc} X_1 \times Y_1 & \xrightarrow{f \times Y_1} & X_2 \times Y_1 \\ {\scriptstyle X_1 \times g}\downarrow \lrcorner & & \downarrow{\scriptstyle X_2 \times g} \\ X_1 \times Y_2 & \xrightarrow{f \times Y_2} & X_2 \times Y_2 \end{array}$$

is always a pullback in $\mathcal{X} \times \mathcal{Y}$.

So any stable functor $S : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ must preserve it.

In particular, if $\mathcal{Y} \equiv [\mathcal{X} \to \mathcal{Z}]$ is an exponential,

for $\mathsf{ev} : \mathcal{X} \times [\mathcal{X} \to \mathcal{Z}] \to \mathcal{Z}$ to be stable, the naturality square

$$\begin{array}{ccc} S_1(X_1) & \xrightarrow{S_1(f)} & S_1(X_2) \\ {\scriptstyle \phi_{X_1}}\downarrow \lrcorner & & \downarrow{\scriptstyle \phi_{X_2}} \\ S_2(X_1) & \xrightarrow{S_2(f)} & S_2(X_2) \end{array}$$

for $\phi : S_1 \to S_2$ at $f : X_1 \to X_2$ must be a pullback.

Then ϕ is called a cartesian transformation.

CCCs of stable domains

For reasons that we shall see later, the slices (down-sets) of *stable* function-spaces inherit the properties of the spaces more closely than in Scott-style domain theory.

So these slices can be more familiar structures:

- Berry's dI-domains, where "d" means distributive and "I" means that the slices by compact elements are finite;
- Girard's qualitative domains and coherence spaces, whose slices are distributive or Boolean algebras.

In fact, for the function-spaces to be boundedly complete, the slices must be distributive.

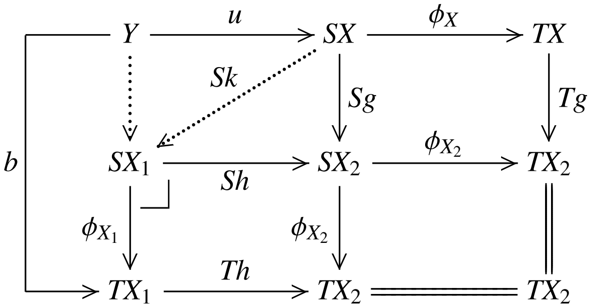
Jean-Yves Girard said that he discovered linear logic from the coherence space model, which is simple graph theory, with obvious linear, multiplicative and list ("exponential") structures. He then used them to model his System F.

In fact, this can also be done with Scott-style domain theory.

Back to category theory (towards my theorem)

When does composition with a natural transformation $\phi : S \to T$ take S-candidates to T-candidates?

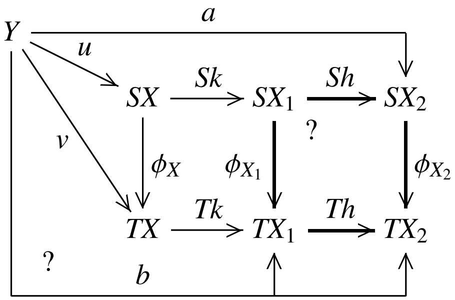
If ϕ is a cartesian transformation then it does:

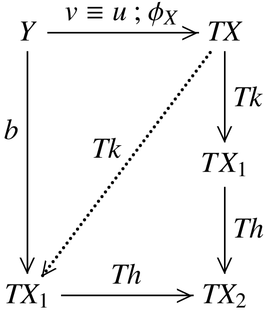


The commutative square tests T-candidacy of $u\,;\phi_X$.
Then b factors through the pullback and S-candidacy gives k, which also serves for T-candidacy. For uniqueness use the pullback and S-candidacy again.

Composition with cartesian transformations

Conversely, if composition with a natural transformation $\phi : S \to T$ takes S-candidates to T-candidates then ϕ is cartesian:





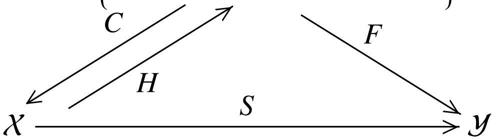
Let a and b test the claim that the bold square is a pullback. Factorise a through an S-candidate u; by its property it factors through SX_1. The lower left triangle commutes by uniqueness of k since $u \,;\, \phi_X$ is a T-candidate.

Trace Factorisation (1)

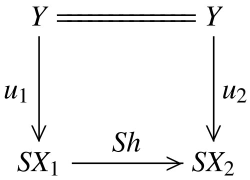
Any stable functor $S : \mathcal{X} \to \mathcal{Y}$ is the composite $S = F \cdot H$ where

- $H : \mathcal{X} \to \mathcal{T}$ is a homomorphism (has a left adjoint C) and
- $F : \mathcal{T} \to \mathcal{Y}$ is an isotomy (equivalence on slices).

$$\mathcal{T} \equiv \left\{\text{candidates} \quad T \equiv (Y \xrightarrow{u} SX)\right\}$$

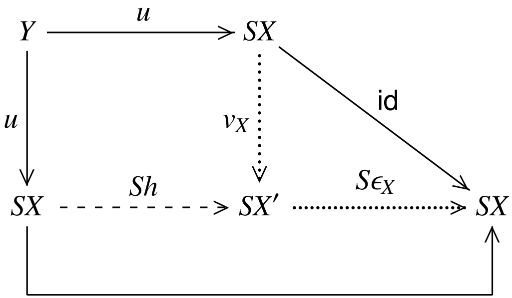


C and F are the projections (the next slides gives H)
F is an isotomy because its vertical maps (h, id) are isos:



Trace Factorisation (2)

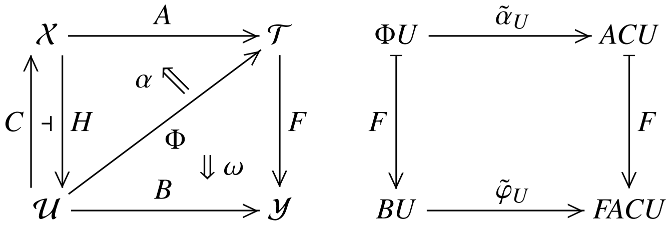
To find the right adjoint H of C,
factorise $\mathsf{id}_{SX} = v_X \,;\, s\epsilon_X$
and form the fill-in $h : X \to X'$:



Then $HX \equiv (SX \xrightarrow{v_X} SX')$ and $\eta_T \equiv (u, h)$.
This has $F(HX) \;=\; SX$ as required.
Also $C\eta_T : \epsilon_{CT} \equiv h \,;\, \epsilon_X = \mathsf{id}$

Trace Factorisation (3)

For a factorisation system we also need a universality property, orthogonality, so
let F be an isotomy, $C \dashv H$ and A, B stable.



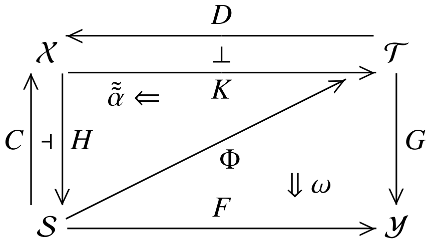
This is a 2-categorical situation, so instead of commutative triangles, we have a bijection beween

- natural transformations $\varphi : B \cdot H \to F \cdot A$ and
- diagonals $\Phi : \mathcal{U} \to \mathcal{T}$ with $\alpha : \Phi \to A \cdot C$ and $\omega : \Phi \cdot F \cong B$.

Assume for simplicity that F is a fibration. Then the adjoint transpose $\tilde{\alpha} : \Phi \to A \cdot C$ is the prone (cartesian) lifting of $\tilde{\varphi} : B \to F \cdot A \cdot F$.

Trace Factorisation (4)

Now factorise two stable functors S and T *via* their traces $\mathcal{S}$ and $\mathcal{T}$



For any cartesian transformation $\phi : S \to T$,
the diagonal functor Φ acts by post-composition with ϕ
and then both triangles commute up to equality.

The functor Φ is both an isotomy and is a left adjoint.

Now suppose S and T were already functors like this.
Then C, H, D, K and Φ would be equivalences
and ϕ an isomorphism.

If further $S = T$, in fact $\phi = \mathsf{id}$ (I need to check this).

Rigid adjunctions are pretty special

The diagonal Φ is both an isotomy and is a left adjoint.
Such functors have very strong properties:

A rigid adjunction is an internal adjunction in the 2-category
of stable categories, stable functors and cartesian transformations.
That is, its unit and counit are cartesian.

Every rigid adjunction is comonadic.

Every stable functor with cartesian $\epsilon : S \to \mathsf{id}$
has unique cartesian $\nu : S \to S \cdot S$ making a comonad.

If $\mathcal{T}$ has and $\Phi : \mathcal{U} \to \mathcal{T}$ preserves pullbacks, and
Φ has a right adjoint whose unit and counit are cartesian,
Then Φ is an isotomy.

In the order case (dI-domains and coherence spaces)
rigid embeddings are simply graph embeddings of traces.

Cartesian closed categories

The trace factorisation of stable functors
and the representation of cartesian transformations between them
make it easy to study slices of function spaces.

Function-spaces can be generalised to dependent products.

At any rate, the stable case is much easier than
the corresponding thing in Scott-style domain theory.

Flavours of CCCs of stable domains

The *essential* property is that functors preserve pullbacks.
(Even that can be weakened: Joyal and Lamarche had models in which pullbacks are only preserved up to epimorphisms.)

Also cofiltered limits and filtered colimits.

Evaluation does not preserves equalisers. (Lamarche)

Otherwise it's like pizza toppings: choose your

- slices: distributive lattices, toposes, ...
- spread: cardinality of multi-colimits, and
- spin: what groups of automorphisms they can have.

The limit–colimit coincidence

Recall from Dana Scott's early work on domain equations:

For any filtered diagram of embedding–projection pairs
(or even just adjoint pairs of continuous functions)
the colimit of the embeddings (left adjoints) is isomorphic to
the limit of the projections (right adjoints).

I called these bilimits (whence bifinite)
but if I had known the usage in 2-category theory,
I would have called them ambilimits.

Then $[\mathsf{bilim}_i \mathcal{X}_i \longrightarrow \mathsf{bilim}_j \mathcal{Y}_j] = \mathsf{bilim}_i \mathsf{bilim}_j [\mathcal{X}_i \longrightarrow \mathcal{Y}_j]$

The same happens with rigid adjunctions.

(There is a pushout–pullback coincidence too.)

The domain of domains

Because of the limit–colimit coincidence,
the category of domains and comparisons looks like a domain.

In Scott-style domain theory it's really a 2-category,
and to address this we would end up with ∞-categories.
But in the stable case the 2-cells are isomorphisms,
possibly just identities.

When I first heard of univalence in HoTT,
I thought, I've seen that before in domain theory.

I suggest HoTT-theorists should look at how to define domains of unrestricted h-level so that there is naturally a domain of domains.

Since we're also using Scott continuity and initial objects,
size issues are not relevant.

That is, except for the spread of the domain of domains,
which is the supremum of those of the domains,
so we must restrict to bounded completeness (spread 0,1)
which requires the slices to be distributive.

Second order polymorphism

Girard's System F allows quantification over types, *e.g.*

$$\forall X. \quad X \longrightarrow (X \to X) \longrightarrow X.$$

We can model this by using the domain of domains.
Then ("first order") dependent products model the quantifier.

This was done in Scott-style domain theory too, *e.g.*
Martin Hyland and Andrew Pitts, "The Theory of Constructions: Categorical Semantics and Topos-Theoretic Models" in AMS Contemporary Mathematics 92.

Calculations in the stable version are *slightly* more tractable.
However, we never managed to find a model in which the type above is interpreted as anything resembling $\mathbb{N}$.
(See Appendix A of *Proofs and Types*.)

If nobody's listening to you

If you have some good results, but nobody else seems to care,

Maybe you need to find another project.

BUT

Make sure you write up your results — including your mistakes.

The subject may become fashionable (again),
and then you'll get the credit!

Youth versus maturity

“No mathematician should ever allow him to forget that mathematics, more than any other art or science, is a young man’s game.”
(G.H. Hardy, *A Mathematician’s Apology*)

“Good general theory does not search for the maximum generality, but for the right generality.”
(Saunders Mac Lane, *Categories for the Working Mathematician*)

What I was younger, I deliberately did the most difficult versions of things.

In category theory it’s easy to find difficult constructions.

But what is more difficult is to judge the significance of your ideas,
to tell a story
in the context of the work of others.

Younger and older mathematicians have different roles to play.