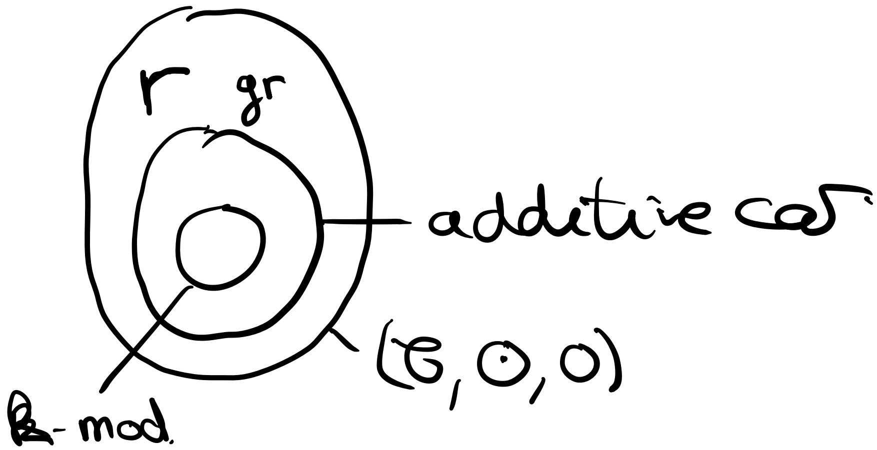
Eilenberg Mac Lane polynomial functors (3/3)

Classification results. $\mathcal{P}ol_d(\mathcal{C}, k) \simeq$



· $\mathcal{C} = (k\text{-mod}, \oplus, 0)$ k comm. ring ; $F : k\text{-mod} \to k\text{-Mod}$

$\mathbb{B} = \Sigma$: finite sets and bijections

$$\mathcal{F}(\Sigma, k) \longrightarrow \prod_{d \in \mathbb{N}} \mathcal{P}ol_d(k, k) \xrightarrow{\oplus} \mathcal{F}(k, k)$$

$$F \longmapsto \left(V \mapsto F(d) \otimes_{\sigma_d} V^{\otimes d}\right) \longmapsto \left(V \mapsto \bigoplus_{d \in \mathbb{N}} F(d) \otimes_{\sigma_d} V^{\otimes d}\right)$$

$$\simeq \left(F(d) \otimes V^{\otimes d}\right)_{\sigma_d}$$

$$\uparrow$$

$$F(d) \otimes V^{\otimes d}$$

$$k\text{-Mod} \underset{\leq d}{\to} k\text{-Mod} \underset{\leq 1}{\to} k\text{-Mod}$$

· $\underline{k = \mathbb{Q}}$

<u>Thm</u>: For $M \in \mathcal{F}(k\text{-mod}, k)$ poly of $\leq d$ then $\exists F \in \mathcal{F}(\Sigma, k)_d$ s.t.

$$M(V) = \bigoplus_{n=0} F(n) \otimes_{\sigma_n} V^{\otimes n}$$

→ we recover the def of poly functors given in [MacDonald 84] [Joyal 86]

analytic functors $\bigoplus_{n \in \mathbb{N}} F(n) \otimes_{\sigma_n} V^{\otimes n}$.

$\mathcal{F}(\mathbb{Q}\text{-mod}, \mathbb{Q})$ is semi-simple.

<u>But</u> no more true in s^{∞}

<u>ex</u>: $\underline{k = \mathbb{F}_2}$

· $\Gamma^2(V) = (V^{\otimes 2})^{\sigma_2}$ not of the previous form

$\mathrm{fdeg}\ \mathfrak{g} \simeq 2$ $\Rightarrow T$

- $\mathcal{F}(\mathbb{F}_2, \mathbb{F}_2)$ is not semi simple

  $\to$ we have non trivial extension.

- Eilenberg–Watts theorem.

$$F = \underbrace{F(0)} \oplus \underbrace{\bar{F}} \qquad \bar{F}(0) = 0$$

$$\mathrm{Pol}_d(R\text{-mod}, k) \simeq k\text{-Mod} \times \mathrm{Add}(R\text{-mod}, k)$$

<u>Thm</u>: (Eilenberg 60, Watts 60)

$$\begin{array}{ccc} \mathrm{Add}(R\text{-mod}, k) & \xrightarrow{\simeq} & (R^{op} \otimes_{\mathbb{Z}} k)\text{-Mod} \\ F & \longmapsto & F(R) \\ M \otimes_R - & \longleftarrow & M \end{array}$$

$\mathrm{Pol}_d(R\text{-mod}, k)$??

- <u>Dold-Kan type thm of Pirashvili</u>

  Ω: category of finite sets and surjections

Thm: (Pirashvili 2000)

$$\mathcal{F}(\Gamma, \mathbb{Z}) \xrightarrow{\simeq} \mathcal{F}(\Omega, \mathbb{Z})$$

finite pointed sets + pointed maps

$$\begin{array}{ccc} F & \longmapsto & \left(\begin{array}{ccc} n & \mapsto & \mathrm{cr}_n F([1], \dots, [1]) \subset F([n]) \\ \downarrow & & \downarrow \\ m & \mapsto & \mathrm{cr}_m F(\qquad\qquad\qquad F([m]) \end{array} \right. \end{array}$$

$$\mathrm{Pol}_d(\Gamma, \mathbb{Z}) \xrightarrow{\simeq} (\mathcal{F}(\Omega, \mathbb{Z}))_{\leq d+1}$$

Ex:

$$\begin{array}{ccc} \Omega & \longrightarrow & \mathbb{k}\text{-Mod} \\ 1 & \longmapsto & M_1 \\ \uparrow & & \uparrow P \\ \circlearrowright 2 & \longmapsto & M_2 \circlearrowleft T \quad T^2 = Id. \end{array}$$

corresponds to a functor in $\mathrm{Pol}_2(\Gamma, k)$

- **Functors on gr**: finitely generated free group $F_n = \langle x_1, \dots, x_n \rangle$.
- $\mathcal{F}(gr, k)$ is not semi-simple.

ex: $0 \to \mathfrak{a}^{\otimes 2} \to P_2 \to \mathfrak{a} \to 0$ is not split.

Thm: [Djament-V. 2015] $k = \mathbb{Z}$ or $\mathbb{Q}$.
There is a recollement diagram

$$
\begin{tikzcd}
\mathrm{Pol}_{d-1}(gr,k) \arrow[r, hook, "i"'] & \mathrm{Pol}_d(gr,k) \arrow[l, bend right, "q_{d-1}"'] \arrow[l, bend left, "p_{d-1}"] \arrow[r, "cr_d"'] & k[\mathfrak{S}_d]\text{-mod} \arrow[l, bend right, "\alpha_d"'] \arrow[l, bend left, "\beta_d"]
\end{tikzcd}
$$

- i is fully faithful and its image is the kernel of cr_d
- [Rem: $\alpha_d(M) = (P_k)^{\otimes d} \otimes_{\mathfrak{S}_d} M$]

  $ab \xrightarrow{a} gr \xrightarrow{k[-]} k\text{-mod}$
- The unit $Id \to cr_d \circ \alpha_d$ is an iso.
- The unit $Id \to p_{d-1} \circ i$ is an iso.

$$\mathrm{Pol}_d(gr,k) / \mathrm{Pol}_{d-1}(gr,k) \simeq k[\mathfrak{S}_d]\text{-mod}.$$

<u>Cor</u> [Djament-V. 2015]

$F \in \mathrm{Pol}_d(gr,k)$ there is a natural filtration

$$0 \subset F_0 \subset F_1 \subset \cdots \subset F_{d-1} \subset F_d = F$$

where $F_k = \mathrm{Ker}(F \twoheadrightarrow q_{k-d-1} F)$

so that $F_r / F_{r-1} \in \mathcal{P}ol_{n-r}(gr, \mathbb{k})$

and is in the image of $\mathcal{F}(ab, \mathbb{k}) \to \mathcal{F}(gr, \mathbb{k})$

$F \mapsto F \circ \mathfrak{a}$.

ex: $0 \to \mathfrak{a}^{\otimes 2} \to S_2 \to \mathfrak{a} \to 0$

$$\begin{array}{ccccccc} 0 & \subset & \mathfrak{a}^{\otimes 2} & \hookrightarrow & S_2 & = & S_2 \\ & & \| & & \| & & \| \\ 0 & \subset & F_0 & \subset & F_1 & \subset & F_2 \end{array}$$

$\underbrace{0 \subset F_0}_{\cong 2}$ $\underbrace{F_0 \subset F_1}_{\cong 1}$ $\underbrace{F_1 \subset F_2}_{\cong 0}$

$S_2 / \mathfrak{a}^{\otimes 2} = \mathfrak{a} \in \mathcal{P}ol_1$

- $\underline{\mathbb{k} = \mathbb{Q}}$ $\quad \mathcal{P}ol_d(ab, \mathbb{Q})$ is semi-simple

$$\underbrace{F = \bigoplus_{n=0}^{d} M(n) \otimes_{\sigma_n} (-)^{\otimes n}}$$ $F \in \mathcal{P}ol_d(ab, \mathbb{Q})$ $\quad M : \Sigma \to \mathbb{k}\text{-Mod}$

- $\underline{\text{Thm}}$ [V. 18]

$$\mathrm{Ext}^i_{\mathcal{F}(gr, \mathbb{Z})}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m}) = \begin{cases} \mathbb{Z}[\mathrm{Surj}(m,n)] & i = m-n \\ 0 & \end{cases}$$

$\Rightarrow S_\lambda, S_\mu \in \mathcal{F}(ab, \mathbb{Q})$

$\mathrm{Ext}^1_{\mathcal{F}(gr, \mathbb{Q})}(S_\lambda \circ \mathfrak{a}, S_\mu \circ \mathfrak{a}) \neq 0 \iff \deg S_\mu = \deg S_\lambda + 1$

Thm: [Powell ArXiv 22] K field of char 0

$$\underbrace{\mathcal{F}_\omega(gr^{op}, K)}_{\text{analytic functors}} \xrightarrow{\sim} \underbrace{\mathcal{F}_{Lie}}_{\text{category of linear functors on Cat Lie}}$$

$\mathcal{P}ol_d(gr^{op}) \to \mathcal{F}(gr^{op})$

pd.

$p_d F \subset p_{d+1} F \subset \cdots \subset F$

$F = \mathrm{colim}_d\, p_d F$

$A_d(n, -)_0$

obj $= \mathbb{N}$

$\mathrm{Cat}\,\mathcal{L}ie(m, n) = \bigoplus_{f \in \mathrm{Fin}(m,n)} \bigotimes_{i=1}^{n} \mathcal{L}ie(|f^{-1}(i)|)$

$\mathcal{L}ie(0) = 0$

$5 \to 2$

$1\ 2 \ \ \mathcal{L}ie(2)\ \ 1$

$3\ 4\ 5 \ \in \mathcal{L}ie(3)\ \ 2$

$F \longmapsto (d \mapsto cr_d(p_d F)(\mathbb{Z}, \ldots, \mathbb{Z}))$

$\mathcal{P}ol_d(gr^{op}, K) \xrightarrow{\sim} (\mathcal{F}_{Lie})_{\leq d}$

Applications

1) [Powell-V] Higher Hochschild homology for a wedge of circle

[Pirashvili 2000] def HH for $X : \Delta^{op} \to Ens$.

$$HH_*(X, \underbrace{\mathcal{L}(A,V)}_{\text{Loday funct.}})$$

(usual HH for $X = S^1$)

[Turchin-Willwacher 2018]

$\llbracket$ $HH_*(\bigvee_d S^1, \mathcal{L}(A_a, A_a))$ $\rightsquigarrow$ rep of $Out(F_n)$

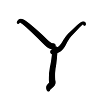
$A_a = \mathbb{Q}[x]/x^2$

$\bigvee_d S^1 = BF_d$

$\rightsquigarrow$ classifying space

$HH_*(B(-), \mathcal{L}(A,A)) : gr \to \mathbb{Q}\text{-Mod}.$

$HH_d(B(-), \mathcal{L}(A,A))$ is poly of degree d.

2) <u>Jacobi diagrams</u> 

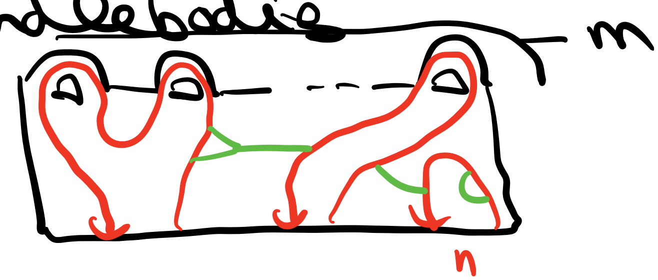
[Habiro-Massuyeau 21]

$\mathcal{A}$: linear category of Jacobi diagrams in handlebodies

$\bigcup\!\!\uparrow$

$\Vert$ $\mathbb{K}[\mathrm{gr}^{op}]$

$$m \downarrow n$$



$\mathcal{A}_d(\cancel{0}\, n, -): \mathrm{gr}^{op} \to \mathbb{K}\text{-Mod}$

↳ Jacobi diag of degree d ⇝ $2d$ vertices

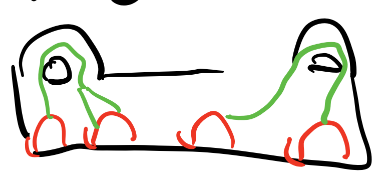
[Katada I & II 21]

$\mathcal{A}_d(0,-)$ is poly of $\leq 2d$.

$\mathcal{A}_d(n,-)$ is not poly.

$\uparrow$

$\mathcal{A}_d(n,-)_0$



<u>Thm:</u> [V] $\mathcal{A}_d(n,-)_0$ also poly of $\leq 2d$.