# Topos theory and measurability

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# The measurability problem

Source / Base probability space $\Omega$, $\omega \in \Omega$ states

**INPUT**

- $X_\omega$ sets
- $a_\omega$ numbers
- $\tau_\omega$ topologies
- $\mathcal{H}_\omega$ Hilbert spaces
- $U_\omega : \mathcal{H}_\omega \to \mathcal{H}_\omega$ operators

Assignment

$\omega \longmapsto \bigcirc$

measurable !

**LOGICAL MANIPULATIONS**

- Theorems
- Lemmas
- Corollaries
- Constructions

**OUTPUT**

- $Y_\omega$ other sets
- $A_\omega$ limits
- $\lambda_\omega$ eigenvalues
- $B_\omega$ optimizers

Assignment

$\omega \longmapsto \bigcirc$

measurable ?

# I. Example: Stochastic optimization

- $\omega \in \Omega$ state in a probability space, $\omega \mapsto C(\omega) \subset \mathbb{R}^d$ is a set-valued map and $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ a bivariate function.

- When is $\omega \mapsto \mathtt{Sol}(\omega) = \inf\{F(\omega, x) : x \in C(\omega)\}$ measurable?

- If $F$ is measurable in $\omega$ for each $x \in \mathbb{R}^d$ and continuous in $x$ for each $\omega \in \Omega$ and $\omega \mapsto C(\omega)$ is measurable and compact-valued, then the answer is YES by a measurable selection theorem.

  Rockafellar & Wets "Variational Analysis"

# II. Example: Ergodic theory

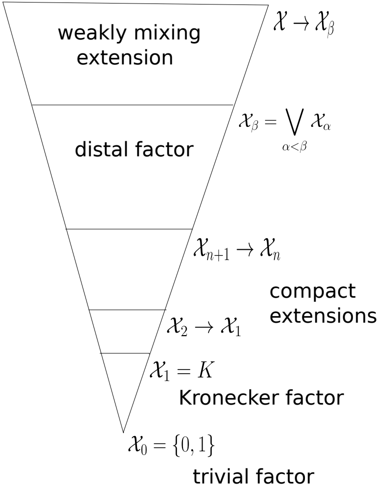
$(X, \mathcal{X}, \mu)$ probability space and $T : X \to X$ measure-preserving.

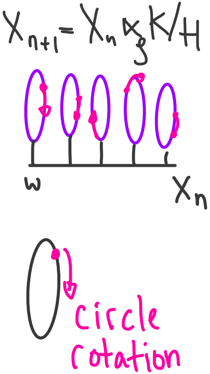
Furstenberg's multiple recurrence theorem: For all $k \geq 1$, every $E \in \mathcal{X}$ with $\mu(E) > 0$ there exists $n \geq 1$ such that

$$\mu(E \cap T^n(E) \cap T^{2n}(E) \cap \ldots \cap T^{(k-1)n}(E)) > 0.$$

Furstenberg "Recurrence in Ergodic Theory and Combinatorial Number Theory"

# Furstenberg-Zimmer structure theorem



# Assumptions of the pointwise approach

Assumptions which guarantee a countable measure-theoretic complexity:

- Range: Polish spaces such as separable Banach spaces, compact metric spaces
- Base probability space: Countably generated, standard Borel, measure-theoretic complete
- Acting groups (in dynamics): Second-countability such as countable discrete groups, Polish groups

Descriptive set theory (Kechris)

# Abstraction and Deletion

$$
\begin{tikzcd}
\mathbf{CncMes} \arrow[d, "\mathtt{Abstract}"] & \mathbf{CncPrb} \arrow[l] \arrow[r, "\mathtt{Null}"] \arrow[d, "\mathtt{Abstract}"] & \mathbf{CncNul} \arrow[l, bend left] \arrow[d, "\mathtt{Abstract}"] \\
\mathbf{AbsMes} \arrow[d, leftrightarrow, "\mathtt{Dual}"] & \mathbf{AbsPrb} \arrow[l] \arrow[r, "\mathtt{Null}"] & \mathbf{AbsNul} \arrow[l, bend right] \arrow[dl, "\mathtt{Delete}"] \\
\mathbf{Bool}_\sigma & \mathbf{PrbAlg}^{\mathrm{op}} \arrow[u, "\mathtt{Include}"] &
\end{tikzcd}
$$

# Basic properties

- If $(X, \mu)$ is **PrbAlg**$^{op}$-space, then $X$ is automatically a complete Boolean algebra.

- Null set issues are excluded for **PrbAlg**$^{op}$-spaces.

- Random variables are **AbsMes**-morphisms $f : X \to \mathcal{B}(\mathbb{R})$ where $\mathcal{B}(\mathbb{R})$ Borel $\sigma$-algebra of $\mathbb{R}$

- Integral: For **AbsMes**-morphisms $f : X \to \mathcal{B}(\mathbb{R})$ with $f^*([0, \infty)) = 1$ define

$$\int f d\mu := \int \mu(f^*(x, \infty))dx$$

Fremlin "Measure algebras"

# Advanced properties

- Fix a base probability algebra $(X, \mu)$.
- Consider $(X, \leq)$ as a small category where $E \leq F$ iff $E \wedge F = E$.
- The map

  $$E \mapsto K(E) = \{(E_n) \text{ countable partition of } E\}$$

  is a Grothendieck basis on $X$.
- Let $\mathbf{Sh}(X)$ be the Grothendieck topos of sheaves on the site $(X, K)$.
- $\mathbf{Sh}(X)$ is Boolean, has a natural numbers object, and satisfies the axiom of choice.
- We have a full mathematical discourse inside $\mathbf{Sh}(X)$ (related to the transfer principle for Boolean-valued models).

# Some examples of sheaves

- For any **AbsMes**-space $K$, the hom-set $\mathtt{Cond}_X(K) := \mathtt{Hom}_{\mathbf{AbsMes}}(X \to K)$ has the structure of a sheaf on the site $(X, K)$.

- If $K$ is the power set of the natural numbers $\mathbb{N}$, then $\mathtt{Cond}_X(\mathbb{N})$ is the natural numbers object in $\mathbf{Sh}(X)$.

- If $K = \mathcal{B}(\mathbb{C}^d)$, then $\mathtt{Cond}_X(K)$ is the $d$-dimensional complex Euclidean space inside $\mathbf{Sh}(X)$.

- If $K$ is the Baire $\sigma$-algebra of a compact Hausdorff group, then $\mathtt{Cond}_X(K)$ is a compact Hausdorff group inside $\mathbf{Sh}(X)$.

# A solution to the measurability problem

$X_\omega$

$\mu_\omega$

$H_\omega$

$\mathcal{U}_\omega$

translate

Sh-sets

Sh-measures

Sh-Hilbert-spaces

etc.

formalize point-free

$\boxed{\text{Internal logic of } \mathrm{Sh}(X)}$

Sh-eigenvalues

Sh-limits

etc.

interpret

solutions of the measurability problem

translate

# Conditional analysis as a tool to interpret the internal discourse of $\mathbf{Sh}(X)$

- Constructing by hand a conditional reasoning from first things by developing a conditional form of set theory first.

- Conditionalizing proofs of uncountable measure-theoretic complexity. For example, a conditional version of the axiom of choice for general conditional sets.

- Based on a conditional form of set theory (naive set theory in $\mathbf{Sh}(X)$), develop a conditional topology, conditional functional analysis and conditional measure theory, etc.

# I. A measurable Bolzano-Weierstraß theorem

**Theorem**

*Let $\Omega$ be a probability space and $L^0$ space of random variables $f : \Omega \to \mathbb{R}$ (identified with respect to almost sure equality). Let $(x_n)$ be a sequence in $L^0$ such that $|x_n| \le M$ almost surely for some $M \in L^0$. Then there exist a sequence of measurable functions $N_k : \Omega \to \mathbb{N}$ with $N_1 < N_2 < \dots$ and $x \in L^0$ such that $x_{N_k} = \sum x_n 1_{N_k = n} \to x$ almost surely.*

**Proof**: Set $x = \lim_{n\to\infty} \inf_{m \ge n} x_m$ almost everywhere. By induction, $N_0 = 0$ and for $k \ge 1$

$$N_k(\omega) = \min\{m \in \mathbb{N} : m > N_{k-1}(\omega), x_m(\omega) \le x(\omega) + 1/n\}.$$

# II. Measurable cohomology

- Let $(X, \mu)$ be an arbitrary probability algebra, $\Gamma$ an arbitrary discrete group, $T : \Gamma \to \mathtt{Aut}(X, \mu)$ a group homomorphism introducing a measure-preserving dynamical structure on $(X, \mu)$, and $K$ be an arbitrary compact Hausdorff abelian group with Baire $\sigma$-algebra $\mathcal{B}a(K)$.
- A family $\rho = (\rho_\gamma)_{\gamma \in \Gamma}$ of elements of $\mathtt{Cond}_X(K)$ is said to be a <u>$K$-valued cocycle</u> if for all $\gamma_1, \gamma_2 \in \Gamma$ it holds that

$$\rho_{\gamma_1+\gamma_2} = \rho_{\gamma_1} \circ T^{\gamma_2} + \rho_{\gamma_2}$$

- A cocycle $\rho$ is a <u>$K$-valued coboundary</u> if there exists $g$ in $\mathtt{Cond}_X(K)$ such that for all $\gamma \in \Gamma$

$$\rho_\gamma = g \circ T^\gamma - g$$

# The uncountable Moore-Schmidt theorem

**Theorem (J. and Tao)**

*A cocycle $\rho$ is a $K$-valued coboundary if and only if $\hat{k} \circ \rho = (\hat{k} \circ \rho_\gamma)_{\gamma \in \Gamma}$ is a $\mathbb{T}$-valued cocycle for all characters $\hat{k} \in \hat{K}$.*

An important ingredient in the proof is the Pontryagin duality between compact and discrete abelian groups in the internal discourse of $\mathbf{Sh}(X)$:

**Theorem (J. and Tao)**

*There is a $\mathbf{Sh}(X)$-Pontryagin duality between the $\mathbf{Sh}(X)$-compact Hausdorff abelian group $\mathtt{Cond}_X(K)$ and the $\mathbf{Sh}(X)$-discrete Hausdorff abelian group $\mathtt{Cond}_X(\hat{K})$.*

# Comments on the proof

- The result is not stated in this way in our paper. We show that the classical Pontryagin duality $\iota : K \to \mathbb{T}^{\hat{K}}$ lifts to

$$\mathtt{Cond}(\iota)(\mathtt{Cond}_X(K)) = \{(\theta_{\hat{k}})_{\hat{k}\in\hat{K}} \in \mathtt{Cond}_X(\mathbb{T})^{\hat{K}} : \theta_{\hat{k}_1+\hat{k}_2} = \theta_{\hat{k}_1} + \theta_{\hat{k}_2} \text{ for all } \hat{k}_1, \hat{k}_2 \in \hat{K}\}$$

- As for the **Sh**$(X)$-Pontryagin duality, need to show how to lift the classical topologies to **Sh**$(X)$-topologies, verify **Sh**$(X)$-compactness (needs **Sh**$(X)$-Tychonoff's theorem), define **Sh**$(X)$-Pontryagin duality and then use the above lifting result to conclude. Heavy use of results in conditional topology. (Unpublished notes)

- It is crucial to use Baire measurability.

# III. Uncountable Furstenberg-Zimmer structure theory

**Theorem (J.)**

*Let Γ be an arbitrary group. Let $(X, \mu, T)$ be an arbitrary probability algebra Γ-dynamical system. Then there are an ordinal number $\beta$ and for each ordinal number $\alpha \leq \beta$ a system $(Y_\alpha, \nu_\alpha, S_\alpha)$ with the following properties.*

(i) $Y_\emptyset = \{0, 1\}$

(ii) *For any successor ordinal $\alpha + 1 \leq \beta$, $(Y_{\alpha+1}, \nu_{\alpha+1}, S_{\alpha+1})$ is a compact extension of $(Y_\alpha, \nu_\alpha, S_\alpha)$.*

(iii) *For any limit ordinal $\alpha \leq \beta$, $Y_\alpha$ is generated by $\bigcup_{\alpha' < \alpha} Y_{\alpha'}$.*

(vi) *$(X, \mu, T)$ is a weakly mixing extension of $(Y_\beta, \nu_\beta, S_\beta)$.*

# Dichotomy between structure and randomness

For any $\Gamma$-system $(X, \mu, T)$, we have the decomposition

$$L^2(X) = \mathtt{WM}_\Gamma(X) \oplus \mathtt{AP}_\Gamma(X)$$

Using Hilbert space and spectral theory we can characterize the structured part $\mathtt{AP}_\Gamma(X)$ in terms of

(a) Finite-dimensional invariant subspaces
(b) Totally bounded orbits
(c) Invariant Hilbert-Schmidt operators
(d) Under ergodicity assumption, translations on homogeneous spaces

# Relative dichotomy between structure and randomness

**Theorem (J.)**

*For an arbitrary group $\Gamma$ and $(X, \mu, T)$ an extension of $(Y, \nu, S)$, we have the decomposition*

$$L^2(X) = \mathtt{WM}_\Gamma(X|Y) \oplus \mathtt{AP}_\Gamma(X|Y)$$

*We can characterize the relatively structured part $\mathtt{AP}_\Gamma(X|Y)$ in terms of*

(a) *Finite-dimensional invariant submodules*

(b) *Conditionally totally bounded orbits*

(c) *Invariant conditional Hilbert-Schmidt operators*

(d) *Under ergodicity assumption, as an isometric group extension $X = Y \times_\rho K/H$*

# $\mathbf{Sh}(Y)$-Hilbert space and spectral theory

- We can form the conditional Hilbert space

$$L^2(X|Y) = \{f \in \mathsf{Cond}_X(\mathbb{C}) : \mathbb{E}(|f|^2|Y) < \infty\}$$

  This is a sheaf on $(Y, K)$ and a Hilbert space in $\mathbf{Sh}(Y)$ with $\mathbf{Sh}(Y)$-inner product $\mathbb{E}(f\bar{g}|Y)$.
- Consider the relative tensor product

$$L^2(X \times_Y X|Y) = \{f \in \mathsf{Cond}_{X\times_Y X}(\mathbb{C}) : \mathbb{E}(|f|^2|Y) < \infty\}$$

  This is a sheaf on $(Y, K)$ and represents the $\mathbf{Sh}(Y)$-Hilbert Schmidt operators on the $\mathbf{Sh}(Y)$-Hilbert space $L^2(X|Y)$.
- The relative dichotomy and the characterizations of compact extensions are the reflection of the classical dichotomy in the internal discourse of $\mathbf{Sh}(Y)$.
- Non-ergodic case requires a much stronger use of the internal logic of $\mathbf{Sh}(Y)$..

# References


- The algebra of conditional sets, and the concepts of conditional topology and compactness, Drapeau, J., Karliczek, and Kupper

- Sheaves and conditional sets, J.

- An uncountable Moore-Schmidt theorem, J. and Tao

- An uncountable Furstenberg-Zimmer structure theory, J.

Thanks for listening!