Sheaf representation of monoidal categories

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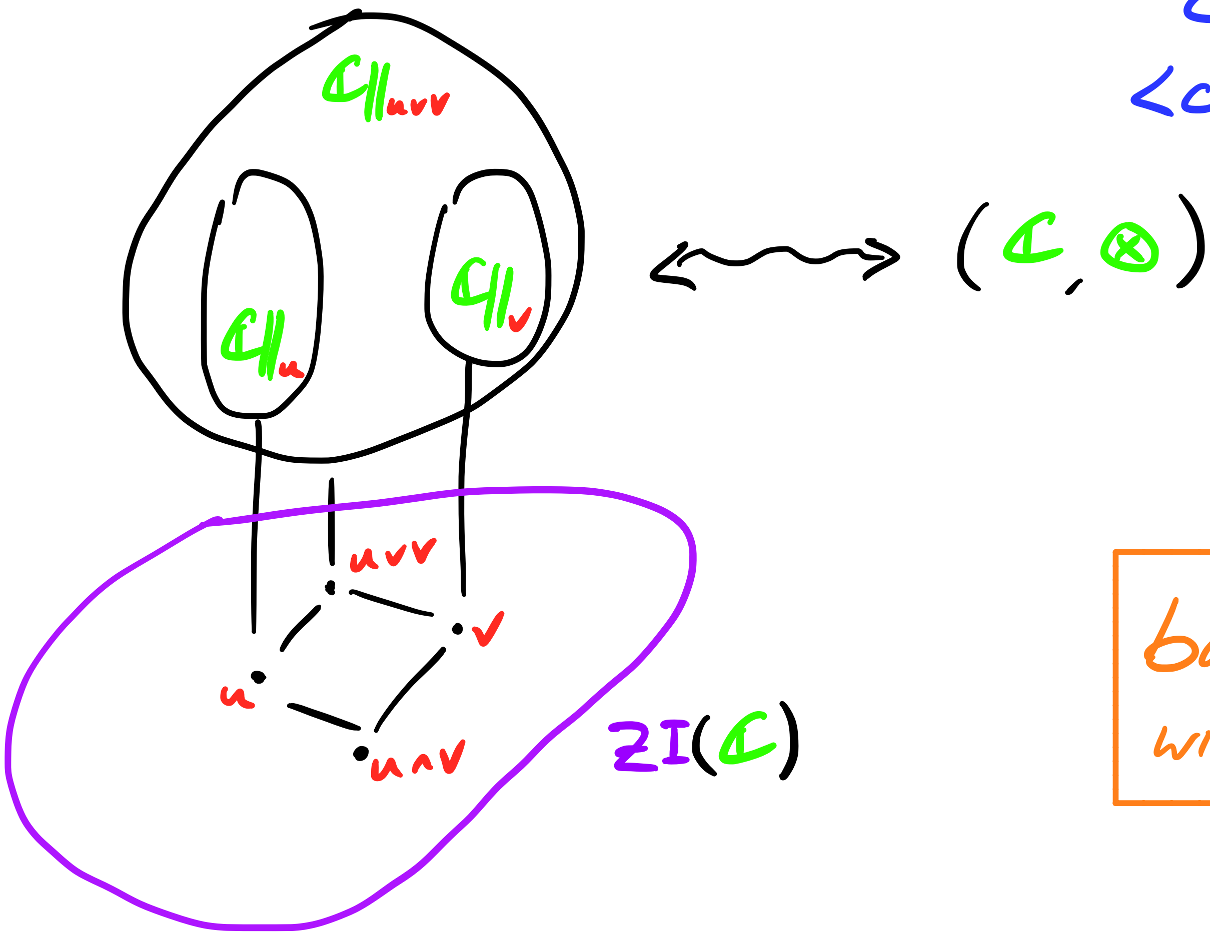


based on arXiv:2106.08896
with Rui Soares Barbosa

"Categories should be nice and easy"

- $\mathbb{V}\mathrm{ect}$ is monoidal, but so is $\mathbb{V}\mathrm{ect} \times \mathbb{V}\mathrm{ect}$

easier: does not decompose as product

Any mon'l cat embeds into a nice one

Any nice mon'l cat is dependent prod of easy ones

overview

$\mathbb{V}\mathrm{ect} \times \mathbb{V}\mathrm{ect} = \prod_{i \in \{0,1\}} \mathbb{V}\mathrm{ect}$ decomposable as $\{0,1\}$ disj. union

can reconstruct
opens as
central idempotents

cat is nice if these behave:

- stiff: form semilattice respected by $\otimes$
- universal joins: complete lattice respected by $\otimes$

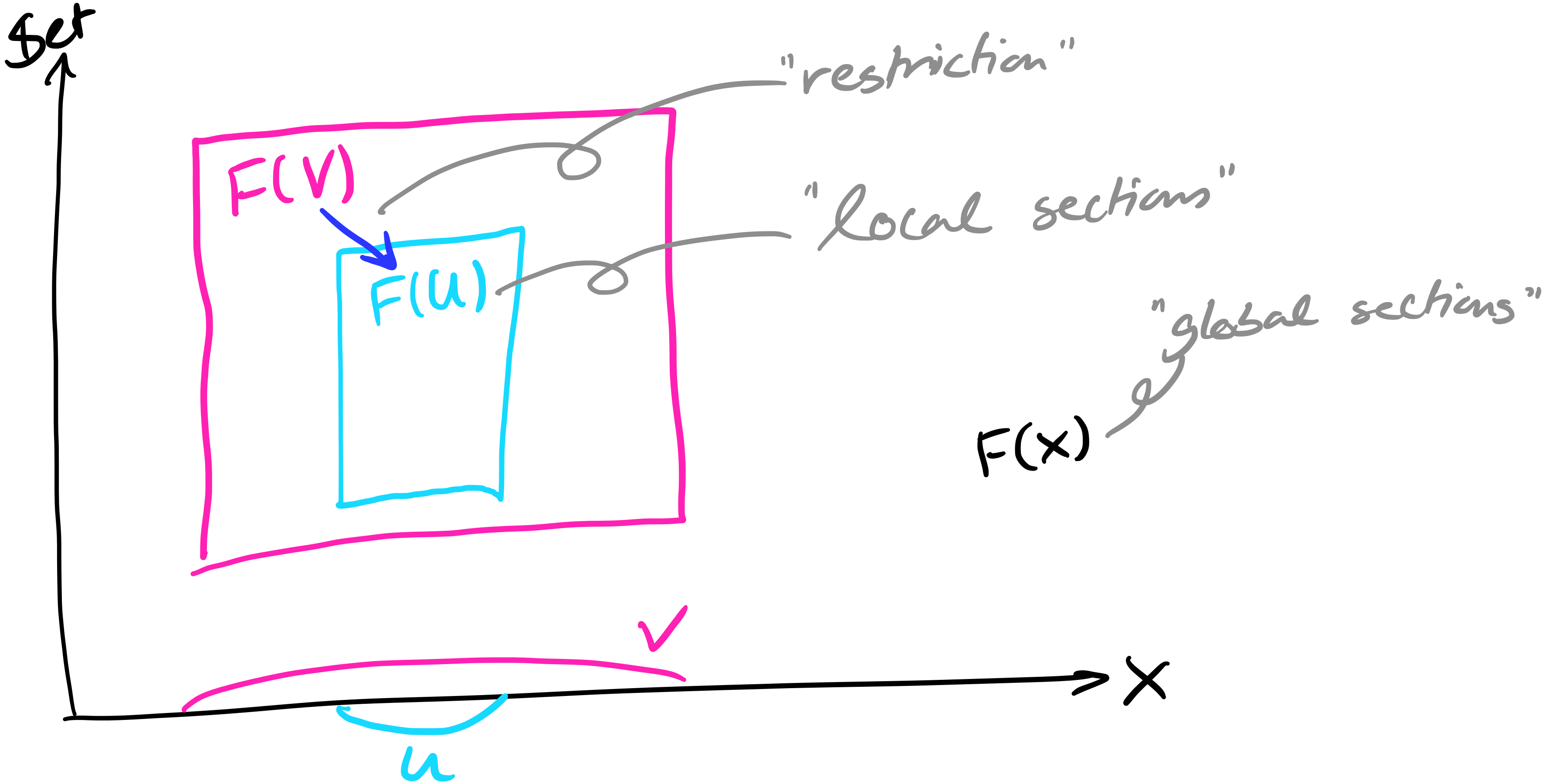
cat is easy if these are:

- (sub) local: any (fin) cover contains open that is covered
- topologically: any net converges to single focal pt
- logically: disjunction property if $A \vee B$ then A or B

Sheaves: continuously parametrised objects

presheaf on X = functor $F : \underbrace{\mathcal{O}(X)^{op}}_{\text{frame of opens}} \longrightarrow \mathbf{Set}$

top. space

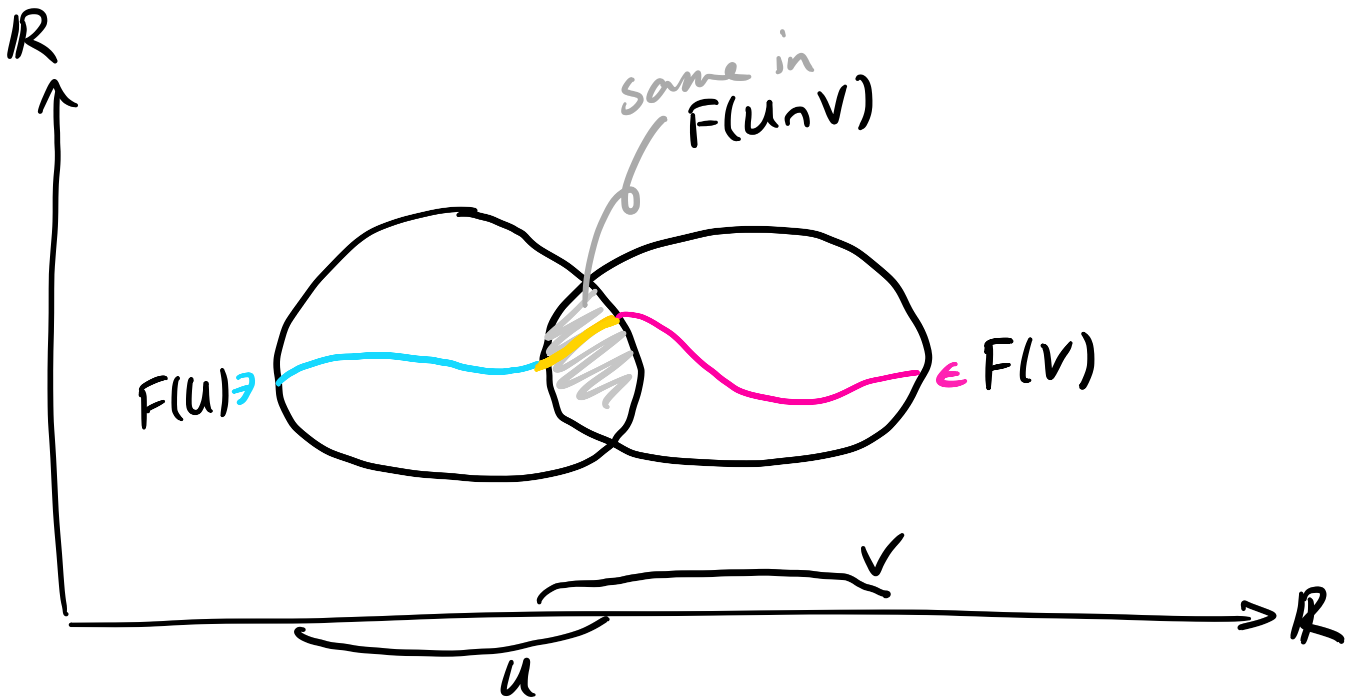


Sheaf = continuous presheaf:

$$\boxed{F(\operatorname{colim} U_i) = \lim F(U_i)}$$

$F(\operatorname{colim} U_i) \cong$ global sections

$\lim F(U_i) \cong \{(s_i) \mid F(U_i \cap U_j \subseteq U_i)(s_i) = F(U_i \cap U_j \subseteq U_j)(s_j)\}$ compatible local sections

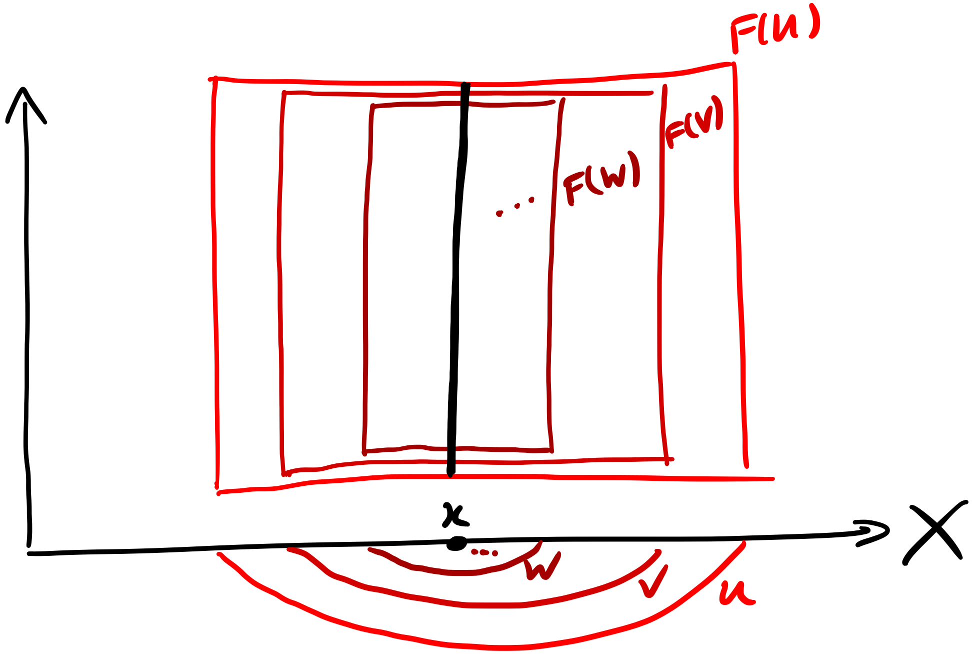


Q: What if values not in Set but in MonCat?

A: sheaf condition becomes equaliser:

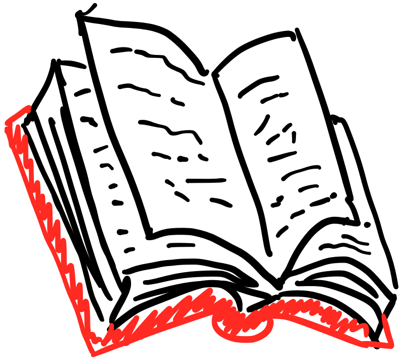
$$
\begin{tikzcd}
F(\bigcup U_i) \arrow[r, "\langle F(U_i \subseteq \bigcup U_i)\rangle_i"] & \prod_i F(U_i) \arrow[r, shift left, "\langle F(U_i \cap U_j \subseteq U_i) \circ \pi_i \rangle_{i,j}"] \arrow[r, shift right, "\langle F(U_i \cap U_j \subseteq U_j) \circ \pi_j \rangle_{i,j}"'] & \prod_{i,j} F(U_i \cap U_j)
\end{tikzcd}
$$

global sections are families of local sections that are pairwise compatible stalk of sheaf F at pt x = $\underset{x \in U}{\operatorname{colim}} F(U)$



"sheaf of #\$@*" = "sheaf whose stalks are #\$@*"

e.g. "sheaf of local rings"



- Boolean algebra = global sections of spaces {0,1}
- ring = global sections of local rings
- topos = global sections of local toposes

mon'l cat w univ joins

=

global sections of local mon'l cats

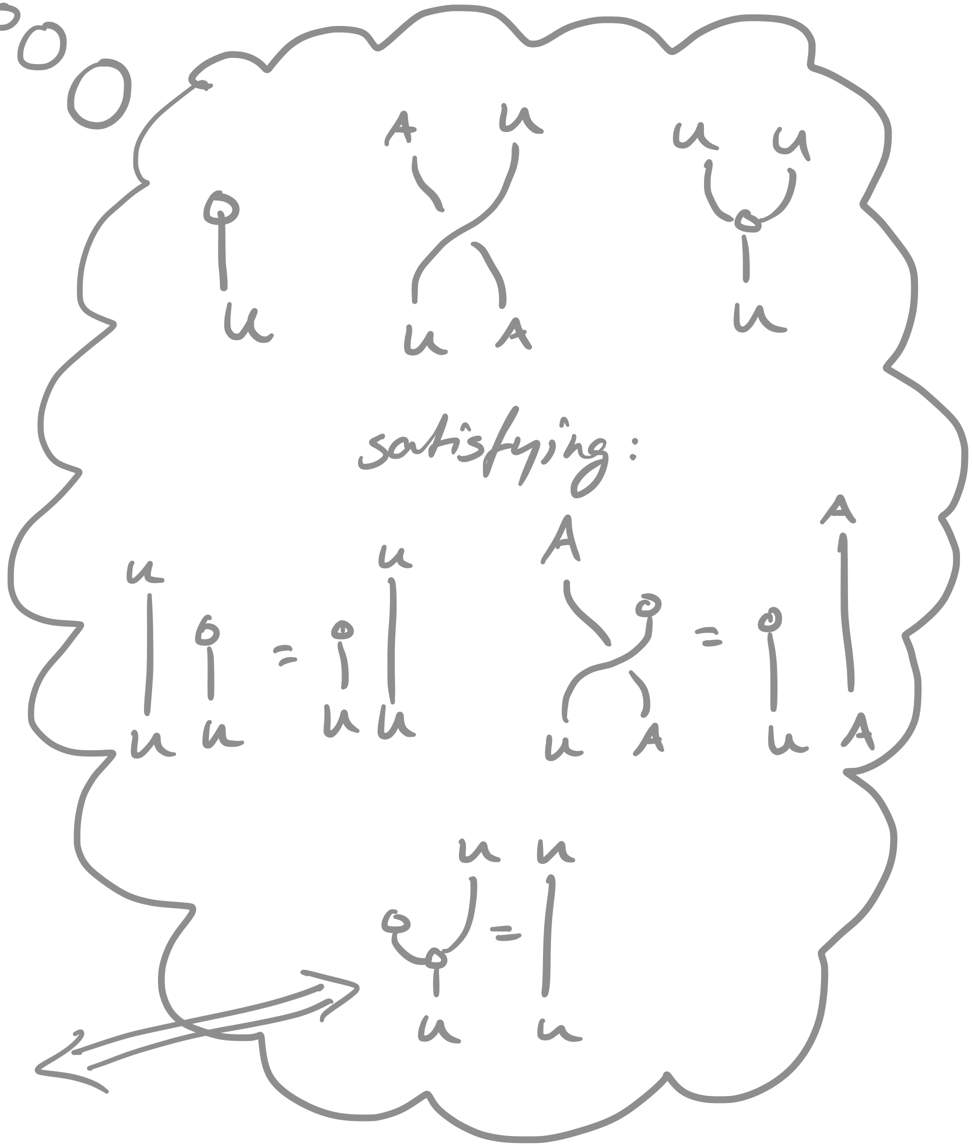
$\Longrightarrow$

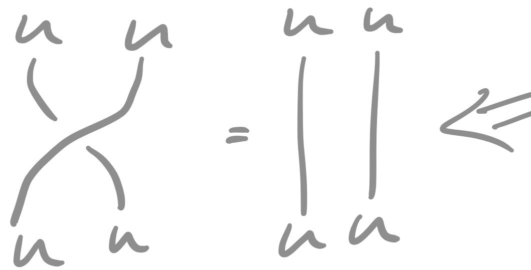
UI

stiff mon'l cat

opens of X = central idempotents of $Sh(X)$

- morphism $U \xrightarrow{u} I$
- half-braiding $U \otimes - \Rightarrow - \otimes U$
- s.t. $U \otimes U \xrightarrow[\simeq]{U \otimes u = u \otimes U} U$





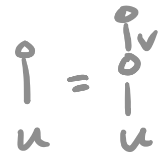
central idempotents

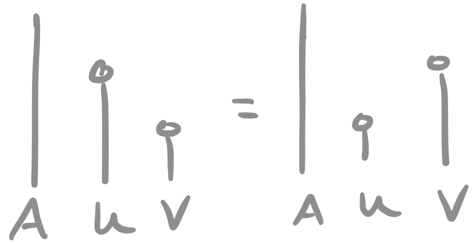
category	central idempotents
cartesian cat	subterminal objs
$Sh(X)$	opens of X
semilattice	everything
quantale	largest subframe $\{q^2 = q \leq 1\}$
Mod_R — comm. ring (nonunital)	idempotent ideals $I^2 = I \subseteq R$ (+...?)
Mod_A — bialgebra over field	central idemp. elts. of A
Hilbert modules over $C_0(X)$	opens of X
endofunctors $C \to C$	trivial

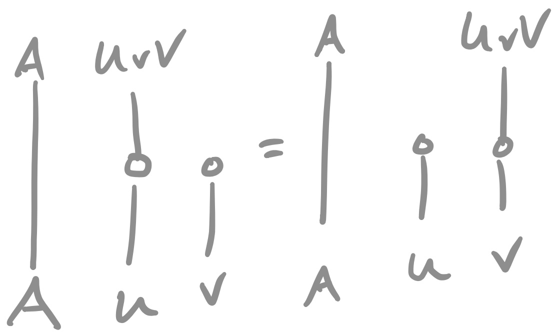
central idempotents

central idempotents form semilattice : $u \wedge v = u \otimes v$

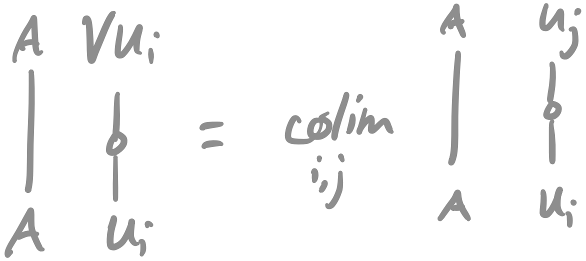
$$\begin{matrix} 1 & = & I \\ u \wedge v & = & u \otimes v \\ 1 & = & I \end{matrix}$$

$u \leq v \iff u = u \wedge v \iff$
$$\begin{tikzcd} v \arrow[dr, "v"] & \\ & I \\ u \arrow[uu, dashed] \arrow[ur, "u"'] & \end{tikzcd}$$
$\iff$ 

- $\mathcal{C}$ is stiff $\iff$  pullback

- $\mathcal{C}$ has univ fin joins $\iff$  distr. lattice

  pullback & pushout and $A \otimes 0 \simeq 0$

- $\mathcal{C}$ has univ joins $\iff$  frame

nice central idempotents

stiff mon'l cat embeds into mon'l cat w univ fin joins

embeds into mon'l cat w univ joins

stiff

univ (fin) joins

$$
\begin{tikzcd}
\mathbb{C} \arrow[rr, "\text{strong mon'l}"] \arrow[drr, "\substack{\text{lax mon'l} \\ \text{pres. central idemp.}}"'] & & D[\mathbb{C}] \arrow[d, dashed, "\substack{\text{lax mon'l} \\ \text{pres. central idemp.} \\ \text{and (fin) joins}}"] \\
& & \mathbb{D}
\end{tikzcd}
$$

univ (fin) joins

$$F(A) \otimes F(u) \xrightarrow{\simeq} F(A \otimes u)$$

$$F(I) \simeq I$$

$$F(\ \ ) = \ \ $$

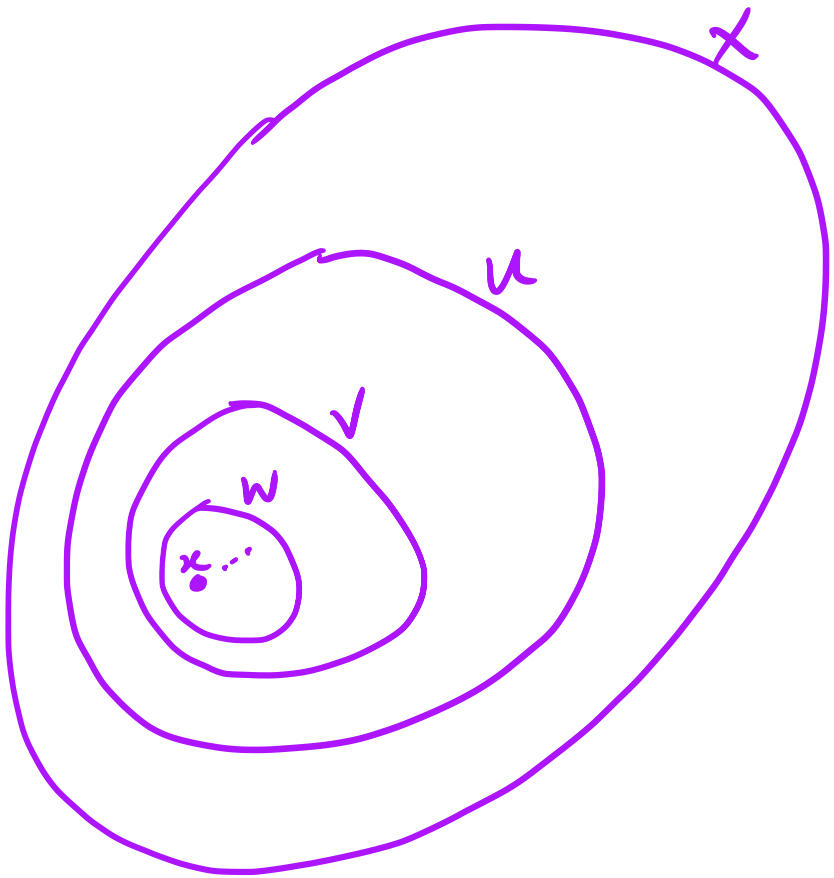
u A

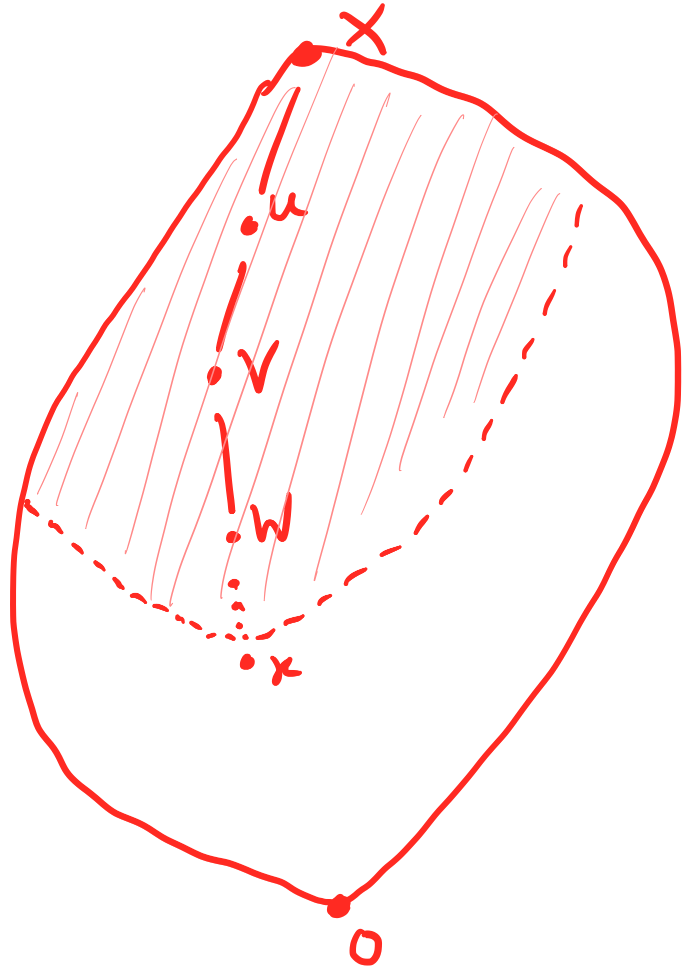
Fu FA

nice(r) central idempotents

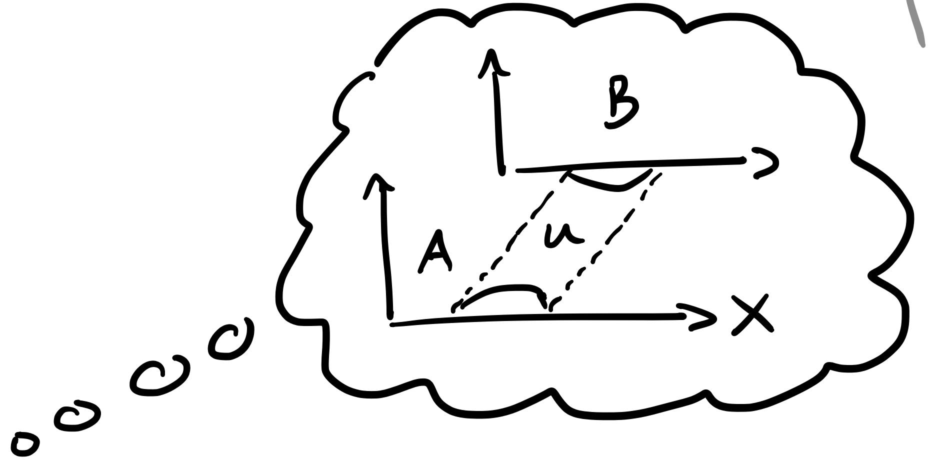
Base space X = Zariski spectrum of central idempotents of $\mathcal{C}$

$= \{$ (completely) prime filters of central idempotents of $\mathcal{C}$ $\}$



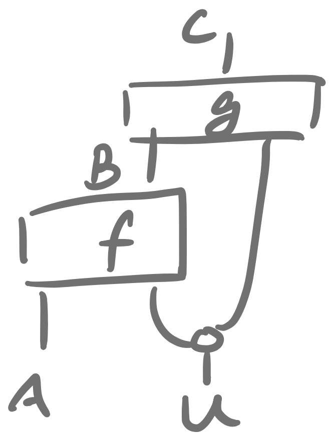


Local sections $F(u) = \mathcal{C}\|_u = \mathrm{Kl}(- \otimes u)$ $\left(\simeq \mathcal{C}/U \text{ if } \otimes = \times \right)$

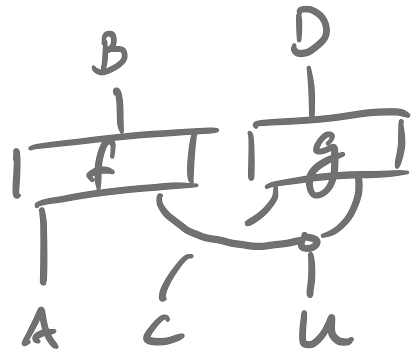


- obj's: as in $\mathcal{C}$
- mor's: $A \otimes U \longrightarrow B$
- identity:

- comp.:



- tensor:



Sheaf condition

binary equaliser enough

$$\begin{tikzcd} \mathrm{Cll}_{u\cup v} \arrow[r] & \mathrm{Cll}_{u} \times \mathrm{Cll}_{v} \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Cll}_{u\cap v} \end{tikzcd}$$

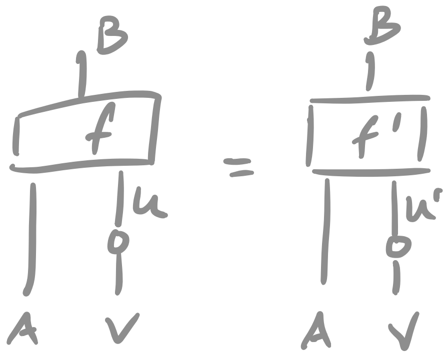
pairs of local sections

that overlap

main thm

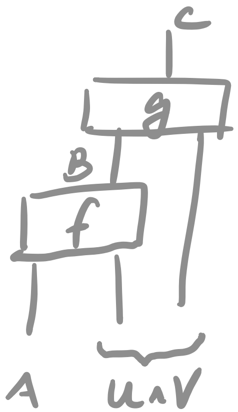
stalks $\mathcal{C}\|_x$ are (sub)local

- obj's: as in $\mathcal{C}$
- mor's: $A \otimes U \xrightarrow{f} B$ for $U \in x$, identified when



"germ"

- comp.: $[V, g] \circ [U, f] =$



main thm

stiff mon'l cat

$\subseteq$

mon'l cat w univ (fin) joins

$=$

global sections of (sub)local mon'l cats

!

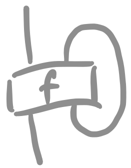
- [Lambek-Moerdijk-Awodey] toposes

$\Longrightarrow$

not toposes
- [Stone] Boolean alg
- [Takahashi] Hilbert modules
- [Pierce] modules over comm. ring

main thm

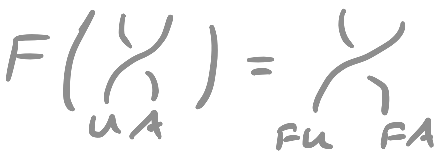
preservation $\Longrightarrow$

	category $\mathcal{C}$	local sections $\mathcal{C}\|_u$	stalks $\mathcal{C}\|_x$
$\dfrac{A\otimes B \to C}{A \to (B \multimap C)}$	closed	closed	closed
	traced	traced	traced
	compact	compact	compact
$u \vee \neg u = 1$, $u \wedge \neg u = 0$	Boolean	Boolean	two-valued
	complete	complete	complete
$u \otimes \mathrm{colim} \cdots = \mathrm{colim}\, u \otimes \cdots$	proj. cocomplete	proj. cocomplete	
	fin. cocomplete		fin. cocomplete

- obj's: mon'l cat $\mathbb{C}$
  w univ (fin) joins
- mor's: $F : \mathbb{C} \longrightarrow \mathbb{D}$ lax mon'l

$F(A) \otimes F(U) \xrightarrow{\cong} F(A \otimes U)$

$F(I) \xrightarrow{\cong} I$



$$\mathbb{M}\mathrm{on}\mathrm{Cat}_{(f)j} \simeq \mathrm{MonScheme}_{(s)l}^{op}$$

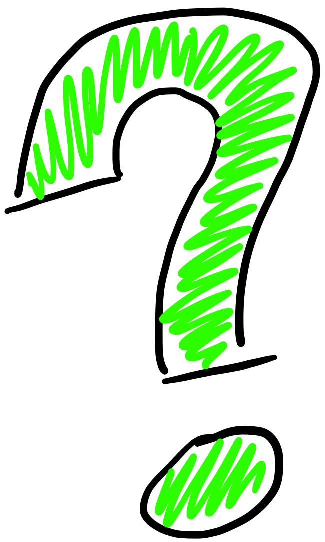
- obj's: $P : O(X)^{op} \to \mathbb{M}\mathrm{onCat}$
  s.t. $P(x)$ (sub)local
  "flabby" $\mathbb{G}\, P(u) \underset{P(u \leq v)}{\overset{\perp}{\rightleftarrows}} P(v)$
- mor's: cts. fn. $\varphi : X \longrightarrow Y$
  nat. transf. $F_v : Q(v) \longrightarrow P(\varphi^{-1}(v))$
  s.t. $F_x : Q(\varphi(x)) \longrightarrow P(x)$
  conservative on central idemp:
  $F_x(v) = 1 \implies v = 1$

corollaries

✓ cleanly separate "spatial" from "temporal" directions

✓ capture more examples

✓ concrete proof



- completeness? coherence?
- linear logic?
- alg/geom?

... causality?

... concurrency? Petri nets?

... localisable monads?

[arXiv:2106.08896] "Sheaf representation of monoidal categories"
R. Soares Barbosa & C. Heunen

[arXiv:2009.12432] "Tensor-restriction categories"
C. Heunen & J.S. Pacaud Lemay
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P. Enrique Moliner & C. Heunen & S. Tull
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