What is “Monoidal Topology”?

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Overview

- Algebras as monoids–Sure! But spaces as monoids?
- Hausdorff's visionary remarks
- Quantale-enriched categories
- (Monad, Quantale)-enriched categories
- The fundamental adjunction
- Equationally defined properties for objects and morphisms
- Trading convergence relations for closure operations
- Comparison with the internal-category approach
- Problems, projects, references

The ubiquity of monoids

Mac Lane, *"Categories for the Working Mathematician"*, 1971,

Chapter VII, "Monoids":

$(M,\ m : M \otimes M \to M,\ e : I \to M)$ in a monoidal category $(\mathcal{C}, \otimes, I)$,

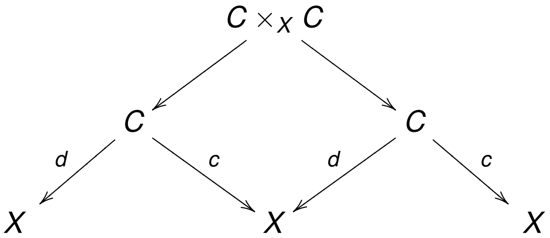
such as:

monoids $(\mathbf{Set}, \times, 1)$, rings $(\mathbf{AbGrp}, \otimes, \mathbb{Z})$, R-algebras $(\mathbf{Mod}_R, \otimes, R)$,

monads $(\mathcal{C}^{\mathcal{C}}, \circ, \mathrm{Id}_{\mathcal{C}})$, *etc, etc*, **...**, and

categories:

Categories as monoids in the bicategory $\mathcal{S}\mathrm{pan}(\mathsf{Set})$



$$C \xrightarrow{(d,c)} X \times X \quad \Longleftrightarrow \quad X \times X \xrightarrow{\hom_C} \mathsf{Set}$$

$$C \times_X C \xrightarrow{m} C \quad \Longleftrightarrow \quad \hom(x,y) \times \hom(y,z) \xrightarrow{m_{x,y,z}} \hom(x,z)$$

$$X \xrightarrow{i} C \quad \Longleftrightarrow \quad 1 \xrightarrow{i_x} \hom(x,x)$$

When $\emptyset$ and 1 are the only values of $\hom_C$, ...

... then $C \hookrightarrow X \times X$ becomes a relation on X,

and the mere existence of i and m becomes properties of the relation:

(R) $1_X \subseteq C$ $\qquad \hom(x,x) = 1$

(T) $C \circ C \subseteq C$ $\qquad \hom(x,y) \wedge \hom(y,z) \leq \hom(x,z)$

Equivalently:

- C is a monoid (!) in $\mathrm{Rel}(X)$, considered as a category with $\subseteq$ as morphisms and monoidal structure $\circ, 1_X$.
- (X, C) is a lax Eilenberg-Moore algebra with respect to the identity monad of the 2-category **Rel** of sets and relations.

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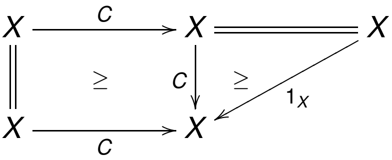

(Pre-)Ordered sets as a precursor to "spaces"?

Felix Hausdorff, *"Grundzüge der Mengenlehre"* (1914), ch. 7, p. 211, on the Metric-Convergence-Neighbourhood trilogy:

"Welchen der drei [...] Grundbegriffe Entfernung, Limes, Umgebung man zur Basis der Betrachtung wählen will, ist bis zu einem gewissen Grade Geschmacksache. [...] Danach scheint die Entfernungstheorie die speziellste, die Limestheorie die allgemeinste zu sein; auf der andern Seite bringt der Limesbegriff sofort eine Beziehung zum Abzählbaren (zu Elementfolgen) in die Theorie hinein, worauf die Umgebungstheorie verzichtet."

"Which of the three [...] fundamental notions, distance, limit, neighbourhood, one wants to choose as the basis of consideration is, to a certain degree, a matter of taste. [...] Accordingly, the distance theory seems to be the most special, the limit theory the most general one; on the other side, the notion of limit brings immediately a connection with countability (with sequences of elements) into the theory, which the neighbourhood theory foregoes."

Haudorff's functional approach to ...

- Order:

$$M \times M \to \{<, >, =\}, \quad M \times M \to \{<, >, =, ||\}$$

- Metric:

$$M \times M \to \mathbb{R}$$

- Convergence:

$$M^{\mathbb{N}} \leftrightarrow M \quad (\iff M^{\mathbb{N}} \times M \to 2 = \{\mathrm{true}, \mathrm{false}\})$$

- Neighbourhood systems:

$$M \to 2^{2^M} \quad (\iff 2^M \times M \to 2 = \{\mathrm{true}, \mathrm{false}\})$$
$$x \mapsto \mathcal{U}(x)$$

... and beyond, ...

"Nun steht einer Verallgemeinerung dieser Vorstellung nichts im Wege, und wir können uns denken, daß eine beliebige Funktion der Paare einer Menge definiert, d. h. jedem Paar (a, b) von Elementen einer Menge M ein bestimmtes Element $n = f(a, b)$ *einer zweiten Menge N zugeordnet sei. In noch weiterer Verallgemeinerung können wir eine Funktion der Elementtripel, Elementfolgen, Elementkomplexe, Teilmengen u. dgl. von M in Betracht ziehen."*

"Now there is no obstacle to generalizing this point of view, and we could think that there be defined an arbitrary function of pairs of a set, i.e., that, to every pair (a, b) of elements in a set M, there be assigned a certain element $n = f(a, b)$ in a second set N. In even further generalization we can consider a function of triples of elements, sequences of elements, complexes [= families] of elements, subsets of M, and the like."

... but he also cautions:

“Eine ganz allgemein gehaltene Theorie dieser Art würde natürlich erhebliche Komplikationen bedingen und wenig positive Ausbeute liefern.”

“If kept very general, a theory of this type would of course cause considerable complications and provide little positive outcome.”

Our general strategy (to keep excessive generality in check) is to put structure on Hausdorff’s set N which is able to capture the syntax used in the key axioms of the theory!

But we must decide early on whether to keep N fixed, or let N “float” (together with M)?

Guided by our principal examples, we’ll keep it fixed, but there is work (by Henriksen, Kopperman, Flagg, and others) that lets N change with M.

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Structure needed on Hausdorff's set N

(Pre-)Ordered sets: $N = 2 = ((\{\mathrm{true}, \mathrm{false}\}, \Longrightarrow), \&, \mathrm{true})$

$$\mathrm{true} \Longrightarrow x \leq x$$
$$x \leq y \;\&\; y \leq z \Longrightarrow x \leq z$$

(Lawvere-)Metric spaces: $N = [0, \infty] = (([0, \infty], \geq), +, 0)$

$$0 \geq d(x, x)$$
$$d(x, y) + d(y, z) \geq d(x, z)$$

(Barr-)Topological spaces: $N = 2$ again, but entries are not just points

$$\mathrm{true} \Longrightarrow \dot{x} \rightsquigarrow x$$
$$\mathfrak{X} \rightsquigarrow \mathfrak{y} \;\&\; \mathfrak{y} \rightsquigarrow z \Longrightarrow \Sigma\mathfrak{X} \rightsquigarrow z$$

with $x, z \in X$, $\mathfrak{y} \in \mathrm{U}X$, $\mathfrak{X} \in \mathrm{UU}X$, $\quad A \in \dot{x} \iff x \in A$,
$$A \in \Sigma\mathfrak{X} \iff \{\mathfrak{x} \in \mathrm{U}X \mid A \in \mathfrak{x}\} \in \mathfrak{X}$$

Quantales – taking the role of Hausdorff's set N

V unital and (for convenience) commutative *quantale*

= complete lattice with a comm. monoid structure, $\mathrm{V} = (\mathrm{V}, \otimes, \mathrm{k})$, s.th.

$$u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i$$

= a small, thin, symmetric monoidal-closed category

= a comm. monoid in the symmetric monoidal-closed category **Sup**

Some examples:

- $\mathrm{V} = 2$ with $u \otimes v = u \,\&\, v$, $\quad \mathrm{k} = \mathrm{true}$ (Boolean 2-chain)
- $\mathrm{V} = [0, \infty]$ with $u \otimes v = u + v$, $\quad \mathrm{k} = 0$ (Lawvere quantale)
- V any frame with $u \otimes v = u \wedge v$, $\quad \mathrm{k} = \top$ (a cartesian quantale)
- $\mathrm{V} = 2^M$, for any commutative monoid M (free quantale over M),

  with $A \otimes B = \{\alpha \cdot \beta \mid \alpha \in A, \beta \in B\}$, $\quad \mathrm{k} = \{\eta\}$, η neutral in M

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One more quantale: $\{$distance distribution functions$\}$

$$[0,\infty] \cong [0,1] = (([0,1],\leq),\cdot,1)$$

$$\Delta \ni \varphi : [0,\infty] \to [0,1] \qquad \varphi(\beta) = \sup_{\alpha<\beta} \varphi(\alpha)$$

$$(\varphi \otimes \psi)(\gamma) = \sup_{\alpha+\beta=\gamma} \varphi(\alpha)\psi(\beta) \qquad \kappa(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha > 0. \end{cases}$$

$$[0,\infty] \xrightarrow{\ \sigma\ } \Delta \xleftarrow{\ \tau\ } [0,1]$$

full embeddings whose values are step functions, defined by

$$\sigma(\alpha)(\gamma) = \begin{cases} 0 & \text{if } \gamma \leq \alpha, \\ 1 & \text{if } \gamma > \alpha, \end{cases} \qquad \tau(u)(\gamma) = \begin{cases} 0 & \text{if } \gamma = 0, \\ u & \text{if } \gamma > 0. \end{cases}$$

$\varphi = \sup_{\alpha} \sigma(\alpha) \otimes \tau(\varphi(\alpha))$: Δ as a coproduct in **Qnt** !

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$\varphi = \sup_{\alpha} \sigma(\alpha) \otimes \tau(\varphi(\alpha))$: $\quad \Delta$ as a coproduct in **Qnt** !

Quantale-enriched categories

$(\mathsf{V}, \otimes, \mathsf{k})$ (commutative) quantale

V-*relation*	$r : X \nrightarrow Y$	$r : X \times Y \to \mathsf{V}$	
	$s \cdot r : X \nrightarrow Z$	$s \cdot r(x,z) = \bigvee_y r(x,y) \otimes s(y,z)$	
	$r^\circ : Y \nrightarrow X$	$r^\circ(y,x) = r(x,y)$	
V-*graph*	$f_\circ : X \nrightarrow Y$	$f_\circ(x,y) = \begin{cases} \mathsf{k} & \text{if } f(x) = y \\ \bot & \text{else} \end{cases}$	
	$f^\circ : Y \nrightarrow X$	$f^\circ(y,x) = f_\circ(x,y)$	
V-*category*	$(X, a : X \nrightarrow X)$	$\mathsf{k} \leq a(x,x)$	$1_X^\circ \leq a$
		$a(x,y) \otimes a(y,z) \leq a(x,z)$	$a \cdot a \leq a$
V-*functor*	$(X,a) \xrightarrow{f} (Y,b)$	$a(x,y) \leq b(fx, fy)$	$a \leq f^\circ \cdot b \cdot f_\circ$

Some examples

V	V-**Cat**	
1	**Set**	sets X
2	**Ord**	(pre)ordered sets $(X, \leq)$
$[0, \infty]$	**Met**	(generalized) metric spaces (X, d)
2^M	M-**Ord**	M-scaled (pre)ordered sets $(X, (\leq_\alpha)_{\alpha \in M})$ $x \leq_\eta x \quad$ (η neutral in M) $x \leq_\alpha y,\ y \leq_\beta z \Rightarrow x \leq_{\alpha \cdot \beta} z$
Δ	**ProbMet**	probabilistic (gen'd) metric spaces $(X, (p_\alpha)_{\alpha \geq 0})$ ($p_\alpha(x, y) =$ prob'ty of $d(x, y) < \alpha$, with d random) $p_0(x, x) = 0, \quad (\alpha > 0 \Longrightarrow p_\alpha(x, x) = 1)$ $\alpha + \beta \leq \gamma \Longrightarrow p_\alpha(x, y) \cdot p_\beta(y, z) \leq p_\gamma(x, z)$

Where are Barr's topological spaces?

Recall the Manes-Barr ultrafilter-convergence axioms on a set X:

for all $x, z \in X$, $\mathfrak{y} \in \mathrm{U}X$, $\mathfrak{X} \in \mathrm{UU}X$ one must have :

$$\text{true} \Longrightarrow \dot{x} \rightsquigarrow x, \qquad \mathfrak{X} \rightsquigarrow \mathfrak{y} \ \&\ \mathfrak{y} \rightsquigarrow z \Longrightarrow \Sigma\mathfrak{X} \rightsquigarrow z,$$

where ($A \in \dot{x} \iff x \in A$) and ($A \in \Sigma\mathfrak{X} \iff \{\mathfrak{x} \in \mathrm{U}X \mid A \in \mathfrak{x}\} \in \mathfrak{X}$)

define the (unique) structure of the ultrafilter monad $\mathbb{U}$ on **Set**.

Challenge:

the relation $\rightsquigarrow : \mathrm{U}X \nrightarrow X$ needs to be extended to $\rightsquigarrow : \mathrm{UU}X \nrightarrow \mathrm{U}X$!

The Barr extension:

for $r : X \nrightarrow Y$ given by ($X \xleftarrow{\pi_1} R \xrightarrow{\pi_2} X$), define $\overline{\mathrm{U}}r : \mathrm{U}X \nrightarrow \mathrm{U}Y$ by

$$\overline{\mathrm{U}}r = (U\pi_2)_\circ \cdot (\mathrm{U}\pi_1)^\circ$$

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Lax extensions of **Set**-monads

$\mathbb{T} = (\mathsf{T},\ \mathsf{m}:\mathsf{TT}\to\mathsf{T},\ \mathsf{e}:\mathrm{Id}_{\mathbf{Set}}\to\mathsf{T})$ monad on **Set**, equipped with

$\hat{\mathsf{T}}: \mathsf{V}\text{-}\mathbf{Rel} \to \mathsf{V}\text{-}\mathbf{Rel}$, a *lax extension* of $\mathbb{T}$ (Seal 2004), that is:

- $\hat{\mathsf{T}}$ is a lax 2-functor, coinciding with T on objects;
- $(\mathsf{T}f)_\circ \leq \hat{\mathsf{T}}(f_\circ)$ and $(\mathsf{T}f)^\circ \leq \hat{\mathsf{T}}(f^\circ)$, for every map f of sets;
- $\mathsf{m}_\circ : \hat{\mathsf{T}}\hat{\mathsf{T}} \to \hat{\mathsf{T}}$ and $\mathsf{e}_\circ : \mathrm{Id}_{\mathsf{V}\text{-}\mathbf{Rel}} \to \hat{\mathsf{T}}$ are oplax nat. transformations

NOTE: Such lax extensions are in 1-1 correspondence with *monotone lax distributive laws* $\lambda : \mathsf{TP_V} \to \mathsf{P_V T}$ of the V*-power-set monad*, or *discrete* V*-presheaf monad*, $(\mathsf{P_V}, \mathsf{s}, \mathsf{y})$ over $\mathbb{T}$:

$$\mathsf{P_V}X = \mathsf{V}^X,\ (\mathsf{P_V}f)(\sigma)(y) = \bigvee_{y\in f^{-1}x} \sigma(x),\ (\mathsf{s}_X\Sigma)(x) = \bigvee_{\sigma\in\mathsf{V}^X} \Sigma(\sigma)\otimes\sigma(x)$$

Lax extensions of **Set**-monads

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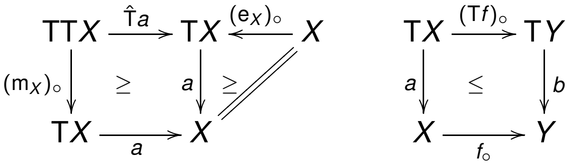
$(\mathbb{T}, \mathsf{V})$-Cat $\quad (X, a : \mathsf{T}X \nrightarrow X) \quad k \le a(e_X(x), x)$

$$\hat{\mathsf{T}}a(\mathfrak{X}, \mathfrak{y}) \otimes a(\mathfrak{y}, z) \le a(m_X(\mathfrak{X}), z)$$

$$(X, a) \xrightarrow{f} (Y, b) \quad a(\mathfrak{x}, y) \le b(\mathsf{T}f(\mathfrak{x}), f(y))$$



Equivalently: $\quad e_X^\circ \le a$

$$a \circ a \le a \quad \text{(Kleisli convolution)}$$

$$a \le f^\circ \cdot b \cdot (\mathsf{T}f)_\circ$$

Kleisli for $r : \mathsf{T}X \nrightarrow Y,\ s : \mathsf{T}Y \nrightarrow Z$: $\ s \circ r := s \cdot \hat{\mathsf{T}}r \cdot m_X^\circ : \mathsf{T}X \nrightarrow Z$

$(\mathbb{T}, \mathsf{V})$-categories

$(\mathbb{T}, \mathsf{V})$-Cat $\quad (X, a : \mathsf{T}X \nrightarrow X) \quad \mathsf{k} \le a(\mathsf{e}_X(x), x)$

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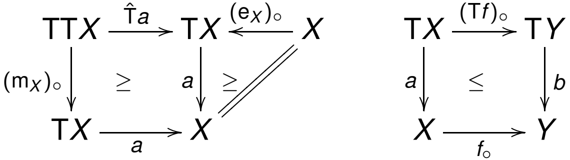

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Some Examples

V, $\mathbb{T}$, $\hat{\mathsf{T}}$	($\mathbb{T}$,V)-**Cat**	
....	....	
V, $\mathbb{I}\mathrm{d}$, Id	V-**Cat**	
2, $\mathbb{P} = \mathbb{P}_2$, $\hat{\mathsf{P}}$ (Seal 2004) $A(\hat{\mathsf{P}}r)B \Leftrightarrow \forall y \in B\, \exists x \in A \quad (x\, r\, y)$	**Cls**	$A \subseteq cA$ $B \subseteq cA \Longrightarrow cB \subseteq cA$
2, $\mathbb{U}$, $\overline{\mathrm{U}}$ (Barr 1971) $\mathfrak{x}(\overline{\mathrm{U}}r)\mathfrak{y} \Leftrightarrow \forall A \in \mathfrak{x}, B \in \mathfrak{y} \exists x \in A, y \in B$	**Top** $(x\, r\, y)$	$\dot{x} \rightsquigarrow x$ $\mathfrak{X} \rightsquigarrow \mathfrak{y}, \mathfrak{y} \rightsquigarrow z \Rightarrow \Sigma\mathfrak{X} \rightsquigarrow z$
$[0, \infty]$, $\mathbb{U}$, $\overline{\mathrm{U}}$ (Clem.-Hofm. 2003)	**App**	$a(\dot{x}, x) = 0$

$$(\overline{\mathrm{U}}r)(\mathfrak{x}, \mathfrak{y}) = \sup_{A \in \mathfrak{x}, B \in \mathfrak{y}} \inf_{x \in A, y \in B} r(x, y)$$

$$\Big(\sup_{\mathcal{A} \in \mathfrak{X}, B \in \mathfrak{y}} \inf_{\mathfrak{x} \in \mathcal{A}, y \in B} a(\mathfrak{x}, y)\Big) + a(\mathfrak{y}, z) \geq a(\Sigma\mathfrak{X}, z)$$

Some Properties

$O : (\mathbb{T}, \mathsf{V})$-**Cat** $\to$ **Set** is *topological*; that is:

O is a faithful Grothendieck bifibration with (large-)complete fibres.

Equivalently:
Every family $f_i : X \to O(Y_i, b_i),\ i \in I,$ has an O-cartesian lifting a;
a is the largest structure on X making all f_i morphisms; explicitly:

$$a = \bigwedge_{i \in I} f_i^\circ \cdot b \cdot (Tf_i)_\circ$$

Consequently:

- O has both adjoints; $O \cong (\mathbb{T}, \mathsf{V})$-**Cat**$(E, -)$, with $E = (\{*\}, e^\circ_{\{*\}})$
- $(\mathbb{T}, \mathsf{V})$-**Cat** is complete and cocomplete, with O (co)continuous

What about V as a $(\mathbb{T}, \mathsf{V})$-category?

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A fundamental adjunction

The **Set**-monad $\mathbb{T}$ with its lax extension $\hat{\mathsf{T}}$ to V-**Rel**
may be considered as a (KZ-)monad on V-**Cat** (T 2009):

$$\mathsf{T}(X, a_0) = (\mathsf{T}X, \hat{\mathsf{T}}a_0)$$

If the Kleisli convolution is associative, then (Clementino-Hofm. 2009):

$$(X, a_0, \xi) \longmapsto (X, a_0 \cdot \xi_\circ)$$

$$(\mathsf{V}\text{-}\mathbf{Cat})^{\mathbb{T}} \underset{M}{\overset{K}{\rightleftarrows}} (\mathbb{T}, \mathsf{V})\text{-}\mathbf{Cat} \qquad (M \dashv K)$$

$$(\mathsf{T}X, \underbrace{\hat{\mathsf{T}}a \cdot m_X^\circ}_{=:\,\hat{a}}, m_X) \longleftarrow\!\!\mid (X, a)$$

In particular (Hofmann 2007): If the V-category $(\mathsf{V}, \mathrm{hom})$ has a "good" $\mathbb{T}$-structure ξ, then K makes V a $(\mathbb{T}, \mathsf{V})$-category; plus: Barr extension!

A fundamental adjunction

The **Set**-monad $\mathbb{T}$ with its lax extension $\hat{\mathsf{T}}$ to V-**Rel** may be considered as a (KZ-)monad on V-**Cat** (T 2009):

$$\mathsf{T}(X, a_0) = (\mathsf{T}X, \hat{\mathsf{T}}a_0)$$

If the Kleisli convolution is associative, then (Clementino-Hofm. 2009):

$$(X, a_0, \xi) \longmapsto (X, a_0 \cdot \xi_\circ)$$

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$M \dashv K$ is a factor of the E-M adjunction

$$(X, a) \longmapsto (X, a \cdot (e_X)_\circ)$$

$$(\mathsf{V}\text{-}\mathbf{Cat})^{\mathbb{T}} \underset{M}{\overset{K}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} (\mathbb{T}, \mathsf{V})\text{-}\mathbf{Cat} \underset{A^\circ}{\overset{A_\circ}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathsf{V}\text{-}\mathbf{Cat}$$

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$\mathbb{T} = \mathbb{U}$, $\mathsf{V} = 2$:

$$\mathbf{OrdCompHaus} \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \mathbf{Top} \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \mathbf{Ord}$$

K topologizes $(X, \leq, \xi)$ by $(\mathfrak{x} \rightsquigarrow y \iff \xi(\mathfrak{x}) \leq y)$; A° = Alexandroff

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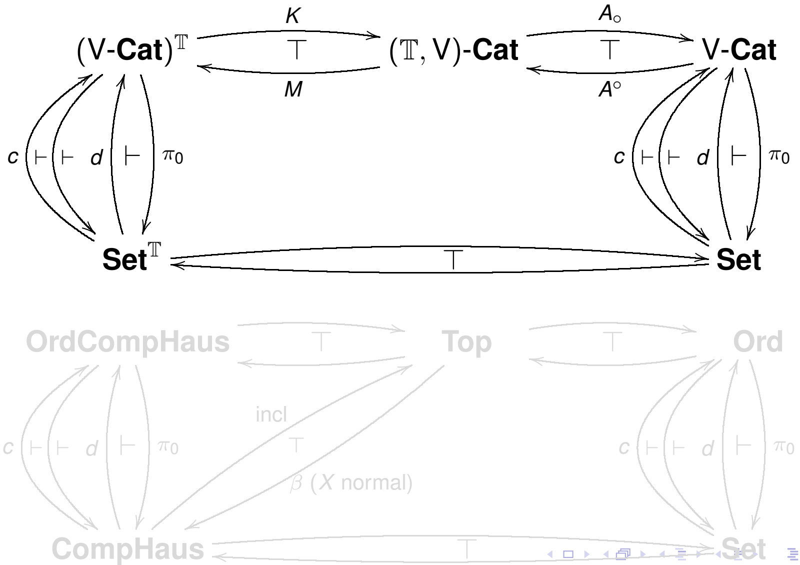
The greater picture (when $\mathbb{T}$ is flat and V integral)



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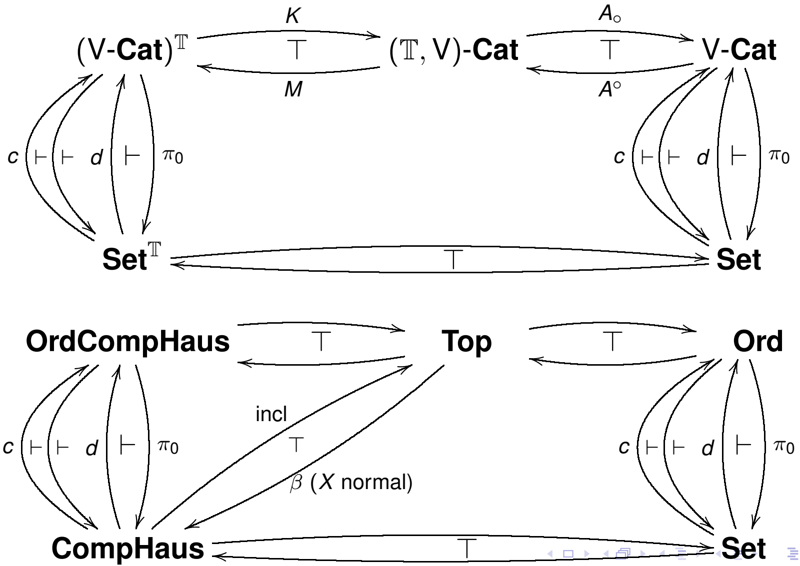

Replacing inequalities by equalities: T_1, core compact

$(X, a : \mathsf{T}X \nrightarrow X)$

(R) $1_X \leq a \cdot (e_X)_\circ$ **$\mathbf{T_1}$:** $1_X \geq a \cdot (e_X)_\circ$

$\mathbb{T} = \mathbb{U}, \mathsf{V} = 2$: $(\dot{x} \rightsquigarrow y \Rightarrow x = y)$

(T) $a \cdot \hat{\mathsf{T}}a \leq a \cdot (m_X)_\circ$ **core compact:** $a \cdot \hat{\mathsf{T}}a \geq a \cdot (m_X)_\circ$

Pisani 1999: $\mathbb{T} = \mathbb{U}, \mathsf{V} = 2$: $\Sigma\mathfrak{X} \rightsquigarrow z \Longrightarrow \exists\, \mathfrak{y}\, (\mathfrak{X} \rightsquigarrow \mathfrak{y} \rightsquigarrow z)$

$\Longleftrightarrow \forall x \in B \subseteq X$ open

$\exists A \subseteq X$ open $(x \in A \ll B)$

$\Longleftrightarrow X$ exponentiable in **Top**

NOTE: If we express
(R) and (T) equivalently as $e_X^\circ \leq a$ and $a \circ a \leq a$ resp., and "strictify" then, *different* properties will emerge: discrete and no condition at all!

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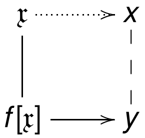
Replacing inequalities by equalities: proper, open

$f : (X, a) \to (Y, b)$

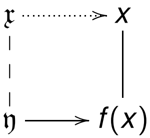
$f_\circ \cdot a \leq b \cdot (\mathsf{T}f)_\circ$ **proper:** $f_\circ \cdot a \geq b \cdot (\mathsf{T}f)_\circ$ $\quad \bigvee_{x \in f^{-1}y} a(\mathfrak{x}, x) \geq b(\mathsf{T}f(\mathfrak{x}), y))$

Manes 1974: $\mathbb{T} = \mathbb{U}, \mathsf{V} = 2$:



$a \cdot (\mathsf{T}f)^\circ \leq f^\circ \cdot b$ **open**: $a \cdot (\mathsf{T}f)^\circ \geq f^\circ \cdot b$ $\quad \bigvee_{\mathfrak{x} \in (\mathsf{T}f)^{-1}\mathfrak{y}} a(\mathfrak{x}, x) \geq b(\mathfrak{y}, f(x))$

Möbus 1981: $\mathbb{T} = \mathbb{U}, \mathsf{V} = 2$:



Some stability properties for proper and open maps

- Isomorphisms are proper/open
- Proper/open maps are closed under composition
- $g \cdot f$ proper/open, g injective $\Longrightarrow f$ proper/open
- $g \cdot f$ proper/open, f surjective $\Longrightarrow g$ proper/open
- f proper/open $\Longrightarrow$ every pullback of f is proper/open

Tychonoff-Frolík-Bourbaki Theorem V completely distributive:

$$f_i : X_i \to Y_i \quad \text{proper} \quad (i \in I) \Longrightarrow \prod_{i \in I} f_i : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i \quad \text{proper}$$

Proof: A straightforward two-line calculation!
Note that, by contrast (*not* by categorical dualization!), one has:

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Replacing inequalities by equalities: T_2, compact

NOTE:
Maps of sets are Lawverian maps in V-**Rel**: $f_\circ \dashv f^\circ$; conversely:
If $r \dashv s$ in V-**Rel**, then $r = f_\circ, s = f^\circ$ for a unique map f, provided that:

- $k = \top > \bot$ (V is "integral" and non-trivial)
- $u \vee v = \top$ and $u \otimes v = \bot \Longrightarrow u = \top$ or $v = \top$ (V is "lean")

In what follows, we also silently assume:

- $k \leq \bigvee_{i \in I} u_i \Longleftrightarrow k \leq \bigvee_{i \in I} u_i \otimes u_i$ (V is "superior")

All three are okay for $V = 2$, $[0, \infty]$, Δ, or any linearly ordered frame, but not for $V = 2^M$.

(X, a) **Hausdorff:** $a \cdot a^\circ \leq 1_X$ $(\bot < a(\mathfrak{z}, x) \otimes a(\mathfrak{z}, y)) \Longrightarrow x = y$
(X, a) **compact:** $1_{TX} \leq a^\circ \cdot a$ $\forall \mathfrak{z} \in TX\ (k \leq \bigvee_{x \in X} a(\mathfrak{z}, x))$

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Compact + Hausdorff = Eilenberg-Moore

$\mathbb{T}$	V	$(\mathbb{T}, \mathsf{V})$-**Cat**$_{\mathrm{Comp}}$	$(\mathbb{T}, \mathsf{V})$-**Cat**$_{\mathrm{Haus}}$
$\mathbb{I}\mathrm{d}$	2	**Ord**	**Set** $\cong$ {discretely ordered sets}
$\mathbb{I}\mathrm{d}$	$[0, \infty]$	**Met**	**Set** $\cong$ {discrete (gen'ed) metric spaces}
$\mathbb{U}$	2	**Comp**	**Haus**
$\mathbb{U}$	$[0, \infty]$	**App**$_{0\text{-Comp}}$	{approach spaces whose induced pseudotopology is Hausdorff}

Manes' Theorem generalized:

$$\mathbf{Set}^{\mathbb{T}} = (T, V)\text{-}\mathbf{Cat}_{\mathrm{CompHaus}} := (T, V)\text{-}\mathbf{Cat}_{\mathrm{Comp}} \cap (T, V)\text{-}\mathbf{Cat}_{\mathrm{Haus}}$$

Proof (Lawvere, Clementino-Hofmann, T)
$(a \cdot a^\circ \leq 1_X$ and $1_{TX} \leq a^\circ \cdot a) \iff a \dashv a^\circ \iff a$ is a map.

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Tychonoff's Theorem

V completely distributive

$(\forall i \in I : X_i = (X_i, a_i) \text{ compact}) \Longrightarrow (X, a) = \prod_{i \in I} X_i \text{ compact}$

Proof (Schubert 2005): For all $\mathfrak{z} \in \mathsf{T}X$:

$$\bigvee_{x \in X} a(\mathfrak{z}, x) = \bigvee_{x \in X} \bigwedge_{i \in I} a_i(Tp_i(\mathfrak{z}), p_i(x)) = \bigwedge_{i \in I} \bigvee_{x_i \in X_i} a_i(Tp_i(\mathfrak{z}), p_i(x)) \geq k$$

The product stability of proper morphisms follows similarly.
No surprise though, since under our silent hypotheses on $\mathbb{T}$, V:

$$(X, a) \to (1, \top) \text{ proper} \iff (X, a) \text{ compact}$$

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Kuratowski-Mrówka Theorem

Under mild hypotheses on $\mathbb{T}$ and V:

Theorem (Clementino-T 2007)

$f:(X,a)\to(Y,b)$ proper $\iff$ • f has compact fibres
• $\mathsf{T}f:(X,\hat{a})\to(Y,\hat{b})$ proper

(in **Top, App, ...**) $\iff$ • f has compact fibres
• f is closed

$\iff$ f is *stably* closed

Corollary

X compact $\iff \forall Z: X\times Z\to Z$ closed (equ'ly: proper)

$(X\xrightarrow{f}Y)$ proper $\iff \forall(Z\to Y):(X\times_Y Z\to Z)$ closed (proper)

Normal and extremally disconnected spaces

Recall:

$$(\mathbb{T},\mathsf{V})\text{-}\mathbf{Cat} \xrightarrow{M} \mathsf{V}\text{-}\mathbf{Cat}^{\mathbb{T}} \to \mathsf{V}\text{-}\mathbf{Cat}, \quad (X,a) \mapsto (\mathsf{T}X, \hat{a}, m_X) \mapsto (\mathsf{T}X, \hat{a}),$$

$$\text{with } \hat{a} = (\mathsf{T}X \xrightarrow{m_X^\circ} \mathsf{T}\mathsf{T}X \xrightarrow{\hat{T}a} \mathsf{T}X)$$

For $\mathbb{T} = \mathbb{U}$, $\mathsf{V} = 2$ and $X \in \mathbf{Top}$, the functor provides $\mathrm{U}X$ with the order

$$\mathfrak{x} \le \mathfrak{y} \iff \forall A \subseteq X \text{ closed} : (A \in \mathfrak{x} \Rightarrow A \in \mathfrak{y})$$

Reminder:

$X \in \mathbf{Top}$ normal $\iff$ disjoint closed sets have disjoint nbhds in X

X extremally disconnected $\iff$ closures of open sets are open in X

Normal and extremely disconnected spaces

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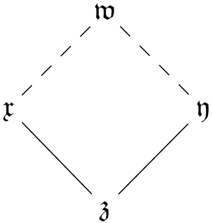
The duality of normal vs extremally disconnected

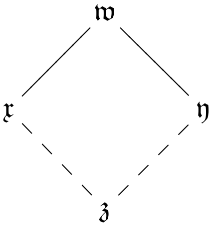
$X \in$ **Top** normal



$(X, a) \in (T, V)$-**Cat** **normal**

$\hat{a} \cdot \hat{a}^\circ \leq \hat{a}^\circ \cdot \hat{a}$

X extremally disconnected



(X, a) **extremally disconnected**

$\hat{a}^\circ \cdot \hat{a} \leq \hat{a} \cdot \hat{a}^\circ$

(X, a) ext. disc. in $(\mathbb{T}, \mathrm{V})$-**Cat** $\iff$ $(\mathrm{T}X, \hat{a})$ ext.disc. in V-**Cat**

$\iff$ $(\mathrm{T}X, \hat{a}^\circ)$ normal in V-**Cat**

Monoidal topology without convergence relations?

Cls	$c : \mathsf{P}X \to 2^X$	(R) $A \subseteq cA$
		(T) $B \subseteq cA \Rightarrow cB \subseteq cA$
Top	c fin'ly additive:	(A) $c(A \cup B) = cA \cup cB$
		$c(\emptyset) = \emptyset$
		(C) $f(c_X A) \subseteq c_Y(fA)$

[Seal 2009]

V-**Cls**	$c : \mathsf{P}X \to \mathsf{V}^X$	(R) $\forall x \in A : \mathsf{k} \le (\mathsf{c}A)(\mathsf{x})$
$= (\mathbb{P}, \mathsf{V})$-**Cat**		(T) $(\bigwedge_{y \in B} (cA)(y)) \otimes (cB)(x) \le (cA)(x)$

[Lai-T 2016]

V-**Top**	c fin'ly additive:	(A) $c(A \cup B)(x) = (cA)(x) \vee (cB)(x)$
		$c(\emptyset)(x) = \bot$
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[Lai-T 2016]

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$c(\emptyset)(x) = \bot$

(C) $(c_X A)(x) \leq c_Y(fA)(fx)$

The cases V $= [0, \infty]$, or Δ, had been studied earlier:

$[0, \infty]$-**Cls**

$\delta : X \times \mathrm{P}X \to [0, \infty]$ (R) $\forall x \in A : 0 \geq \delta(x, A)$

$\delta(x, A) = (cA)(x)$ (T) $(\sup_{y \in B} \delta(y, A)) + \delta(x, B) \geq \delta(x, A)$

$[0, \infty]$-**Top** $=:$ **App** [Lowen 1989] (but his condition (T) is different!):

δ finitely additive (A) $\delta(x, A \cup B) = \min\{\delta(x, A), \delta(x, B)\}$

$\delta(x, \emptyset) = \infty$

$f : X \to Y$ (C) $\delta_X(x, A) \geq \delta_Y(fx, fA)$

Δ-**Top** $=:$ **ProbApp** [Jäger 2015]

How to reconcile closure and ultrafilter convergence?

[Clementino-T 2003] For V completely distributive and $r : X \nrightarrow Y$, define $\overline{\mathrm{U}}r : \mathrm{U}X \nrightarrow \mathrm{U}Y$ by

$$\overline{\mathrm{U}}r(\mathfrak{x}, \mathfrak{y}) := \bigwedge_{A \in \mathfrak{x}, B \in \mathfrak{y}} \bigvee_{x \in A, y \in B} r(x, y)$$

$\mathbb{P}$ and $\mathbb{U}$ interact via the V-relation $\varepsilon_X : \mathrm{P}X \nrightarrow \mathrm{U}X$:

$$\varepsilon_X(A, \mathfrak{x}) = \begin{cases} \mathrm{k} & \text{if } A \in \mathfrak{x} \\ \bot & \text{else} \end{cases}$$

Obtain $\quad A_\varepsilon : (\mathbb{U}, \mathrm{V})\text{-}\mathbf{Cat} \to (\mathbb{P}, \mathrm{V})\text{-}\mathbf{Cat}, \ (X, a) \mapsto (X, c_a = a \circ \varepsilon_X)$

$$(c_a A)(y) = \bigvee_{\mathfrak{x} \ni A} a(\mathfrak{x}, y)$$

A_ε has a right adjoint $(X, c) \mapsto (X, a_c)$, with $a_c(\mathfrak{x}, y) = \bigwedge_{A \in \mathfrak{x}} (cA)(y)$

$(\mathbb{U}, \mathsf{V})$-**Cat** $\cong$ V-**Top**

Theorem [Lai-T 2016]

Let V be completely distributive. Then:

$$A_\varepsilon : (\mathbb{U}, \mathsf{V})\text{-}\mathbf{Cat} \hookrightarrow (\mathbb{P}, \mathsf{V})\text{-}\mathbf{Cat} = \mathsf{V}\text{-}\mathbf{Cls}$$

is a full coreflective embedding; its image is V-**Top** $\cong (\mathbb{U}, \mathsf{V})$-**Cat**.

Corollaries:

[Clementino-Hofmann 2003]

App $\cong (\mathbb{U}, [0, \infty])$-**Cat**

$0 \geq a(\dot{x}, x)$

$(\sup_{\mathcal{A} \in \mathfrak{X}, B \in \mathfrak{y}} \inf_{\mathfrak{x} \in \mathcal{A}, y \in B} a(\mathfrak{x}, y)) + a(\mathfrak{y}, z) \geq a(\Sigma \mathfrak{X}, z)$

[Jäger 2016],[Lai-T 2016]

ProbApp $\cong (\mathbb{U}, \Delta)$-**Cat**

(R) $\kappa \leq a(\dot{x}, x)$

(T) ...

$(\mathbb{U}, \mathsf{V})$-**Cat** $\cong$ V-**Top**

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Let V be completely distributive. Then:

$$A_\varepsilon : (\mathbb{U}, \mathsf{V})\text{-}\mathbf{Cat} \hookrightarrow (\mathbb{P}, \mathsf{V})\text{-}\mathbf{Cat} = \mathsf{V}\text{-}\mathbf{Cls}$$

is a full coreflective embedding; its image is V-**Top** $\cong (\mathbb{U}, \mathsf{V})$-**Cat**.

Corollaries:

[Clementino-Hofmann 2003]

App $\cong (\mathbb{U}, [0, \infty])$-**Cat**

$0 \geq a(\dot{x}, x)$

$(\sup_{\mathcal{A} \in \mathfrak{X}, B \in \mathfrak{y}} \inf_{\mathfrak{x} \in \mathcal{A}, y \in B} a(\mathfrak{x}, y)) + a(\mathfrak{y}, z) \geq a(\Sigma\mathfrak{X}, z)$

[Jäger 2016],[Lai-T 2016]

ProbApp $\cong (\mathbb{U}, \Delta)$-**Cat**

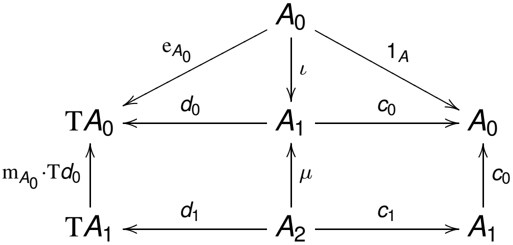
(R) $\kappa \leq a(\dot{x}, x)$

(T) ...

Away from **Set**: Burroni's $\mathbb{T}$-categories

$\mathcal{C}$ category with pullbacks, $\mathbb{T} = (\mathrm{T}, \mathrm{m}, \mathrm{e})$ monad on $\mathcal{C}$; form

Cat($\mathbb{T}$):



Morphisms $f = (f_0, f_1) : A \to B$ preserve the structure, i.e. d_0, c_o, ι, μ

Ord($\mathbb{T}$): $\mathbb{T}$-categories with (d_0, c_0) jointly monic

EM($\mathbb{T}$): $\mathbb{T}$-categories with $A_1 = \mathrm{T}A_0,\ d_0 = 1_{\mathrm{T}A_0}$

Obtain: EM($\mathbb{T}$) $\longrightarrow$ Ord($\mathbb{T}$) $\longrightarrow$ Cat($\mathbb{T}$) $\longrightarrow \mathcal{C}$

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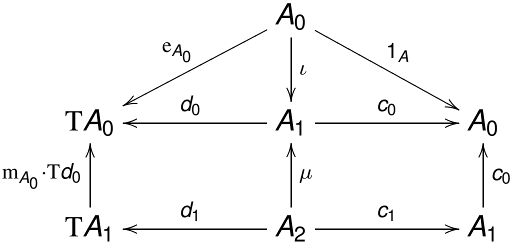

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Obtain: $\mathrm{EM}(\mathbb{T}) \longrightarrow \mathrm{Ord}(\mathbb{T}) \longrightarrow \mathrm{Cat}(\mathbb{T}) \longrightarrow \mathcal{C}$

Burroni 1971:

$\mathcal{C} =$ **Set**, $\mathbb{T}$ laxly extended to **Rel** á la Barr. Then:

$$\mathrm{Ord}(\mathbb{T}) \cong (\mathbb{T}, 2)\text{-}\mathbf{Cat}$$

In particular: $\mathrm{Ord}(\mathbb{U}) \cong \mathbf{Top}$

But what about arbitray quantales V, rather than 2?
For example: Is V-**Cat** of the form $\mathrm{Ord}(\mathbb{T})$, for some $\mathbb{T}$? ...

While for every $\mathbb{T}$, laxly extended to V-**Rel**, one can find a monad Π laxly extendable to **Rel** (encoding both, $\mathbb{T}$ and V) such that (Lowen-Vroegrijk 2008, Hofmann 2014)

$$(\mathbb{T}, \mathsf{V})\text{-}\mathbf{Cat} \cong (\Pi, 2)\text{-}\mathbf{Cat},$$

the lax extension of Π is *not* á la Barr, and $(\Pi, 2)\text{-}\mathbf{Cat} \not\cong \mathrm{Ord}(\Pi)$.

Comparison with $(\mathbb{T}, \mathsf{V})$-Cat

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Questions, To-Do list

- Is every topological category over **Set** of the form $(\mathbb{T}, \mathsf{V})$-**Cat**?
- If not, characterize the latter categories amongst the former.
- Apply $(\mathbb{T}, \mathsf{V})$-category theory in case of "probabilistic" quantales.
- To which extent are $(\mathbb{T}, \mathsf{V})$-categories covered by Burroni?
- Pursue monoidal topology in Burroni's context.
- Apply the emerging theory in particular in topological algebra.

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