Topological Inspiration for Infinity Modular Operads

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A really fast introduction to a lot of cool math:

- Let $Gal(\mathbb{Q})$ denote the absolute Galois group of $\mathbb{Q}$.
- This is a large profinite group:

$$\widehat{G} = \lim G/H$$

but we don't even know the finite quotients of $Gal(\mathbb{Q})$!

Idea : Identify $g \in Gal(\mathbb{Q})$ with a pair

$$(\chi(g), f_g) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$$

← free group on 2 letters

- $\chi(g)$ is the **cyclotomic character**.
- $\widehat{F}_2 = \pi_1(\mathcal{M}_{0,4}) \cong \widehat{\Gamma}_{0,4}$.

A slightly easier group: $\widehat{\mathrm{GT}}$

Drinfeld 94

Notation: For any homomorphism of profinite groups

$$\widehat{F}_2 \longrightarrow G$$

$$(x, y) \longmapsto (a, b)$$

we write $f(a, b)$ for the image of any $f \in \widehat{F}_2$. For example:

- Given $id : \widehat{F}_2 \to \widehat{F}_2$, we have $f = f(x, y)$;
- Given the map $\widehat{F}_2 \to \widehat{F}_2$ which swaps generators x and y we have $f \mapsto f(y, x)$.

A slightly easier group: $\widehat{\mathrm{GT}}$

The **Grothendieck-Teichmüller group** $\widehat{\mathrm{GT}}$ is the group of pairs

$$(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$$

satsfying the property that

$$x \mapsto x^\lambda \quad \text{and} \quad y \mapsto f^{-1} y^\lambda f$$

induce an automorphism of $\widehat{F}_2$ and :

(I) $f(x,y)f(y,x) = 1$,

(II) $f(x,y)x^m f(z,x) z^m f(y,z) y^m = 1$ where $xyz = 1$ and $m = (\lambda - 1)/2$,

(III) $f(x_{34}, x_{45}) f(x_{51}, x_{12}) f(x_{23}, x_{34}) f(x_{45}, x_{51}) f(x_{12}, x_{23}) = 1$ in $\hat{\Gamma}_{0,5}$ where x_{ij} is a Dehn twist of boundaries i and j.

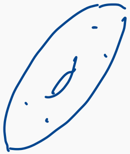
Theorem (Ihara)
There is an injection $Gal(\mathbb{Q}) \hookrightarrow \widehat{\mathrm{GT}}$.

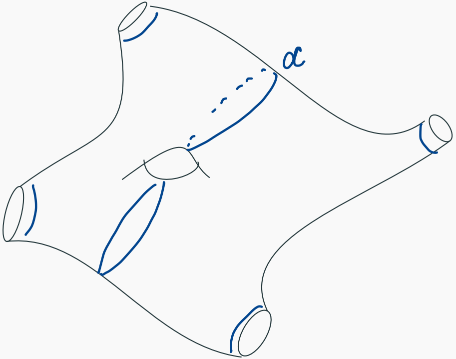
So our question becomes: What is $\widehat{\mathrm{GT}}$?

- $\widehat{F}_2 \cong \widehat{\Gamma}_{0,4}$
- relations in $\widehat{\mathrm{GT}}$ are coming from mapping class groups.
- The mapping class group has a presentation

$$\Gamma_{g,n} = \langle \alpha_1, \ldots, \alpha_k \mid (C), (B), (D), (L) \rangle .$$

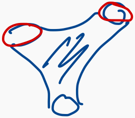
Hatcher–Thurston
80's

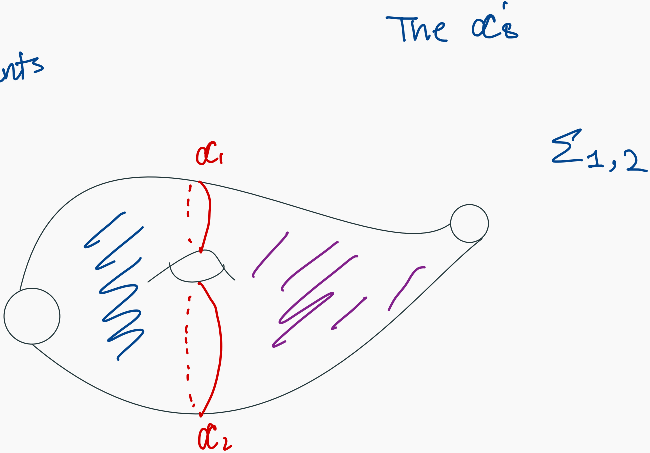




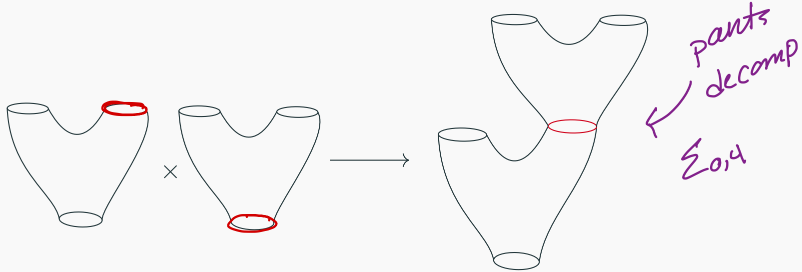
Pants Decompositions

A **pants decomposition** of $\Sigma_{g,n}$ is a collection of simple closed curves that cuts $\Sigma_{g,n}$ into *pairs of pants* (i.e. $\Sigma_{0,3}$).



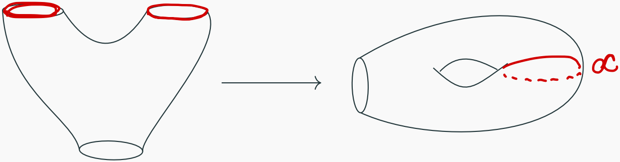


Notice that a **pants decomposition** looks like the result of a composition.

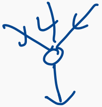


We break Mapping Class Groups into Pants Decompositions

Notice that a **pants decomposition** looks like the result of a composition.



Modular Operads

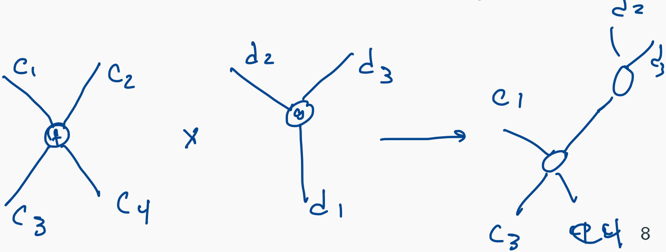


A $\mathfrak{C}$-coloured **cyclic operad** is an algebraic structure consisting of:

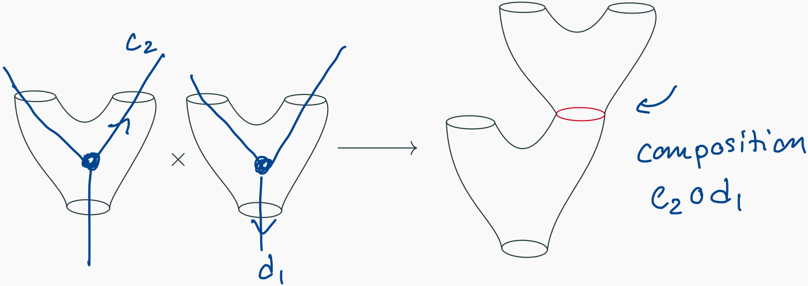
- an involutive set of colours $\mathfrak{C}$; ← objects $\mathfrak{C} \xrightarrow{i} \mathfrak{C}$, $i^2 = id$
- for each $c_1, \ldots, c_n \in \mathfrak{C}$ a Σ_n-set $\mathrm{P}(c_1, \ldots, c_n)$;
- a family of equivariant, associative and unital composition operations

$$\mathrm{P}(c_1, \ldots, c_n) \times \mathrm{P}(d_1, \ldots, d_m) \longrightarrow \mathrm{P}(c_1, \ldots, \hat{c}_i, \ldots, d_1, \ldots \hat{d}_j, \ldots, d_m),$$

when $c_i = d_j^\dagger$.



$*$-autonomous categories examples of cyclic operads



Modular Operads

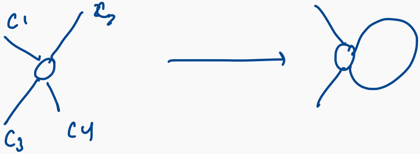
A $\mathfrak{C}$-coloured **modular operad** is a cyclic operad which also has

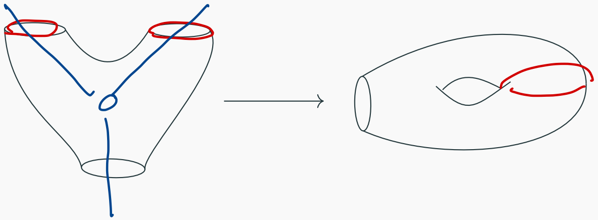
- a family of equivariant contraction operations

$$P(c_1, \ldots, c_n) \longrightarrow P(c_1, \ldots, \hat{c}_i, \ldots, \hat{c}_j, \ldots c_n),$$

  when $c_i = c_j^\dagger$.

satisfying some axioms.

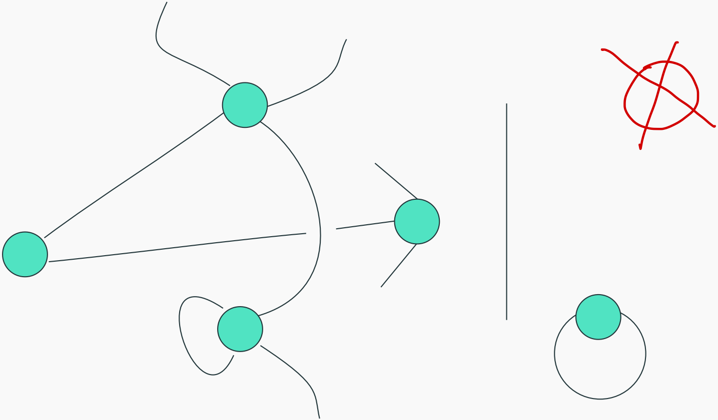




A modular operad is almost the right thing for studying $\widehat{\Gamma}_{g,n}$

There is an equivalence of categories:

$$\text{ModOp} \xrightarrow[\cong]{N} \mathbf{Set}^{\mathsf{U}^{op}}_{Segal}.$$



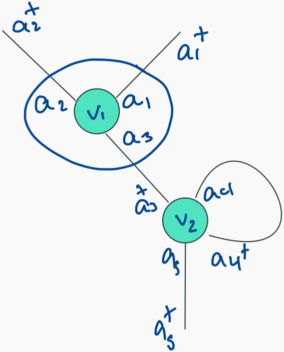
A **graph** G is a diagram of finite sets:

Joyal · Kock

$$i \circlearrowright A \xleftarrow{s} D \xrightarrow{t} V$$

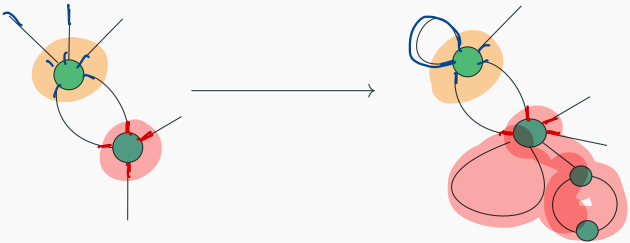
A: half edges; D: domain = half edges attached to a vertex; V: set of vertices

- i is a free involution;
- s is a monomorphism.



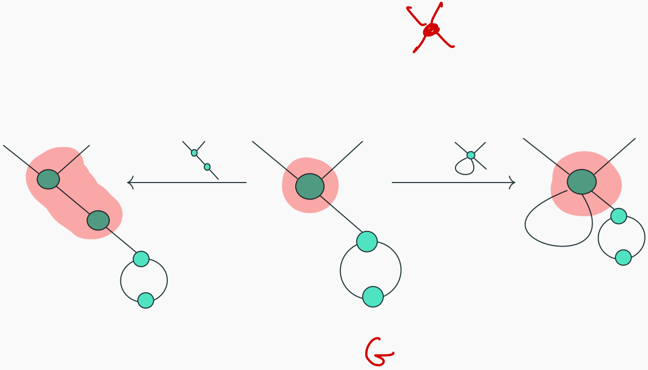
$A = \{a_1, a_1^+, \ldots a_5, a_5^+\}$

$D = \{a_2, a_1, a_3, a_4, a_3^+, a_5, a_4^+\}$



Inner coface maps

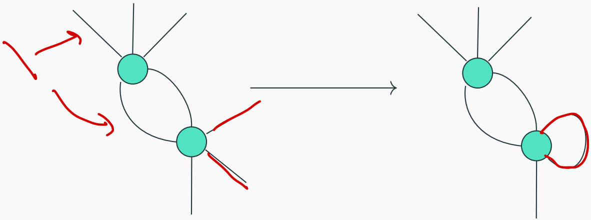
An **inner coface map** $d_v : G \to G'$ is a graphical map defined by "blowing-up" a single vertex v in G by a graph which has precisely **one** internal edge.



Outer coface maps

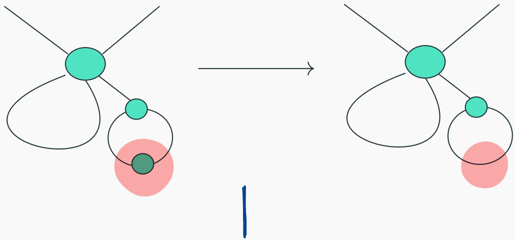
An **outer coface map** is either:

- an **embedding** $d_e : G \to G'$ in which G' has precisely **one** more internal edge than G or
- an **embedding** $\updownarrow \to \bigstar_n$.



Codegeneracy maps

A **codegeneracy map** $s_v : G \to G'$ is a graphical map defined by "blowing-up" a vertex v in G by $\updownarrow$.



The graphical category

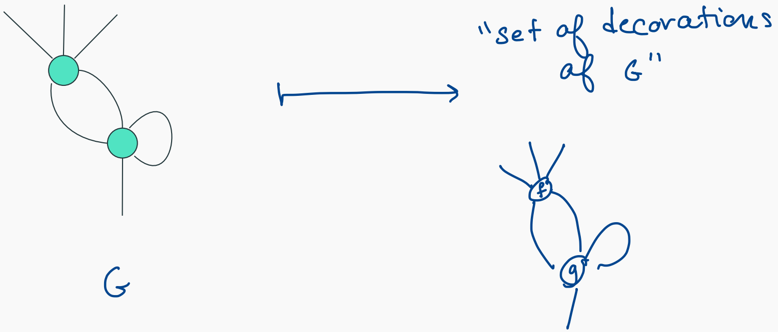
The **graphical category** U is the category whose objects are connected graphs. The morphisms are composites of inner coface maps, outer coface maps, codegeneracies and isomorphisms.

Graphical Sets

The category of **graphical sets** is $\mathrm{Set}^{\mathrm{U}^{op}}$.

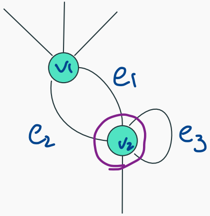
$\mathrm{U}^{op} \xrightarrow{X} \mathrm{Set}$

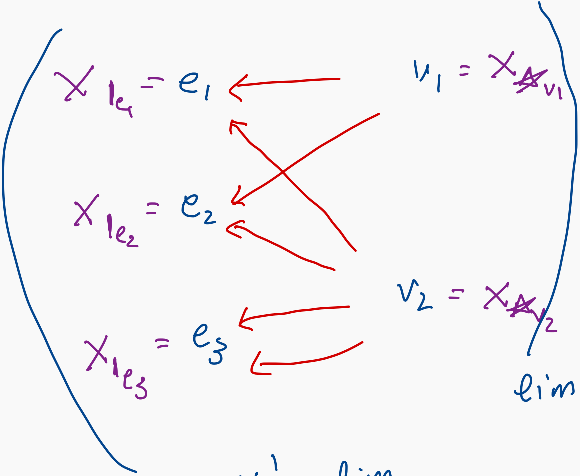
- X_G: evaluation of X at $G \in \mathrm{U}$.
- $\varphi : G \to G' \Rightarrow \varphi^* : X_{G'} \to X_G$.



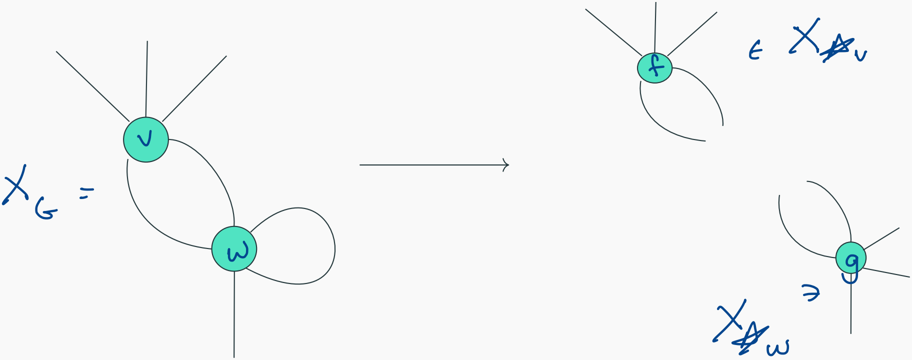
A special graphical set made of internal edges: $X^1_G = \lim\limits_{\star_v \leftarrow \updownarrow \rightarrow \star_w}$

Let $X : U^{op} \longrightarrow Set$

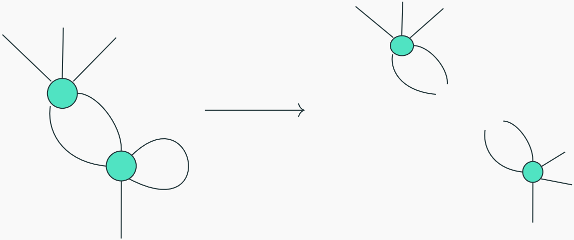




$X^1_G = \lim\limits_{\star_v \leftarrow 1 \rightarrow \star_w}$



$f \circ g \in X'_G$



The embeddings $\bigstar_v \hookrightarrow G$ induce a Segal map

$$X_G \longrightarrow X_G^1 \subseteq \prod_{v \in V(G)} X_{\bigstar_v}$$

which factors through X_G^1.

Segal graphical sets

A graphical set $X \in \mathrm{Set}^{\mathrm{U}^{op}}$ is strictly **Segal** if the Segal map

$$X_G \longrightarrow X_G^1 \subseteq \prod_{v \in V(G)} X_{\star_v}$$

is a bijection for each G in U.

Theorem (HRY20b)
There is an equivalence of categories:

$$\mathrm{ModOp} \xrightarrow[\cong]{N} \mathbf{Set}^{\mathrm{U}^{op}}_{Segal}.$$

Segal graphical sets

A graphical set $X \in \mathrm{sSet}^{\mathsf{U}^{op}}$ is weakly **Segal** if the Segal map

$$X_G \longrightarrow X_G^1 \subseteq \prod_{v \in V(G)} X_{\star_v}$$

is a weak homotopy equivalence for each G in U.

Weak Segal Modular Operads Take Us Back To $Gal(\mathbb{Q})$

A modular operad of Seamed Surfaces

The goupoid $\mathcal{S}_{g,n}$:

- objects are surfaces $P := (\Sigma_{g,n}, P, Q)$ together with a "atomic" **pants decomposition**;
- morphisms are $\pi_0\mathrm{Diff}^+(\Sigma_{g,n}, \partial, \sigma)$.

Σ_n acts freely on $\mathcal{S}_{g,n}$ by permuting the labels of boundaries $\Rightarrow$

$$B\mathcal{S}_{g,n} \simeq B\Gamma_{g,n}.$$

A modular operad of Seamed Surfaces

Operations:

$$\mathcal{S}_{g,n} \times_{ij} \mathcal{S}_{h,k} \xrightarrow{\circ_{ij}} \mathcal{S}_{g+h,n+k-2}$$

and

$$\mathcal{S}_{g,n} \xrightarrow{\xi_{ij}} \mathcal{S}_{g+1,n-2}$$

can be defined on objects by gluing surfaces and on morphisms as the "combination" of the maps on the subsurfaces.

These are well-defined, associative operations and thus

$$\mathcal{S} = \{\mathcal{S}_{g,n}\}$$

assembles into a **modular operad in groupoids**. $\Rightarrow$

$$N\mathcal{S} = \{N\mathcal{S}_{g,n}\} \in \mathsf{Set}^{\mathsf{U}^{op}}_{Segal}.$$

The genus 0 case

Proposition (BHR)

$$\widehat{\mathsf{GT}} \cong \pi_0 \operatorname{map}^h(N\widehat{\mathcal{S}_0}, N\widehat{\mathcal{S}_0})$$

But to get back to our comparison with the mapping class groups:

Theorem (BHR)

$$\widehat{\mathsf{GT}} \cong \pi_0 \operatorname{map}^h(B\widehat{N\mathcal{S}_0}, B\widehat{N\mathcal{S}_0})$$

∞-cyclic operads

Gal $\mathbb{Q}$

The genus 0 case

Proposition (BHR)

$$\widehat{\mathrm{GT}} \cong \pi_0 \,\mathrm{map}^h(N\widehat{\mathcal{S}_0}, N\widehat{\mathcal{S}_0})$$

But to get back to our comparison with the mapping class groups:

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$$\widehat{\mathrm{GT}} \cong \pi_0 \,\mathrm{map}^h(B\widehat{N\mathcal{S}_0}, B\widehat{N\mathcal{S}_0})$$

Point: Here we can see how $Gal(\mathbb{Q})$ acts.

Groups Related to $\widehat{\mathsf{GT}}$: The higher genus case

There is a subgroup $\Lambda \subseteq \widehat{\mathsf{GT}}$:

Schneps - Hatcher

Λ

Theorem (BR)
There is an isomorphism

$$\Lambda \cong \mathrm{End}_0(N\widehat{\mathcal{S}}).$$

Theorem (BR - In Progress)
There is an isomorphism

$$\Lambda \cong \pi_0\, \mathrm{map}^h(B\widehat{N\mathcal{S}}).$$

Question: Can weak Segal modular operads give us even more information about $Gal(\mathbb{Q})$?

① the involution on colours $\mathbb{C}$
doesn't play a role here in my example

② Dan Petersen ~~Be~~ using modular operads
coloured in groupoids
for studying bundles
on $M_{g,n}$

③ compact closed categories
traced $*$-autonomous categories } modular operads

$$sSet^{\mathcal{U}^{op}}$$

$$Map^h(Sc[G], X) \simeq X_1^G$$

Thanks!