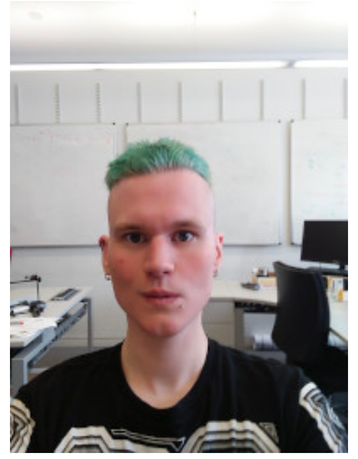
Algebraic theories with string diagrams

Pawel Sobocinski, Compositionality Group, Tallinn University of Technology
(joint work with Ivan Di Liberti, Fosco Loregian and Chad Nester)

Topos Institute, September 30, 2021

Plan of the talk

- **Universal algebra basics**
- Lawvere and cartesian categories
- Props, string diagrams and Fox’s theorem
- Partial Lawvere theories
- Variety theorem and ongoing and future work

Theory of Commutative Monoids

$$(\ \underline{\{m,\ e\}},\ \{\ \underline{m(m(x,y),z) = m(x,m(y,z)),\ m(x,y) = m(y,x),\ m(e,x) = x}\ \}\ )$$

signature consisting of
operation symbols

equations

$\{m,\ e\}$

pairs of **terms** over some set of variables
implicit universal quantification

arity 2

arity 0

Universal Algebra basics 1

- A signature is a pair Σ = (S,α) where S is a set of **operation symbols** together with an **arity function** α : S → **N**
- A Σ-algebra is a pair (A,[-]) where A is a set and [-] is a function that sends operation symbols to functions [σ] : $A^{\alpha(\sigma)}$ → A
- A Σ-algebra homomorphism is the obvious thing

$$
\begin{tikzcd}
A^n \arrow[r, "f^n"] \arrow[d, "\llbracket t_n \rrbracket_A"'] & B^n \arrow[d, "\llbracket t_n \rrbracket_B"'] \\
A \arrow[r, "f"'] & B.
\end{tikzcd}
$$

- Given a set of variables V, the **term** Σ-algebra T_V is
  - T_V ::= V | t_0 | $t_1(T_V)$ | $t_2(T_V,T_V)$ | ... | $t_n(T_V,...,T_V)$ | ...
- The term Σ-algebra satisfies a universal property, any v : V → A extends to a unique homomorphism v* : T_V → A

Universal Algebra basics II

- An equation is a pair (s,t) $\in T_V \times T_V$
- A theory is a pair (Σ, E) where Σ is a signature and E is a set of equations.
  - Example: the theory of commutative monoids
- A model is a Σ-algebra where every equation $e \in E$ holds
- A model homomorphism is a Σ-algebra homomorphism
- The class of models of a theory is called a **variety**
- **Theorem** (Birkhoff 1935) A class of Σ-algebras is a variety iff it is closed under homomorphic images, subalgebras and products.

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Lawvere and cartesian categories

- There are several things to be unhappy about with classical universal algebra
  - taking theory=presentation isn’t great: different presentations yield the same varieties
  - reliance on ad-hoc standards, e.g. inductively defined terms over a fixed countable set of variables
  - Lawvere’s functorial semantics clarifies and simplifies all of this beautifully

Finite products

- The **category with free finite products on one object** is FinSet^{op}
- FinSet^{op} has (up to equivalence) an alternative "operational" description
  - objects: natural numbers, we think of $m = \{x_1, x_2, \ldots, x_m\}$
  - arrows $m \to n$: n-tuples of variables in $\{x_1, x_2, \ldots, x_m\}$, e.g.
    - there is exactly one arrow $1 \to 2$: (x_1, x_1)
    - there are two arrows $2 \to 1$: (x_1) and (x_2)
    - composition is **substitution**: e.g. $(x_1, x_1);(x_2) = x_1$

Finite products ctd

- The category with **free finite products on a signature Σ** has a similar operational description
  - objects: natural numbers, we think of $m = \{x_1, x_2, \ldots, x_m\}$
  - arrows $m \to n$: n-tuples of terms in $T_{\{x1,x2,\ldots,xm\}}$, e.g. for the sig of monoids
    - there is an arrow $1 \to 2$: (x_1, e)
    - there is an arrows $2 \to 1$: (m)
    - composition is **substitution**: e.g. $(x_1, e); (m) = m(x_1, e)$

Terms demystified!
The algebra of terms and substitution is simply a convenient description of a category with free products

Abstract universal algebra

- Equate a theory with a category **L** with finite products (single sorted: with one generating object)
- doesn't suffer from reliance on particular presentations
- e.g. for commutative monoids, take the free category generated by {m,e}, quotient by least congruence generated by eqs
- A (classical) model is a product preserving functor **L** → **Set**
- Model homomorphisms are natural transformations
- Simple, beautiful, easily generalisable

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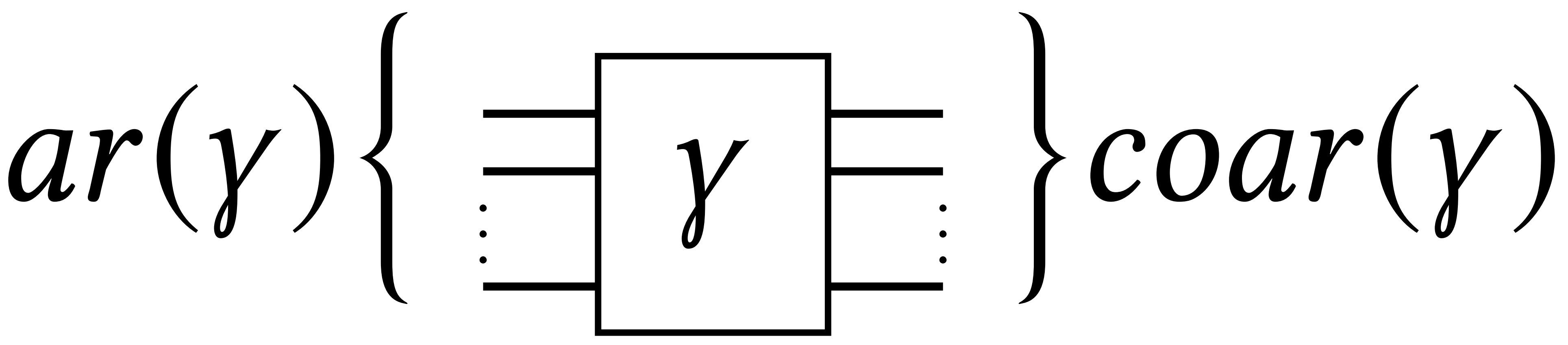
Props

- A prop is a symmetric strict monoidal category with
  - objects: natural numbers
  - monoidal product on objects is addition, i.e. $m \otimes n := m+n$
- homomorphism of props: identity on objects symmetric strict monoidal functor
- e.g. the prop **F** of functions.
  - Think of m as $\underline{m}=\{1,2,\ldots,m\}$. The arrows of F are simply functions $m \to n$.
  - The monoidal product on arrows is the obvious thing, stacking graphs of functions on top of one another.

Monoidal theories

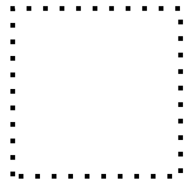
- A monoidal signature Γ = (G, *ar*, *coar*) where G is a set of operations
    - *ar* : G → **N** is gives arities
    - *coar* : G → **N** gives **coarities**



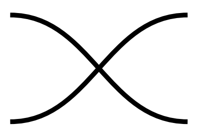
Terms for props

- Classical terms are a description of a free category with products
- Terms for props should be a description of a free prop - string diagrams!

$$\frac{}{\gamma : (ar(\gamma), coar(\gamma))}$$

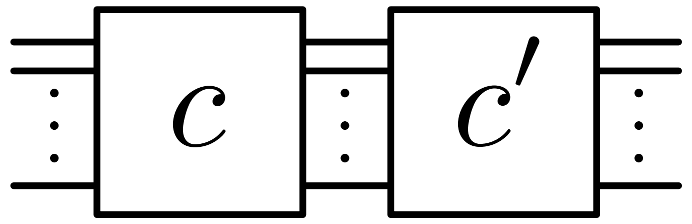
 $: (0, 0)$

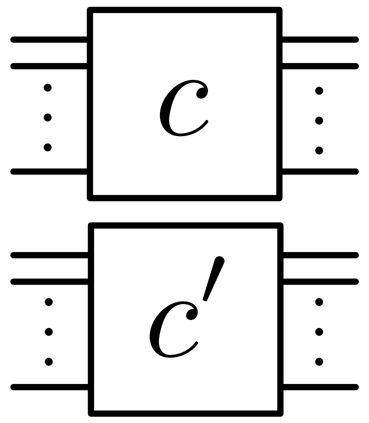
 $: (1, 1)$

 $: (2, 2)$

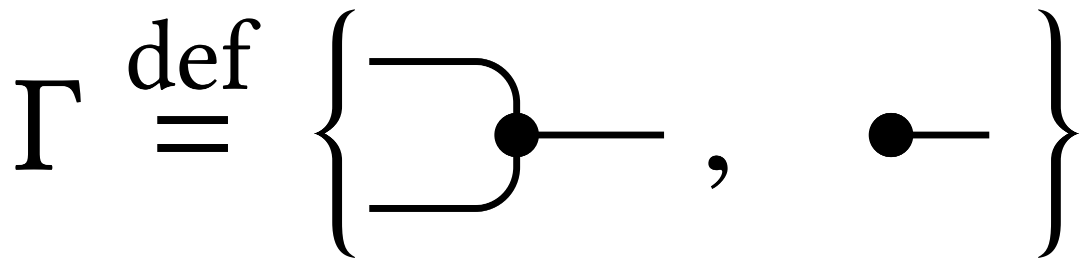
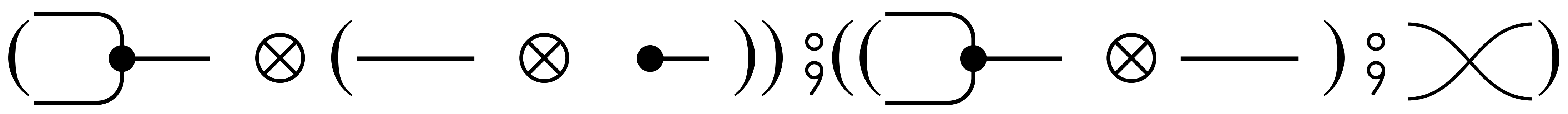
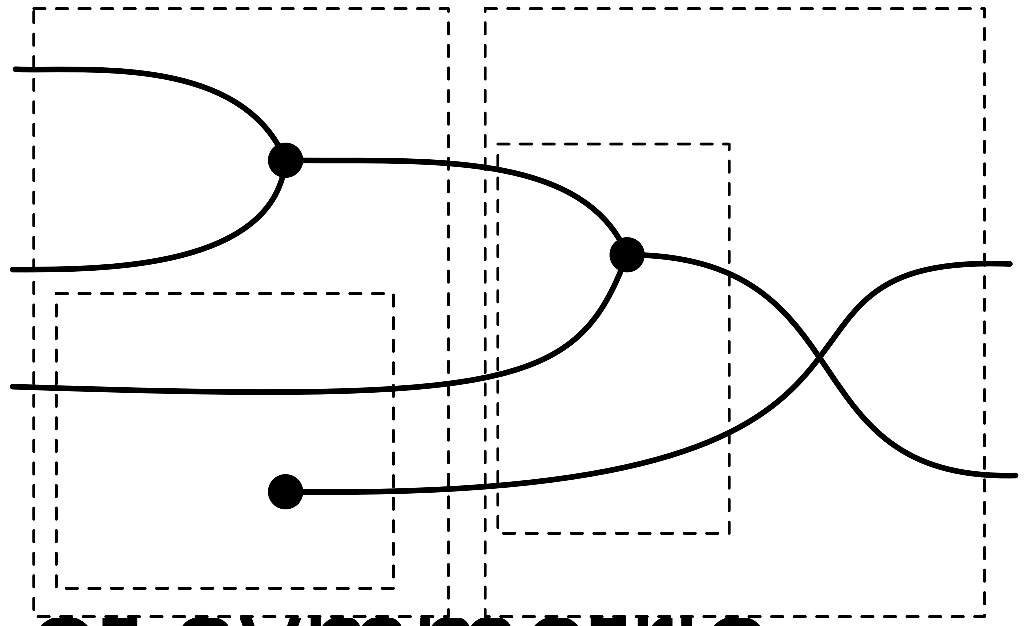
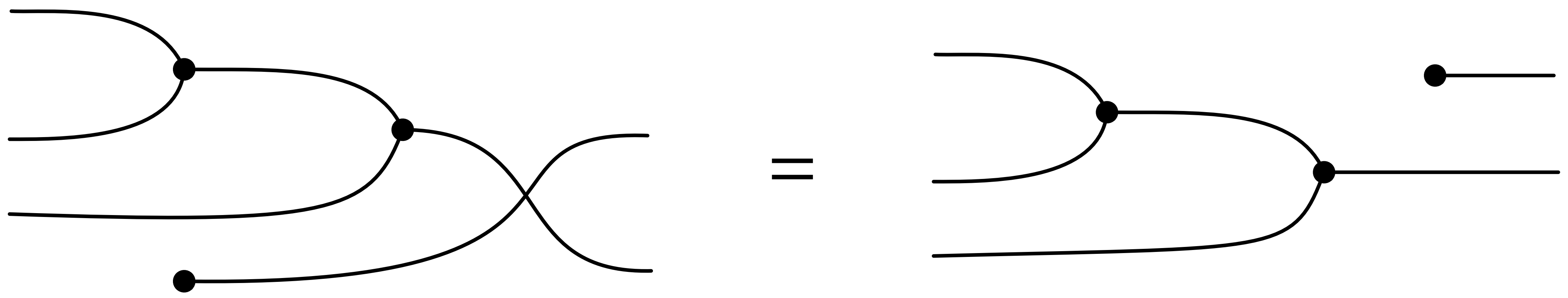
$$\frac{c : (n, z) \quad d : (z, m)}{c \mathbin{;} d : (n, m)}$$

$$\frac{c : (n, m) \quad d : (r, z)}{c \otimes d : (n+r, m+z)}$$

$c \mathbin{;} c'$ is drawn 

$c \otimes c'$ is drawn 

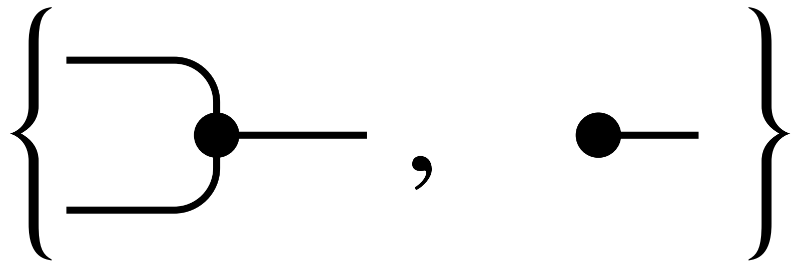
From terms to string diagrams

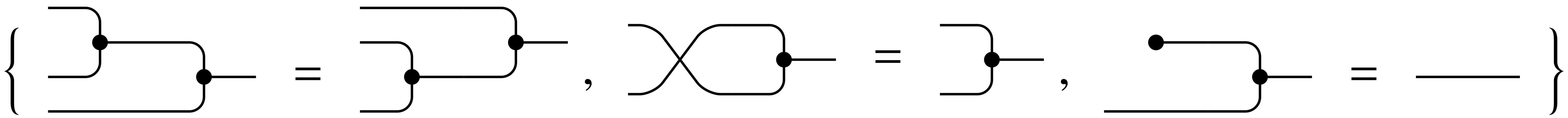
- Consider 
  - then  is drawn 
- to go to string diagrams we need to quotient wrt the laws of symmetric strict monoidal cats. This means that:
  - erasing the dotted lines
  - “only connectivity matters” 

String diagrams over Γ are a convenient description of the free prop X_Γ on Γ

Monoidal theories

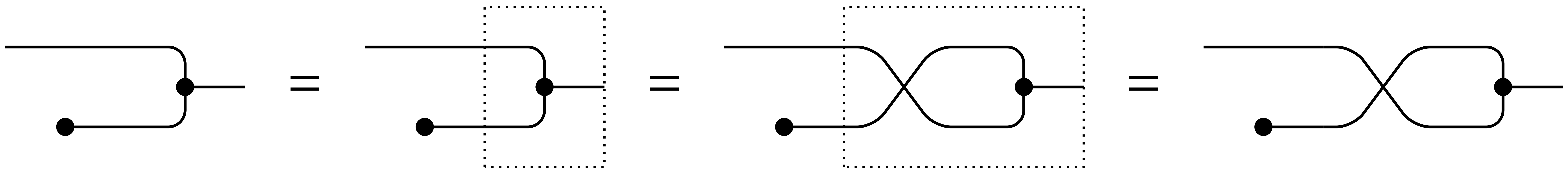
- Now that we know what terms are, we can talk about a monoidal theory
- For a signature Γ, a Γ-equation is a pair (s,t) ∈ $\mathbf{X}_{\Gamma}[m,n]$ for some m,n
- A presentation is a pair (Γ, F) where F is a set of Γ-equations. The induced prop is the monoidal theory.
- e.g. The monoidal theory of commutative monoids is isomorphic to **F**.





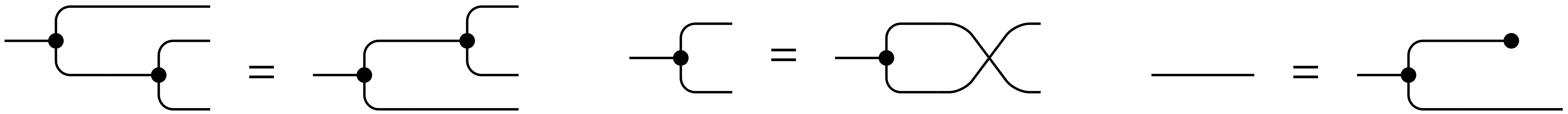
Diagrammatic reasoning

- is the name of equational reasoning with string diagrams, e.g.

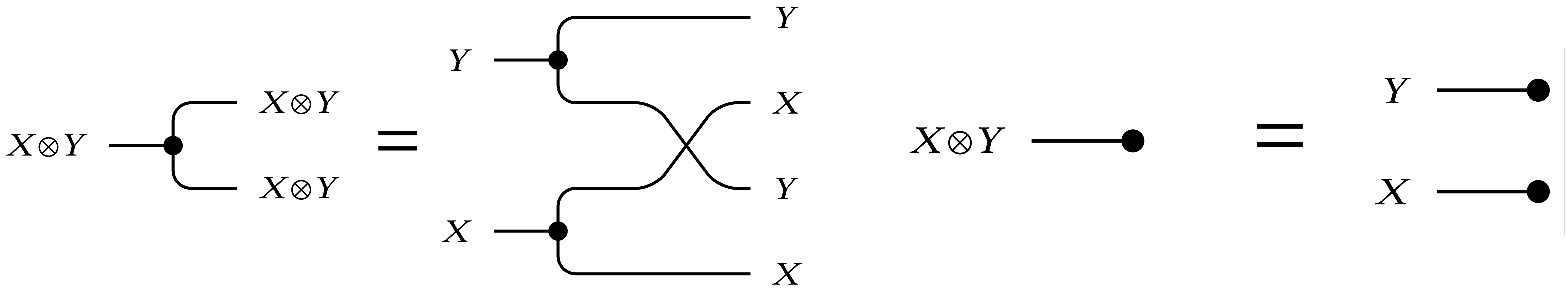


Fox's Theorem

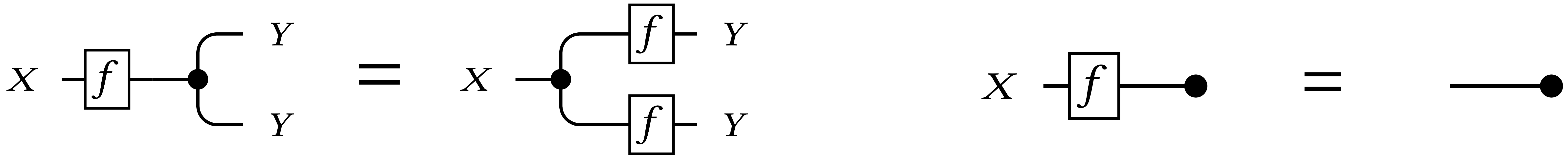
- Theorem (Fox 1976): A symmetric monoidal category is cartesian iff every object can be equipped with a commutative comonoid structure which is **coherent** and **natural**.



coherent:



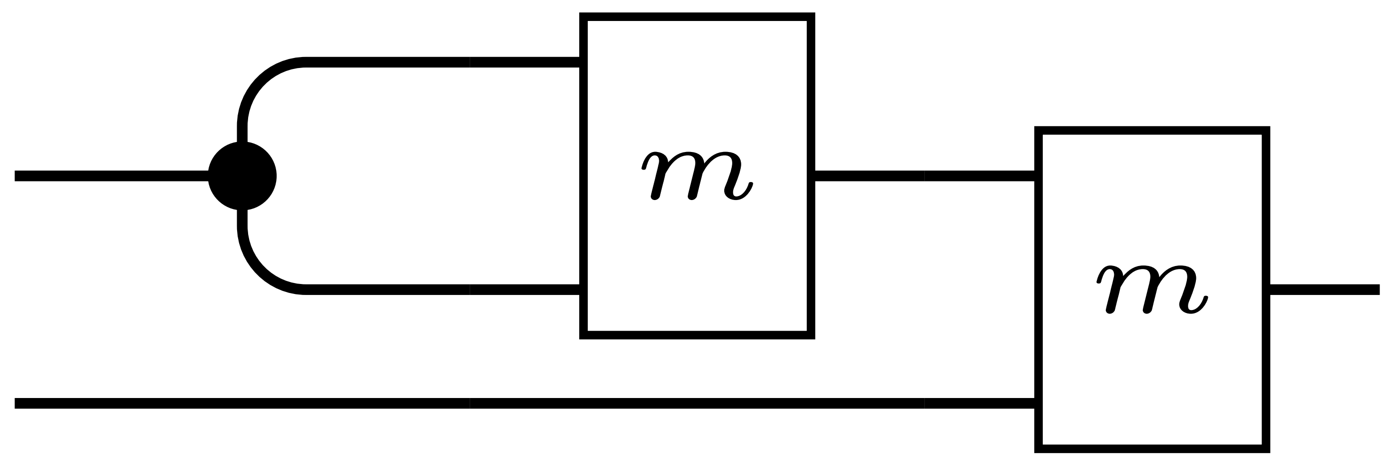
natural:



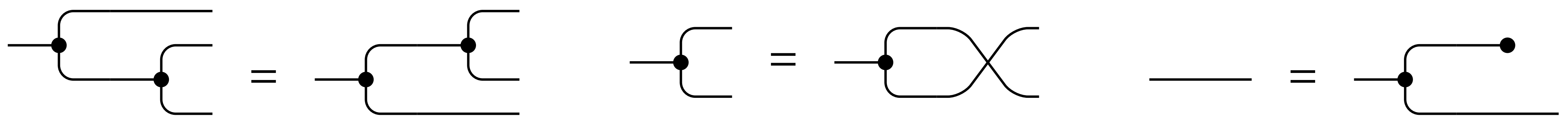
Lawvere with string diagrams

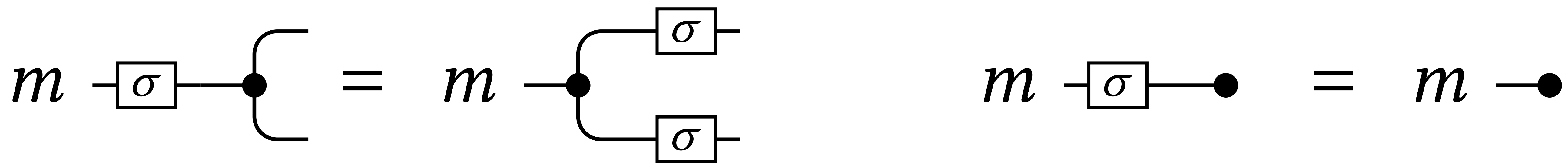
- **A single sorted Lawvere theory is a cartesian prop**
  - i.e. a prop where the monoidal product is the categorical product
- We already have one concrete description of the free cartesian category on a signature - arrows: classical terms, composition: substitution
- We now have a second: string diagrams!

$$m(m(x,x),y) \quad \Leftrightarrow$$



A recipe

- Turn a theory into a monoidal theory in two easy steps
  - Generators: $\Gamma \stackrel{\text{def}}{=} \Sigma +$ 
  - Equations: E (as string diagrams) $+$ 

$+$ 

e.g. as props, the Lawvere theory of commutative monoids is isomorphic to the monoidal theory of commutative bialgebras!

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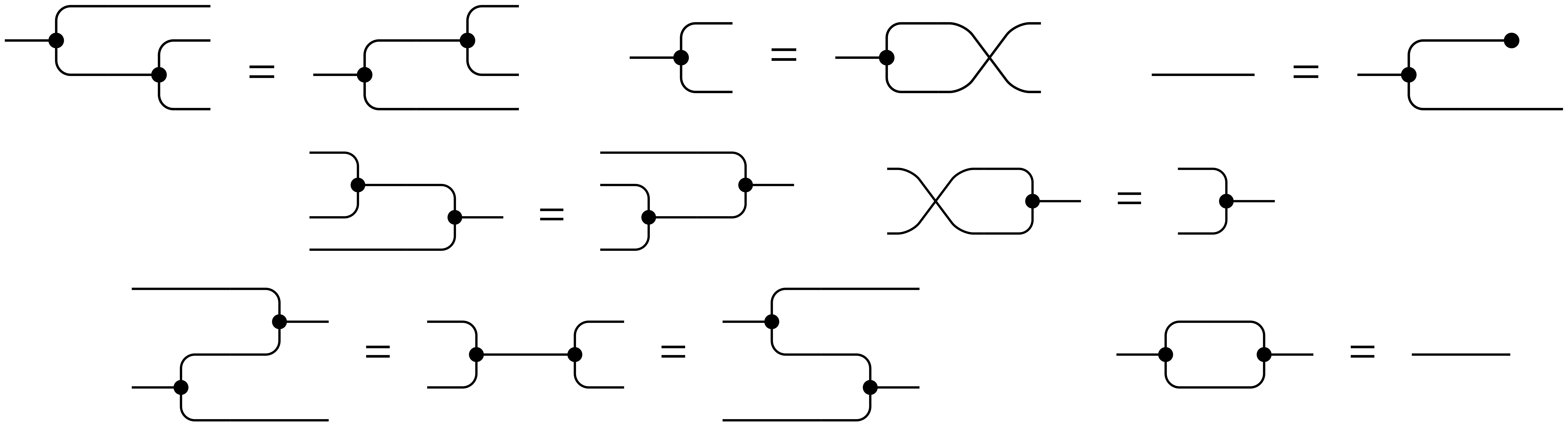
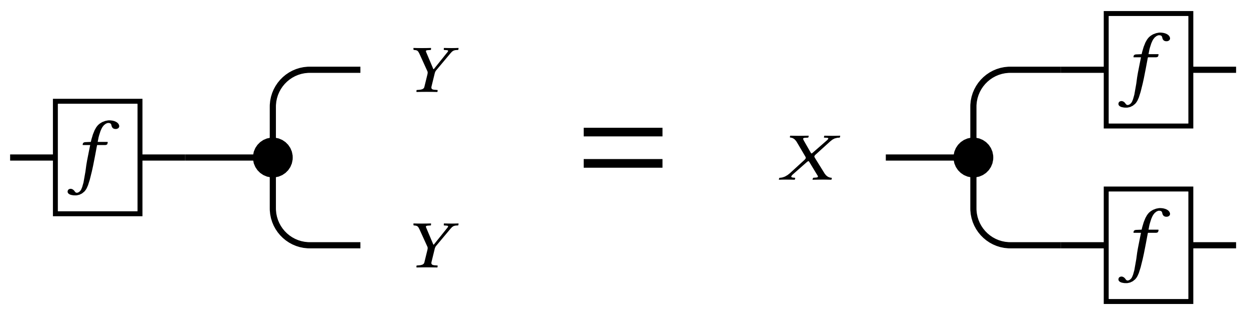
Partial theories and DCR categories

- Partial theories: we want to replace **Set** with **Par** as the universe of models
- Lawvere identified cartesian categories as the categorical structure of interest for algebraic theories
- For partial theories, the corresponding categorical structure is given by **discrete cartesian restriction categories (dcr categories)**
  - **Par** is a DCR category. If **C** has finite limits, Par(**C**) is a DCR category.
- Instead of delving into the details, we can characterise them using a result similar to Fox's theorem

“Fox’s theorem” for DCR categories

- **Theorem.** A DCR category category is a symmetric monoidal category where every object is equipped with a coherent **partial Frobenius algebra** structure, such that the comultiplication is natural.





Some consequences

- The free DCR category on an object is Par($\mathbf{F}^{op}$)

Given a signature Σ, we obtain a syntax for equations!

- Syntax = concrete description of the free DCR category on Σ in terms of string diagrams with partial Frobenius structure

- A presentation is then, as usual, the pair of a signature and equations

- its partial **Lawvere theory is the induced DCR prop**

Models and homomorphisms

- Suppose that **L** is a partial Lawvere theory. A model is a cartesian restriction functor L → **Par**

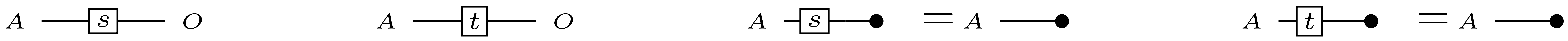
- A homomorphism is a lax natural transformation

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\leq"] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd}$$

Even though Par is the universe, this implies all homomorphism are total functions

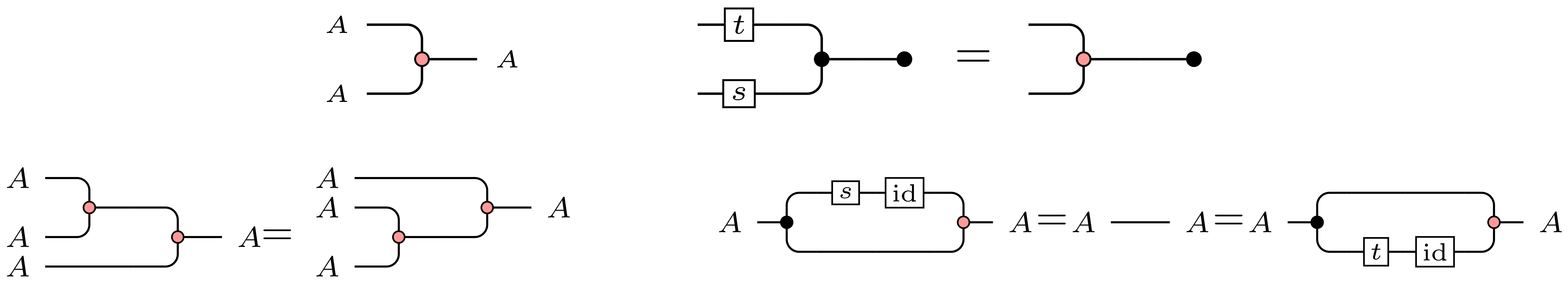
Examples (2 sorted)

directed graphs



reflexive graphs

categories



+ monoidal categories, cartesian restriction categories, DCR categories, cartesian categories, cartesian closed categories, …

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The Variety Theorem

Partial varieties are exactly the locally finitely presentable categories

- **DCRC**$^{\leq}$ - 2 category of DCR categories, restriction functors, and lax transformations
- **Lex** - 2-category of small categories with finite limits, functors, and natural transformations
- **LFP** - the 2-category of locally finitely presentable categories, right adjoints preserving directed colimits, and adjoint homomorphisms

Theorem. LFP^{op} is reflective in **DCRC**$^{\leq}$

$$\begin{tikzcd}
\mathsf{DCRC}^{\leq} \arrow[r, bend left, "\mathsf{Split}_{\mathcal{R}}"] \arrow[r, phantom, "\perp"] & \mathsf{DCRC}_s^{\leq} \arrow[l, bend left] \arrow[r, bend left, "\mathsf{Total}"] \arrow[r, phantom, "\simeq"] & \mathsf{Lex} \arrow[l, bend left, "\mathsf{Par}"] \arrow[r, bend left, "\mathsf{Lex}(-,\mathsf{Set})"] \arrow[r, phantom, "\simeq"] & \mathsf{LFP}^{\mathrm{op}} \arrow[l, bend left, "\mathsf{LFP}(-,\mathsf{Set})"]
\end{tikzcd}$$

A string diagrammatic calculus for finite limits

- We can give a string diagrammatic treatment of the category with free finite limits on a signature
- We want to capture the total maps after splitting idempotents
- Objects: carved out by some diagram of c
- Arrows:

Relational theories

- What if we want to take models in **Rel**, instead of **Par** or **Set**?
- Then the right categorical notion is a **cartesian bicategory** (of relations) of Carboni and Walters.
  - Aurelio Carboni and RFC Walters, “Cartesian Bicategories I”, JPAA 49:11–32, 1987
  - Filippo Bonchi, Dusko Pavlovic, P.S. “Functorial Semantics for Relational Theories” https://arxiv.org/abs/1711.08699
- Chad Nester recently proved a variety theoem: the class of varieties is the class of **definable** categories
  - Chad Nester. A Variety Theorem for Relational Universal Algebra, RAMICS 2021 (to appear).

Takeaway

- Smooth extension of Lawvere’s functorial semantics methodology to partial theories
- The story continues to relational theories
- In each case (including the classical) string diagrams give us a calculus. Ordinary terms only give a satisfactory calculus in the classical case.