

ZEN AND THE ART OF ∞ -CATEGORIES

OR

HOW I LEARNT TO STOP WORRYING
AND LOVE ∞ -CATEGORY THEORY*

DOMINIC VERITY

CENTRE OF AUSTRALIAN CATEGORY THEORY (CoACT)
MACQUARIE UNIVERSITY

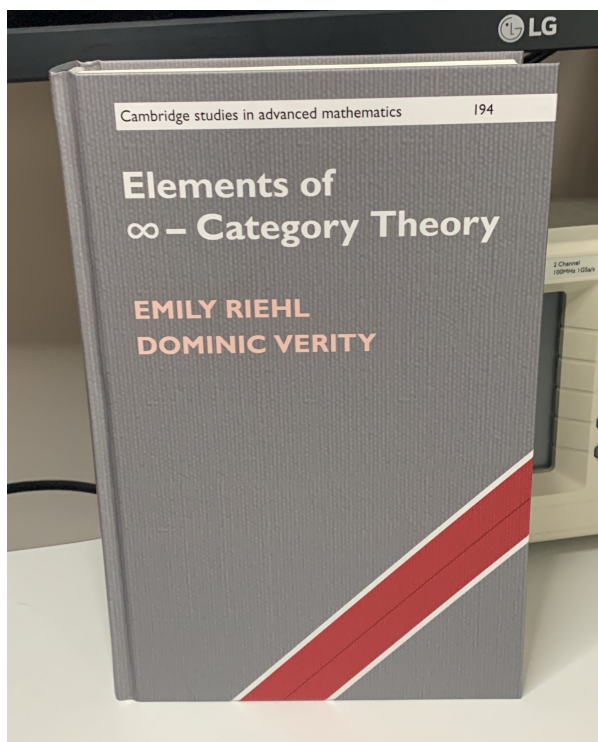
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MATHEMATICAL SCIENCES INSTITUTE (MSI)
AUSTRALIAN NATIONAL UNIVERSITY

<https://dom-verity.github.io>

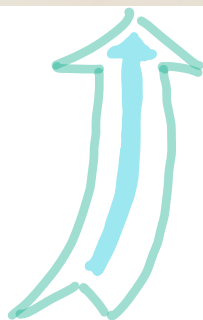
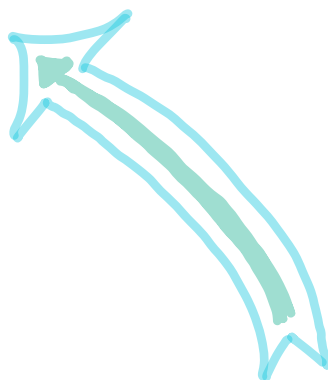
I acknowledge the traditional owners of the land on which this talk takes place, the Wattamattagal clan of the Darug nation, whose cultures and customs have nurtured this land since the Dreamtime.

I pay my respects to Elders past, present and emerging.

THE OBLIGATORY PLUG



EVERYTHING YOU
WANTED TO KNOW
ABOUT MODEL
AGNOSTIC ∞ -CATEGORY
THEORY, BUT WERE
TOO AFRAID TO ASK!!



EMILY RIEHL (JHU)

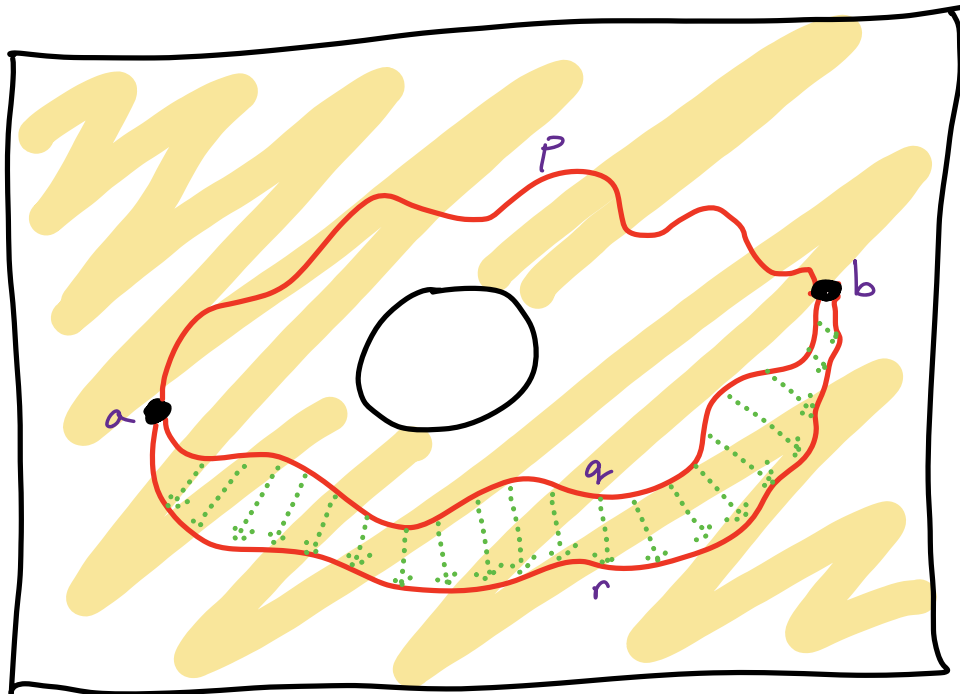


DOM VERITY (MQ)

PARTNERS IN
 ∞ -CRIME

BUILDING UP TO ∞ -CATS

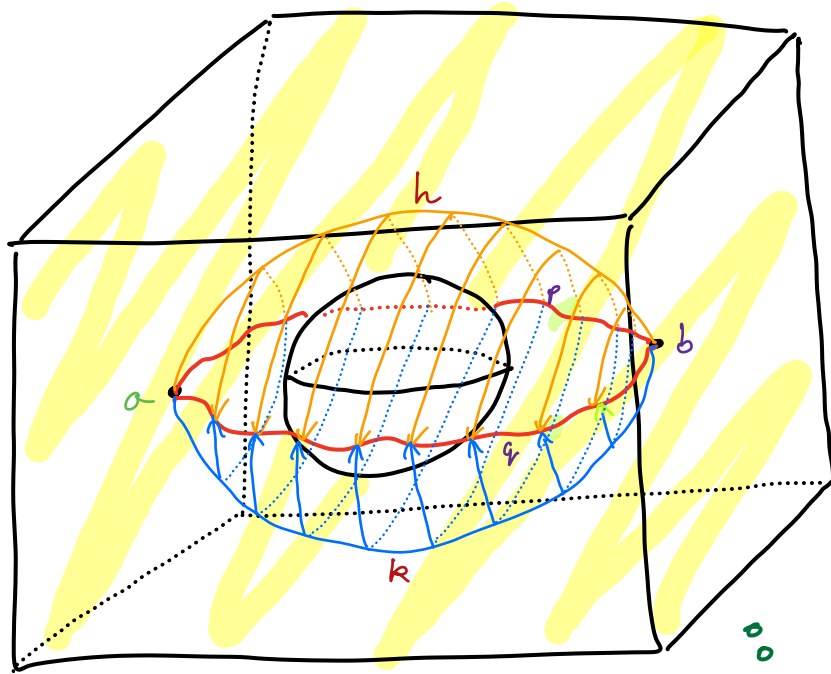
THE FUNDAMENTAL ∞ -GROUPOID



A
FAMILIAR
STORY

FUNDAMENTAL GROUPOID $\Pi_1(X)$

-) OBJECTS POINTS IN THE SPACE
-) ARROWS BASED HOMOTOPY CLASSES OF CONTINUOUS PATHS.
-) COMPOSITION CONCATENATION OF PATHS.
-) INVERSES GIVEN BY TRAVELING PATHS IN THE REVERSE DIRECTION



HERE THE PATHS p AND q ARE
HOMOTOPIC VIA TWO DISTINCT HOMOTOPIES
 h AND k WHICH CIRCUMNAVIGATE
THE CENTRAL HOLE.

FUNDAMENTAL ∞ -GROUPOID $\Pi(X)$

-) OBJECTS POINTS IN THE SPACE
-) ARROWS CONTINUOUS PATHS BETW POINTS
-) HOMOTOPIES BETWEEN PATHS
-) 2-HOMOTOPIES BETWEEN HOMOTOPIES
-) 3-HOMOTOPIES BETWEEN 2-HOMOTOPIES

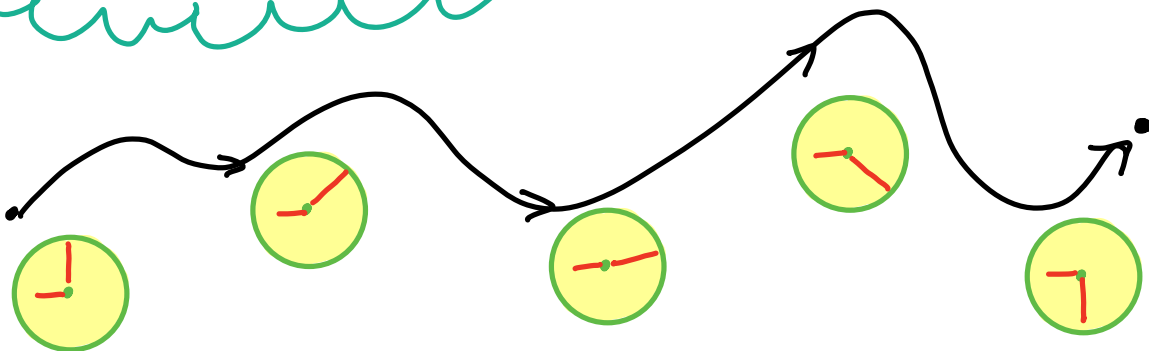
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DIRECTED SPACES & THE FUNDAMENTAL $(\infty, 1)$ -CATEGORY

IN PHYSICS AND COMPUTER SCIENCE IT IS COMMON TO MODEL SYSTEMS USING STATE SPACES. PATHS IN SUCH SPACES REPRESENT HOW THE SYSTEM EVOLVES THROUGH TIME.

BUT MANY SYSTEMS ARE IRREVERSIBLE

PATHS SHOULD POINT FORWARDS IN TIME



IT DOESN'T MAKE SENSE TO TRAVERSE THESE "TIME-LIKE" PATHS BACKWARDS SO THEY MAY NOT HAVE INVERSES.

SUCH CONSIDERATIONS LEAD US TO
DIRECTED SPACES . °

THESE INCORPORATE SOME KIND
OF ORDER, REPRESENTING NOTIONS
SUCH AS CAUSALITY OR
RESOURCE USE

ON APPLYING THE "PATHS CONSTRUCTION"
TO A DIRECTED SPACE, RESTRICTING
OURSELVES TO CONSIDER DIRECTED PATHS ONLY,
WE OBTAIN AN ∞ -CATEGORY.

THE FUNDAMENTAL
 ∞ -CATEGORY $\Pi(X)$

CLAIM WE MAY FRUITFULLY STUDY $\Pi(X)$
USING GENERALISATIONS OF STANDARD
CATEGORICAL TOOLS....

BUT HOW?

WHEN WE START LOOKING...

... EXAMPLES OF ∞ -CATEGORIES ARE ABUNDANT IN "NATURE".

-) DERIVED ALGEBRAIC GEOMETRY
 -) FORMAL FOUNDATIONS AND TYPE THEORY
 -) ALGEBRAIC TOPOLOGY
 -) QUANTUM FIELD THEORY
+ QUANTUM COMPUTATION
 -) CATEGORY THEORY ITSELF
- • • AND MUCH MORE.

SLOGAN

∞ -CATEGORIES \equiv
○
○
○

CATEGORIES
+
EQUALITY AS HOMOTOPY
COHERENT STRUCTURE

THEY MAY BE
UBIQUITOUS BUT YOU
HAVEN'T ACTUALLY DEFINED WHAT
THEY ARE! *

* AND I WON'T!!

AN INTUITIVE MODEL

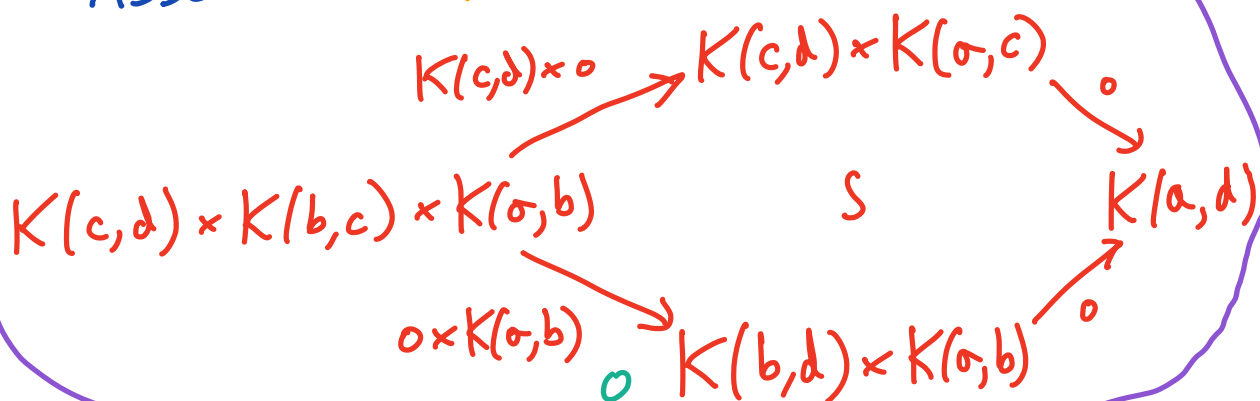
$(\infty, 1)$ -CATEGORY \approx CATEGORY WEAKLY ENRICHED IN SPACES
 A

SET OF OBJECTS
 $ob(A)$

HOM SPACES
 $A(\sigma, b)$

CONTINUOUS COMPOSITION OPERATIONS
 $A(b, c) \times A(\sigma, b) \xrightarrow{\circ} A(\sigma, c)$

ASSOCIATIVITY HOMOTOPIES



\equiv CONTINUOUS FAMILY OF PATHS
 $h \circ (g \circ f) \sim (h \circ g) \circ f$
IN $K(a, d)$

MACLANE PENTAGON "HIGHER" HOMOTOPY

$$\begin{array}{ccc} & (k \circ h) \circ (g \circ f) & \\ \sim & & \sim \\ k \circ (h \circ (g \circ f)) & \Rightarrow & ((k \circ h) \circ g) \circ f \\ \wr & & \int \\ k \circ ((h \circ g) \circ f) & \sim & (k \circ (h \circ g)) \circ f \end{array}$$

A HOMOTOPY
OF HOMOTOPIES

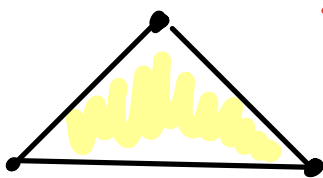
+ EVEN HIGHER COHERENCE
HOMOTOPIES AT ALL LEVELS
ABOVE THIS... UGGHHH!

FORMALISING

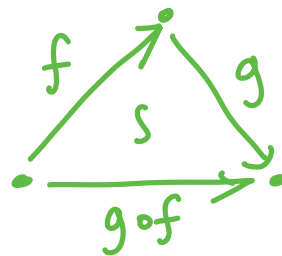
... WE CAN MAKE PROGRESS ON FORMALISING THIS IDEA WITH TWO KEY OBSERVATIONS:

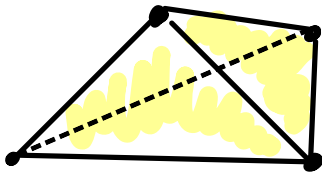
1) GIVEN COMPOSABLE ARROWS $g, f \in A$ IT MAY NOT MAKE GOOD SENSE TO SPEAK OF THE COMPOSITE $g \circ f$ BECAUSE ANY ARROW HOMOTOPIC TO IT HAS EQUAL RIGHT TO THAT CROWN.

2) HIGHER ASSOCIATIVITY LAWS ARISE FROM THE ALGEBRAIC STRUCTURE OF THE HIGHER SIMPLICES.

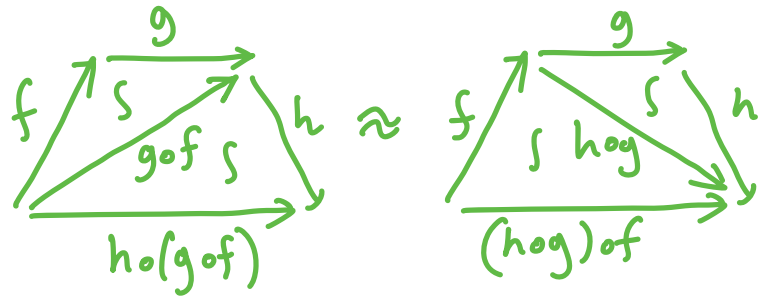


2-SIMPLEX





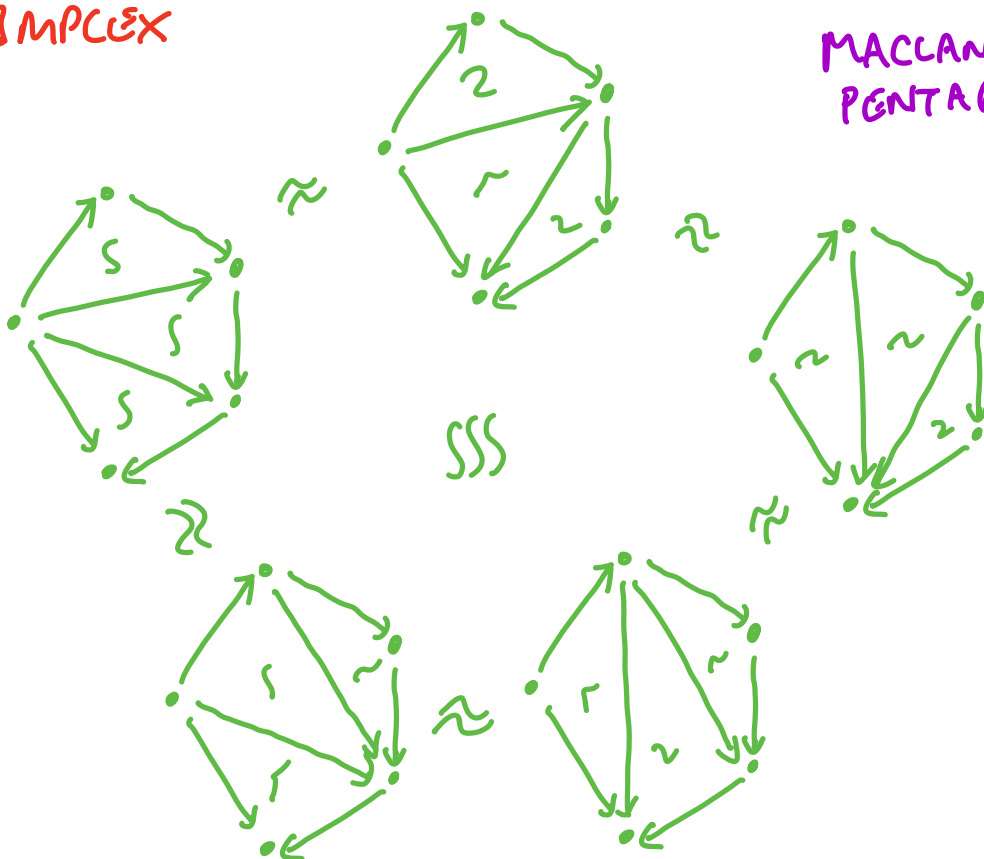
3-SIMPLEX



FOR THIS TO BE A
WELL DEFINED 3-SIMPLEX
THESE MUST BE OPPOSITE
TO BE EQUAL

4-SIMPLEX

MACLANE
PENTAGON



VARIOUS METHODS HAVE BEEN
EXPLORED TO DIRECTLY FORMALISE
OUR INTUITIVE MODEL USING
THESE IDEAS.

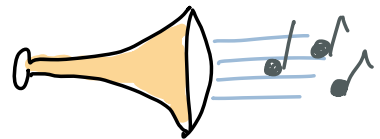
THESE HAVE GIVEN RISE TO
A NUMBER OF FORMAL MODELS
OF THE ∞ -CATEGORY
ZEITGEIST, FOR EXAMPLE:

-) (COMPLETE) SEGAL SPACES
-) SEGAL CATEGORIES
-) TAMSAMANI CATEGORIES

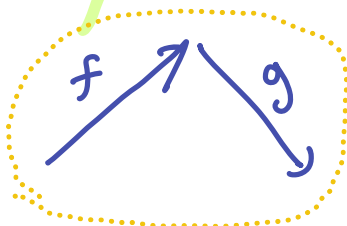
WE'LL LOOK, VERY BRIEFLY,
AT A SLIGHTLY DIFFERENT KIND
OF MODEL:

QUASI-CATEGORIES

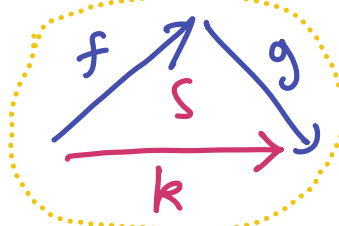
A QUASI-CATEGORY A IS A
SIMPLICIAL SET WHICH ADMITS
FILLERS FOR INNER HORNS.



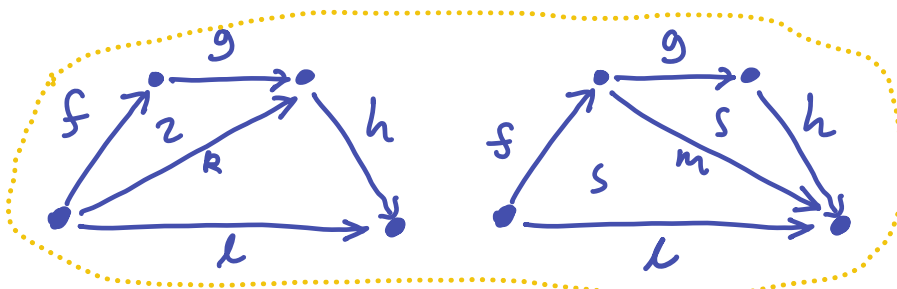
A COMBINATORIAL
SIMPLICIAL SPACE



FILL

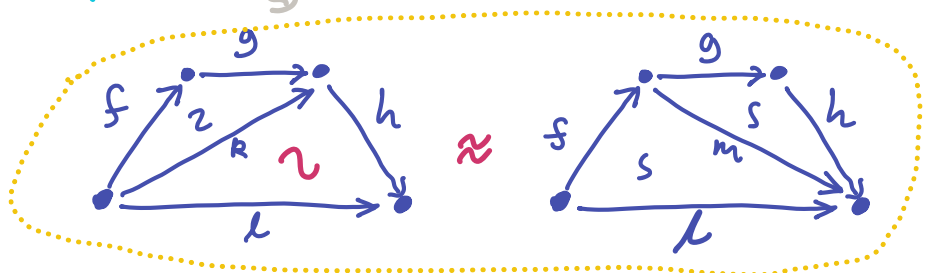


$$\Lambda'[2] \subseteq \Delta[2]$$



$$\Lambda'[3] \subseteq \Delta[3]$$

FILL



OK...

... SO NOW WE HAVE SOME (ARGUABLY TOO MANY) FORMAL STRUCTURES THAT CLAIM TO MODEL OUR INTUITION ABOUT ∞ -CATEGORIES ...

... BUT YOU PROMISED WE'D DO SOME CATEGORY THEORY WITH THEM ...

... HOW DOES THAT WORK?



... AND WHAT MODEL SHOULD WE USE?

∞ - COSMOI

LET'S NOT PICK A FAVOURITE MODEL

DOM'S AN AUSTRALIAN
CATEGORY THEORIST, SO
WE'LL HUMOUR HIM AND
DO THINGS 2 -CATEGORICALLY

DOESN'T THAT MEAN THAT
WE'LL HAVE TO INVENT
 $(\infty, 2)$ -CATEGORIES? SOUNDS
HARD!!

NOT NECESSARILY, WE CAN USE
A SEMI-STRUCT $(\infty, 2)$ -MODEL LIKE
 $QUASI$ -CATEGORY ENRICHED
CATEGORIES

GOOD IDEA, I CAN
MANAGE THAT 😊

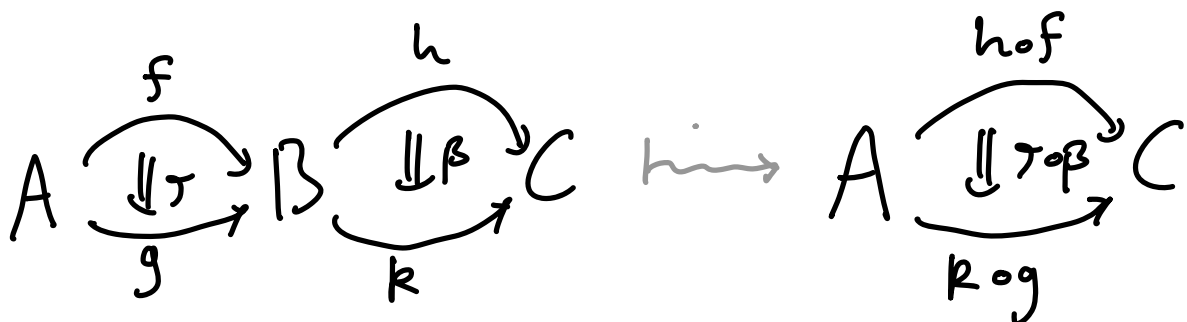
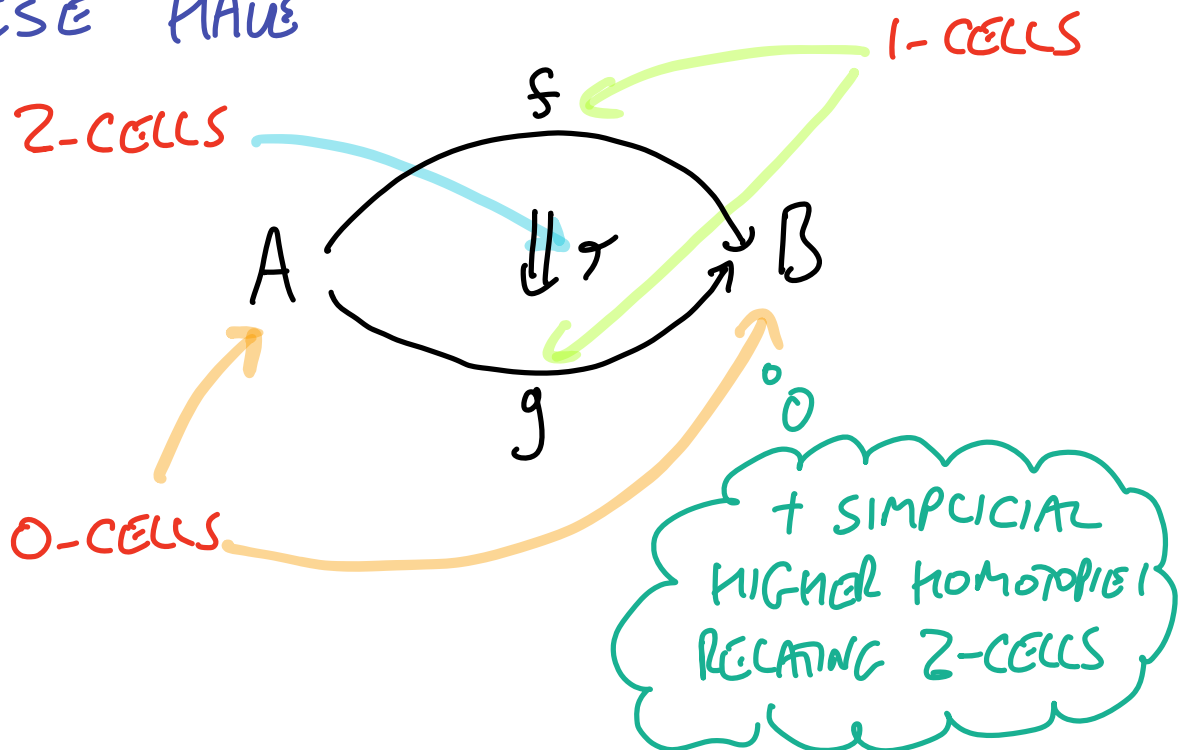
COOL, DRINK YOUR
COLD COFFEE AND
WE'LL SET TO
WORK



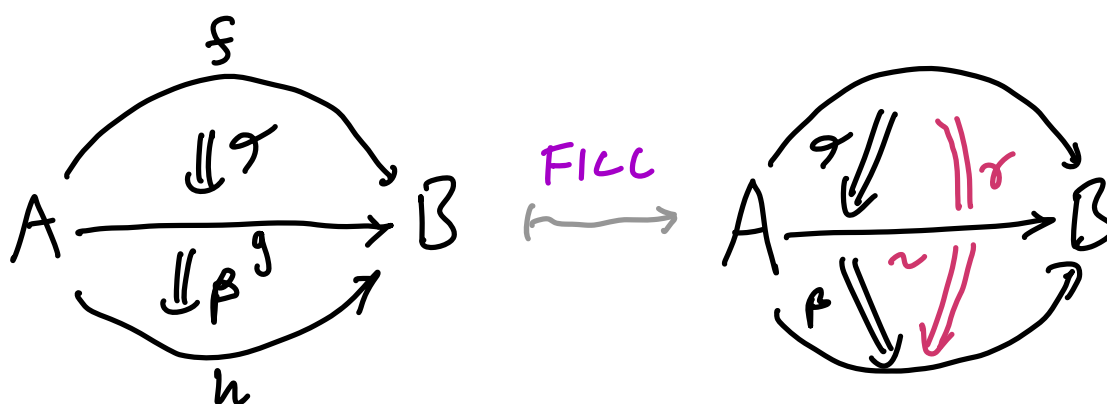
TECHNICALLY SPEAKING

∞ -COSMOS \equiv CATEGORY OF FIBRANT
 \mathcal{K} OBJECT ENRICHED IN THE
JOYAL MODEL STRUCTURE
ON SIMPLICIAL SETS

THESE HAVE



HORIZONTAL COMPOSITION IS A STRICTLY
ASSOCIATIVE OPERATION



VERTICAL COMPOSITION GIVEN BY FILLING
INNER HOLMS.

THIS IS STILL A LITTLE
MORE COMPLICATED THAN
WE MIGHT HOPE FOR

FOR A SURPRISINGLY LARGE
RANGE OF PURPOSES IT
SUFFICES TO WORK IN A
QUOTIENT OF \mathcal{K} CALLED
ITS HOMOTOPY 2-CATEGORY

HOMOTOPY 2-CATEGORIES

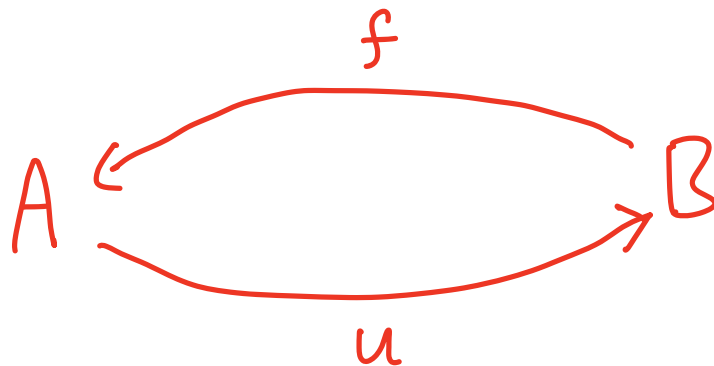
THE HOMOTOPY 2-CATEGORY h_*K OF AN ∞ -COSMOS K HAS

-) 0-CELLS AND 1-CELLS THOSE OF K
-) 2-CELLS HOMOTOPY CLASSES OF PARALLEL 2-CELLS IN K
-) HORIZONTAL COMPOSITION AS IN K
-) VERTICAL COMPOSITION USE FILLED TO COMPOSE REPRESENTATIVES OF HOMOTOPY CLASSES.

THIS IS A GOOD, OLD FASHIONED, STRICT 2-CATEGORY

QUESTION WHAT DO WE GET IF WE STUDY STANDARD 2-CATEGORICAL STRUCTURES IN THE HOMOTOPY 2-CATEGORY h_*K ?

CASE STUDY: ADJUNCTIONS



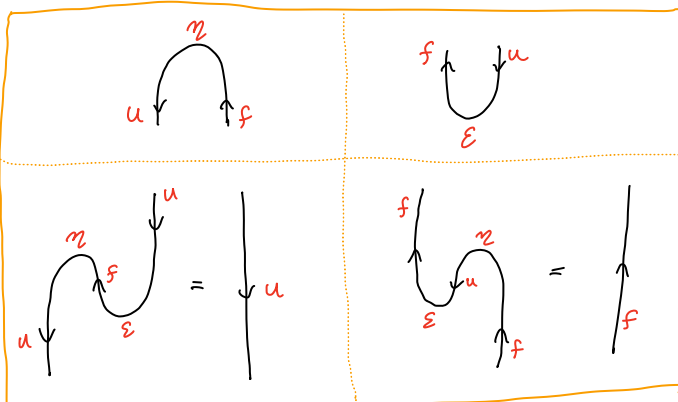
$$[\eta]: id_B \Rightarrow uf$$

$$[\varepsilon]: fu \Rightarrow id_A$$

$$+ [\varepsilon]f \cdot f[\eta] = id_f$$

$$u[\varepsilon] \cdot [\eta]u = id_u$$

TRIANGLE IDENTITIES.

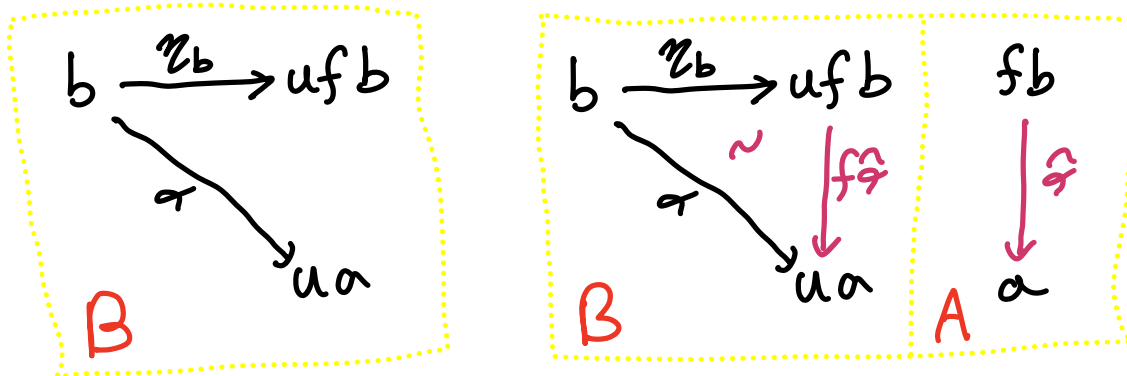


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IT TURNS OUT THAT WHEN INTERPRETED
IN VARIOUS ∞ -MODELS THIS ADJUNCTION
NOTION COINCIDES WITH ALL OTHER
SUCH NOTIONS IN THE LITERATURE

IN QUASI-CATEGORIES, FOR EXAMPLE,
SUCH ADJUNCTIONS ARE CHARACTERISED
BY A HIGHER UNIVERSAL PROPERTY

EXPRESSED AS A KIND OF
"OUTER" HORN FILLING PROPERTY



+ HIGHER VERSIONS OF THIS PROPERTY
AT ALL HIGHER DIMENSIONS.

THIS IS A LITTLE SURPRISING,
WE GET HIGHER UNIVERSAL PROPERTIES
FROM A 2-DIMENSIONAL NOTION
INTERPRETED IN h_*K

AS AN ADDED BENEFIT,
WE FIND THAT ANY 2-CATEGORICAL
PROPERTY OF ADJUNCTIONS HOLDS
FOR ADJUNCTIONS IN **ANY** ∞ -COSMOS

THE 2-CATEGORICAL PROOFS OF
THESE ARE OFTEN MUCH SIMPLER
THAN THE ∞ -CATEGORICAL ONES
TO BE FOUND IN THE LITERATURE
ON SPECIFIC MODELS

HOW MUCH...

... CAN WE CAPTURE IN THIS WAY



THE MONAD STORY CANNOT ENTIRELY BE UNDERTAKEN IN h_*K , IT NEEDS A HOMOTOPY COHERENCE RESULT FOR ADJUNCTIONS.

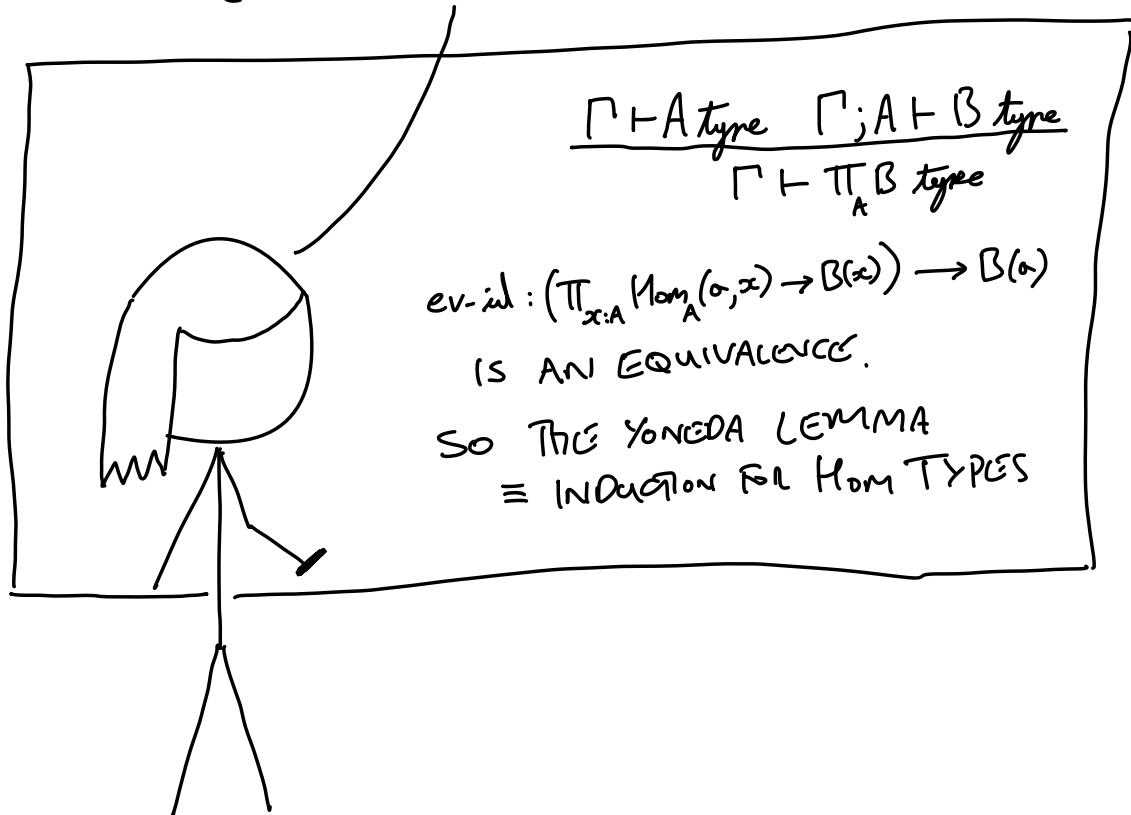
APPENDIX

∞ -CATEGORIES IN HOMOTOPY TYPE THEORY (HoTT)

HoTT IS DESIGNED SO
THAT **EQUALITIES** NATIVELY
BEHAVE LIKE **HOMOTOPIES**

THEY ARE **STRUCTURES**, NOT
RELATIONS, AND THEY SUPPORT
HIGHER STRUCTURES

TERMS IN THESE STRUCTURES
COLLECT INTO AN ∞ -GROUPOID



IN RIEHL AND SHULMAN'S DIRECTED
TYPE THEORY ALL TYPES A SUPPORT
A FAMILY OF HOM TYPES

$\text{Hom}_A(x, y)$ over $x, y : A$

THAT TYPE IS SAID TO BE AN
 ∞ -CATEGORY IF

- EVERY PAIR OF ARROWS $f : \text{Hom}_A(x, y)$
AND $g : \text{Hom}_A(y, z)$ HAS A
UNIQUE COMPOSITE DEFINING A
TERM $g \circ f : \text{Hom}_A(x, z)$
- PATHS IN A ARE EQUIVALENT TO
ISOMORPHISMS

BY PATHS WE MEAN TERMS
IN AN IDENTITY TYPE
 $\text{Id}_A(x, y)$ FOR $x, y : A$

THINK OF THIS AS
THE SPACE OF PATHS
(HOMOTOPIES) FROM x
TO y .

THE UNIQUENESS OF THIS
COMPOSITE IS A LITTLE SURPRISING!!

BUT, HOMOTOPY THEORY IS BUILT
INTO THE VERY NOTION OF EQUALITY IN
THE UNDERLYING TYPE THEORY.

SO UNIQUENESS DECODES
TO GIVE THE ENTIRELY
PALATABLE STATEMENT THAT:

"THE ∞ -GROUPOID (SPACE)
OF COMPOSITES IS CONTRACTIBLE"

