Università di Genova | DIMA DIPARTIMENTO DI MATEMATICA

When an elementary quotient completion is a quasitopos

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Joint work with Maria Emilia Maietti and Fabio Pasquali

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TOPOS INSTITUTE

Overview: logic in category theory

Hyperdoctrines

Cloven faithful fibrations with fibred finite products and products in the base

Prop-categories

Doctrines

F.W. Lawvere

Adjointness in foundations *Dialectica* 1969, available also in *Repr. Theory Appl.Categ.*

Equality in hyperdoctrines and the comprehension schema as an adjoint functor *Proc.AMS Symp.Pure Math.* 1970

B. Jacobs

Categorical Logic and Type Theory North-Holland 1999

A.M. Pitts

Categorical logic *Handbook of Logic in Computer Science, vol. 5, 2000*

M.E. Maietti, G. R.

Quotient completion for the foundation of constructive mathematics *Log.Univer.* 2013

Doctrines

Definition

$\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$

- $\mathcal{B}$ has finite products

$\mathcal{E} \xrightarrow{p} \mathcal{B}$ faithful fibration

- $\mathcal{B}$ has finite products
- if $p(d \xrightarrow[s]{\sim} e) = \mathrm{id}_{p(e)}$ then $s = \mathrm{id}_e$

Doctrines

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$\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$

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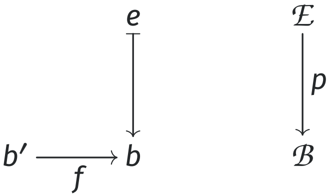

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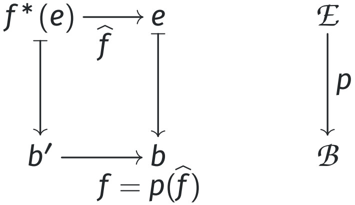

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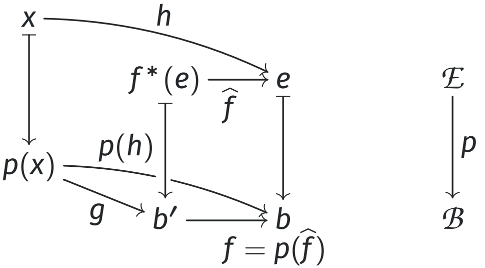

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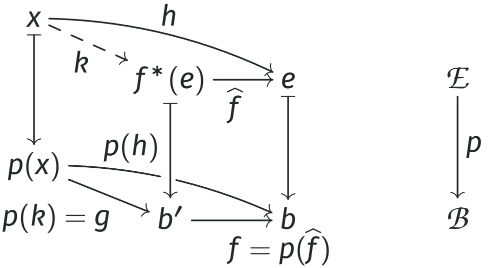

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$$\mathcal{B}^{\mathrm{op}} \xrightarrow{p^{-1}} \mathcal{P}os \longleftarrow\!\!\!\!\mapsto \mathcal{E} \xrightarrow{p} \mathcal{B}$$

the category $p^{-1}(b)$:

e s.t. $p(e) = b$

$\downarrow v$ — $v \colon e \to d$ in $\mathcal{E}$ s.t. $p(v) = \mathrm{id}_b$

e' s.t. $p(d) = b$

Doctrines

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$$\mathcal{B}^{\mathrm{op}} \xrightarrow{p^{-1}} \mathcal{P}os \longleftarrow\!\!\!\!\longmapsto \mathcal{E} \xrightarrow{p} \mathcal{B}$$

the functor $p^{-1}(b) \xrightarrow{p^{-1}(f\colon b' \to b)} p^{-1}(b')$

$$e \longmapsto f^*(e)$$

$$\begin{array}{ccc} f^*(e) & \xrightarrow{\widehat{f}} & e \\ \downarrow & & \downarrow \\ b' & \xrightarrow{f = p(\widehat{f})} & b \end{array}$$

Doctrines

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$$\mathcal{B}^{\mathrm{op}} \xrightarrow{p^{-1}} \mathcal{P}os \longleftarrow\!\!| \; \mathcal{E} \xrightarrow{p} \mathcal{B}$$

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os \longmapsto \textstyle\int P \xrightarrow{\pi_1} \mathcal{B}$$

the category $\int P$: $\underset{\text{s.t. } \varphi \in P(b)}{(b, \varphi)} \xrightarrow[\substack{f : b \to b' \text{ in } \mathcal{B} \\ \text{s.t. } \varphi \leq P(f)(\varphi')}]{f} \underset{\text{s.t. } \varphi' \in P(b')}{(b', \varphi')}$

The powerset doctrine

Definition

$\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$

- $\mathcal{B}$ has finite products

$\mathcal{E} \xrightarrow{p} \mathcal{B}$ faithful fibration

- $\mathcal{B}$ has finite products
- if $p(d \xrightarrow[s]{\sim} e) = \mathrm{id}_{p(e)}$ then $s = \mathrm{id}_e$

Example

$\mathcal{S}et^{\mathrm{op}} \xrightarrow{\wp} \mathcal{P}os$

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \cap & & \cap \\ S & \xrightarrow{f} & S' \end{array} \;\xmapsto{\text{cod}}\; S \xrightarrow{f} S'$$

The doctrine of subobjects

Definition

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$$

- $\mathcal{B}$ has finite products

$$\mathcal{E} \xrightarrow{p} \mathcal{B} \quad \text{faithful fibration}$$

- $\mathcal{B}$ has finite products
- if $p(d \xrightarrow[s]{\sim} e) = \mathrm{id}_{p(e)}$ then $s = \mathrm{id}_e$

Example

$\mathcal{A}$ with pullbacks (and finite products)

$$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathcal{P}os$$

$$\begin{array}{ccc} \bullet & \xrightarrow{[g]} & \bullet' \\ {\scriptstyle [m]}\big\downarrow & & \big\downarrow{\scriptstyle [m']} \\ a & \xrightarrow{f} & a' \end{array} \quad \xmapsto{\mathrm{cod}} \quad a \xrightarrow{f} a'$$

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$\mathcal{A}$ with pullbacks (and finite products)

$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathcal{P}os$

$$\begin{array}{ccc} x & \xrightarrow{g} & x' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ a & \xrightarrow{f} & a' \end{array} \quad \xmapsto{\mathrm{cod}} \quad a \xrightarrow{f} a'$$

The doctrine of variations

Definition

$\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$

- $\mathcal{B}$ has finite products

$\mathcal{E} \xrightarrow{p} \mathcal{B}$ faithful fibration

- $\mathcal{B}$ has finite products
- if $p(d \underset{s}{\xrightarrow{\sim}} e) = \mathrm{id}_{p(e)}$ then $s = \mathrm{id}_e$

Example

$\mathcal{A}$ with weak pullbacks (and finite products)

$$\begin{array}{ccc} \mathcal{A}^{\mathrm{op}} & \xrightarrow{\quad \mathrm{Vrn} \quad} & \mathcal{P}os \\ a & \longmapsto & (\mathcal{A}/a)_{\mathrm{po}} \end{array}$$

$$\begin{array}{ccc} \bullet & \xrightarrow{\exists g} & \bullet' \\ {\scriptstyle [h]}\downarrow & & \downarrow{\scriptstyle [h']} \\ a & \xrightarrow{f} & a' \end{array} \quad \overset{\mathrm{cod}}{\longmapsto} \quad a \xrightarrow{f} a'$$

M. Grandis
Weak subobjects and the epi-monic completion of a category *J.Pure Appl.Algebra* 2000

The doctrine of variations

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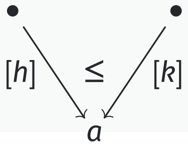
Example

$\mathcal{A}$ with weak pullbacks (and finite products)

$$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Vrn}} \mathcal{P}os$$
$$a \longmapsto (\mathcal{A}/a)_{\mathrm{po}}$$

the poset $(\mathcal{A}/a)_{\mathrm{po}}$:



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the poset $(\mathcal{A}/a)_{\mathrm{po}}$: $x \xrightarrow{\exists t} y$, with $h\colon x \to a$, $k\colon y \to a$

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Weak subobjects and the epi-monic completion of a category *J.Pure Appl.Algebra* 2000

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Example

$\mathcal{A}$ with weak pullbacks (and finite products)

$$\begin{array}{ccc} \mathcal{A}^{\mathrm{op}} & \xrightarrow{\mathrm{Vrn}} & \mathcal{P}os \\ a & \longmapsto & (\mathcal{A}/a)_{\mathrm{po}} \\ \uparrow f & & \uparrow f^* \\ a' & \longmapsto & (\mathcal{A}/a')_{\mathrm{po}} \end{array}$$

the reindexing f^*:

$$\begin{array}{ccc} f^*x & \longrightarrow & x \\ h' \downarrow & \text{w.pb.} & \downarrow h \\ a & \xrightarrow{f} & a' \end{array}$$

M. Grandis
Weak subobjects and the epi-monic completion of a category *J.Pure Appl.Algebra* 2000

Triposes are doctrines

Definition

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Example

H a complete Heyting algebra

$$\mathcal{S}et^{\mathrm{op}} \xrightarrow{H^{(-)}} \mathcal{P}os$$

$$\begin{array}{ccc} S & \longmapsto & H^S \\ \uparrow f & & \uparrow - \circ f \\ S' & \longmapsto & H^{S'} \end{array}$$

A.M. Pitts
Tripos theory in retrospect *Math.Structures Comput.Sci.* 2002

Triposes are doctrines

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Example

A a relative ordered partial combinatory algebra

$\mathcal{S}et^{\mathrm{op}} \xrightarrow{P_A} \mathcal{P}os$

P.J.W. Hofstra
All realizability is relative *Math.Proc.Cambridge Philos.Soc.* 2006
J. van Oosten
Realizability: An Introduction to its Categorical Side Elsevier 2008
J. Zoethout
Computability Models and Realizability Toposes *PhD thesis*, Utrecht 2022

The doctrine of a logical theory

Definition

$\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$

- $\mathcal{B}$ has finite products

$\mathcal{E} \xrightarrow{p} \mathcal{B}$ faithful fibration

- $\mathcal{B}$ has finite products
- if $p(d \xrightarrow[s]{\sim} e) = \mathrm{id}_{p(e)}$ then $s = \mathrm{id}_e$

Example

T a many-sorted theory in (a fragment of) first order logic

$\mathcal{C}tx_T^{\mathrm{op}} \xrightarrow{\mathrm{Wff}^T} \mathcal{P}os$

$\mathcal{C}tx_T$: $(x_1 : S_1, \ldots, x_n : S_n) \xrightarrow{(t_1, \ldots, t_m)} (y_1 : S'_1, \ldots, x_m : S'_m)$

$\mathrm{Wff}^T(x_1 : S_1, \ldots, x_n : S_n)$: $x_1 : S_1, \ldots, x_n : S_n \mid \varphi \vdash_T \psi$

Primary doctrines

Definition

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$$

- $\mathcal{B}$ has finite products
- each *fibre* $P(b)$ has finite products
- each *reindexing* $P(f)\colon P(b') \longrightarrow P(b)$ preserves them

Examples

$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathcal{P}os$ for $\mathcal{A}$ with finite limits

$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Vrn}} \mathcal{P}os$ for $\mathcal{A}$ with weak pullbacks (and finite products)

$\mathcal{C}tx_T^{\mathrm{op}} \xrightarrow{\mathrm{Wff}^T} \mathcal{P}os$ for T a first order theory

Triposes

The internal logic of a primary doctrine

$\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ a primary doctrine

The theory T_P of the doctrine P is in the $=\wedge$-fragment and has

sorts: the objects of $\mathcal{B}$

operations: $x : b \vdash \ulcorner f \urcorner(x) : b'$ for each arrow $f \colon b \longrightarrow b'$

$x_1 : b_1, x_2 : b_2 \vdash \langle x_1, x_2 \rangle : b_1 \times b_2$ for each product $b_1 \times b_2$ (with projections $\pi_1 \colon b_1 \times b_2 \to b_1$, $\pi_2 \colon b_1 \times b_2 \to b_2$)

atomic formulas: $x : b \vdash \ulcorner \varphi \urcorner(x)$ for each $\varphi \in P(b)$

axioms:

$$x : b \vdash \ulcorner \mathrm{id}_b \urcorner(x) = x$$

$$x : b \vdash \ulcorner g \urcorner(\ulcorner f \urcorner(x)) = \ulcorner g \circ f \urcorner(x)$$

$$x_1 : b_1, x_2 : b_2 \vdash \ulcorner \pi_1 \urcorner(\langle x_1, x_2 \rangle) = x_1 \wedge \ulcorner \pi_2 \urcorner(\langle x_1, x_2 \rangle) = x_2$$

$$x_1, x_3 : b_1, x_2, x_4 : b_2 \mid x_1 = x_3 \wedge x_2 = x_4 \vdash \langle x_1, x_2 \rangle = \langle x_3, x_4 \rangle$$

$$x : b \mid \ulcorner \varphi \urcorner(x) \vdash \ulcorner \psi \urcorner(x) \quad \text{for } \varphi \leq \psi \text{ in } P(b)$$

$$\top \vdash \ulcorner \top \urcorner$$

$$x : b \mid \ulcorner \varphi_1 \urcorner(x) \wedge \ulcorner \varphi_2 \urcorner(x) \vdash \ulcorner \varphi_1 \wedge \varphi_2 \urcorner(x) \quad \text{for } \varphi_1, \varphi_2 \text{ in } P(b)$$

Elementary doctrines

Definition

$\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$

- $\mathcal{B}$ has finite products
- each *fibre* $P(b)$ has finite products, and each *reindexing* $P(f)\colon P(b') \longrightarrow P(b)$ preserves them
- for every object b in $\mathcal{B}$ there is $\delta_b \in P(b \times b)$ such that for every arrow of the form $\quad a \times b \xrightarrow{\langle \pi_1, \pi_2, \pi_2 \rangle} a \times b \times b \quad$ the functor

  $$e \mapsto P(\langle \pi_1, \pi_2 \rangle)(e) \wedge P(\langle \pi_2, \pi_3 \rangle)(\delta_b)\colon P(a \times b \times b) \longrightarrow P(a \times b)$$

  gives a left adjoint to the reindexing $P(\langle \pi_1, \pi_2, \pi_2 \rangle)\colon P(a \times b) \longrightarrow P(a \times b \times b)$

A characterisation of elementary doctrines

Proposition

Let $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ *be a primary doctrine and let* T_P *be the theory of* P

The doctrine $\mathcal{C}tx_{T_P}^{\mathrm{op}} \xrightarrow{\mathrm{Wff}^{T_P}} \mathcal{P}os$ *is elementary*

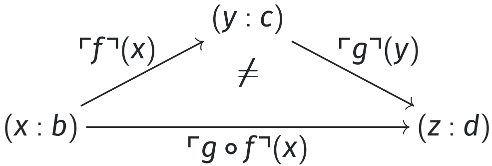
Theorem

The primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ *is elementary if and only if the "Erasure" of the right-angled quotes can be extended to a homomorphism of doctrines*

$$\begin{array}{c} \mathcal{C}tx_{T_P} \\ {\scriptstyle \mathrm{E}}\big\downarrow \\ \mathcal{B} \end{array} \qquad \begin{array}{ccc} \mathcal{C}tx_{T_P}^{\mathrm{op}} & \xrightarrow{\mathrm{Wff}^{T_P}} & \\ {\scriptstyle \mathrm{E}^{\mathrm{op}}}\big\downarrow & {\scriptstyle \varepsilon}\big\downarrow\cdot & \mathcal{P}os \\ \mathcal{B}^{\mathrm{op}} & \xrightarrow[P]{} & \end{array}$$

Adapting the characterisation

The category $\mathcal{Ctx}_{T_P}$ is "bigger" than the original category $\mathcal{B}$
For instance



But $\quad x : b \vdash \ulcorner g \urcorner(\ulcorner f \urcorner(x)) = \ulcorner g \circ f \urcorner(x)$ in T_p

Quotient arrows $\quad (x_1 : S_1, \ldots, x_n : S_n) \underset{(t'_1, \ldots, t'_m)}{\overset{(t_1, \ldots, t_m)}{\rightrightarrows}} (y_1 : S'_1, \ldots, x_m : S'_m)$

in $\mathcal{Ctx}_{T_P}$ with respect to the equivalence relation ext given by

$$x_1 : S_1, \ldots, x_n : S_n \vdash_{T_P} t_1 = t'_1 \wedge \ldots \wedge t_m = t'_m$$

Adapting the characterisation

Proposition

Let $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ *be a primary doctrine and let* T_P *be the theory of* P

The doctrine $\mathcal{C}tx_{T_P}^{\mathrm{op}} \xrightarrow{\mathrm{Wff}^{T_P}} \mathcal{P}os$ *factors through the quotient*

$$\begin{array}{ccc} \mathcal{C}tx_{T_P}^{\mathrm{op}} & & \\ \downarrow & \searrow^{\mathrm{Wff}^{T_P}} & \\ \left(\mathcal{C}tx_{T_P}/\,\mathrm{ext}\right)^{\mathrm{op}} & \xrightarrow[\mathrm{Wff}^{T_P}]{} & \mathcal{P}os \end{array}$$

and the factorisation is elementary

Adapting the characterisation

Theorem

The primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ *is elementary if and only if the "Erasure" of the right-angled quotes can be extended to a homomorphism of doctrines*

$$
\begin{array}{c}
\mathcal{C}tx_{T_P}/\,\mathrm{ext} \\
\mathrm{E}\,\Big\Vert\Big\vert \\
\mathcal{B}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\left(\mathcal{C}tx_{T_P}/\,\mathrm{ext}\right)^{\mathrm{op}} & \xrightarrow{\ \mathrm{Wff}^{T_P}\ } & \\
\mathrm{E}^{\mathrm{op}}\,\Big\Vert\Big\vert & \varepsilon \downarrow\cdot & \mathcal{P}os \\
\mathcal{B}^{\mathrm{op}} & \xrightarrow[\ P\ ]{} &
\end{array}
$$

Adapting the characterisation

Remark

When $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is elementary we get that

$$\varepsilon_{(x_1,x_2:b)}(x_1 = x_2) = \delta_b \qquad \text{in } P(b \times b)$$

But the formulas

$$(x_1 = x_2) \quad \text{and} \quad \ulcorner\delta_b\urcorner(x_1, x_2) \quad \text{are different in } \mathrm{Wff}_{T_P}(x_1 : b, x_2 : b)$$

Write T_P^+ for the theory T_P plus the axioms

$$x_1 : b, x_2 : b \mid \ulcorner\delta_b\urcorner(x_1, x_2) \vdash x_1 = x_2$$

Forcing internally equal arrows to be identitical

Definition

An elementary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ has *comprehensive diagonals* when for $f, f' \colon b \longrightarrow c$ in $\mathcal{B}$

$$x : b \vdash_{T_P^+} \ulcorner f \urcorner(x) = \ulcorner f' \urcorner(x) \quad \text{implies} \quad f = f'$$

Like before, quotient arrows $b \overset{f}{\underset{f'}{\rightrightarrows}} c$ in $\mathcal{B}$ with respect to the equivalence relation ext given by

$$x : b \vdash_{T_P^+} \ulcorner f \urcorner(x) = \ulcorner f' \urcorner(x)$$

Forcing internally equal arrows to be identitical

Examples

$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathcal{P}os$ for $\mathcal{A}$ with finite limits

$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Vrn}} \mathcal{P}os$ for $\mathcal{A}$ with weak pullbacks and finite products

$\mathcal{C}tx_T^{\mathrm{op}} \xrightarrow{\mathrm{Wff}^T} \mathcal{P}os$ for T a first order theory

Triposes on $\mathcal{S}et$

Forcing internally equal arrows to be identitical

Proposition

Let $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ *be an elementary doctrine*

The doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{\quad P \quad} \mathcal{P}os$ *factors through the quotient*

$$\begin{array}{ccc} \mathcal{B}^{\mathrm{op}} & & \\ \downarrow & \searrow^{P} & \\ \left(\mathcal{B}_{/\,\mathrm{ext}}\right)^{\mathrm{op}} & \xrightarrow[P]{} & \mathcal{P}os \end{array}$$

and the factorisation is with comprehensive diagonals

Equivalence relations and quotients

$\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ an elementary doctrine

Definition

A P-*equivalence relation* is an object $\sim$ in $P(b \times b)$ such that

- $x : b \vdash x \sim x$
- $x_1, x_2 : b \mid x_1 \sim x_2 \vdash x_2 \sim x_1$
- $x_1, x_2, x_3 : b \mid (x_1 \sim x_2) \wedge (x_2 \sim x_3) \vdash x_1 \sim x_3$

A *quotient* of the equivalence relation $\sim \in P(b \times b)$ is $q\colon b \longrightarrow c$ in $\mathcal{B}$ such that

- $x_1, x_2 : b \mid x_1 \sim x_2 \vdash q(x_1) = q(x_2)$
- for every $g\colon b \longrightarrow x$ such that $x_1, x_2 : b \mid x_1 \sim x_2 \vdash g(x_1) = g(x_2)$

  there is a unique commutative filling $\begin{array}{ccc} b & \xrightarrow{q} & c \\ & {}_{g}\searrow & \downarrow{\scriptstyle k} \\ & & x \end{array}$

Well behaved quotients

A quotient $q\colon b \longrightarrow c$ of the equivalence relation $\sim \in P(b \times b)$ is

effective if $x_1, x_2 : b \mid q(x_1) = q(x_2) \vdash x_1 \sim x_2$

stable if in every pullback

$$\begin{array}{ccc} p & \xrightarrow{q'} & a \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ b & \xrightarrow[q]{} & c \end{array}$$

the arrow $q'\colon p \longrightarrow a$ is a quotient

Examples

$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathcal{P}os$ for $\mathcal{A}$ with pullbacks (and finite products) has stable effective quotients if and only if $\mathcal{A}$ is exact

$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Vrn}} \mathcal{P}os$ for $\mathcal{A}$ with weak pullbacks (and finite products) has stable effective quotients if and only if $\mathcal{A}$ is exact

Well behaved quotients

A quotient $q\colon b \longrightarrow c$ of the equivalence relation $\sim\, \in P(b \times b)$ is

descent if

$$\begin{array}{ccc} P(c) & \xrightarrow{P(q)} & P(b) \\ & \underset{\text{full}}{\searrow} & \downarrow \\ & & \mathrm{Des}_q \end{array}$$

where

$$\mathrm{Des}_q := \left\{ \varphi \in P(b) \;\middle|\; x_1, x_2\colon b \mid \varphi(x_1) \wedge q(x_1) = q(x_2) \vdash_{T_P^+} \varphi(x_2) \right\}$$

effective descent if

$$\begin{array}{ccc} P(c) & \xrightarrow{P(q)} & P(b) \\ & \overset{\sim}{\searrow} & \downarrow \\ & & \mathrm{Des}_q \end{array}$$

Completing with quotients to an elementary doctrine

Theorem

Let $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ *be an elementary doctrine*

The following data determine an elementary doctrine $\mathcal{R}_P^{\mathrm{op}} \xrightarrow{\mathrm{Des}} \mathcal{P}os$ *with descent quotients of equivalence relations:*

objects: $(b, \sim)$ *where* $\sim \in P(b \times b)$ *is a P-equivalence relation*

arrows: $f\colon (b,\sim) \longrightarrow (c,\approx)$ *is* $f\colon b \longrightarrow c$ *in* $\mathcal{B}$ *such that*
$x_1, x_2 : b \mid x_1 \sim x_2 \vdash_{T_P^+} f(x_1) \approx f(x_2)$

fibres: $\mathrm{Des}(b,\sim) = \left\{ \varphi \in P(b) \,\middle|\, x_1, x_2 \colon b \mid \varphi(x_1) \wedge x_1 \sim x_2 \vdash_{T_P^+} \varphi(x_2) \right\}$

which is free among elementary doctrines with such quotients

Moreover quotients in $\mathcal{R}_P^{\mathrm{op}} \xrightarrow{\mathrm{Des}} \mathcal{P}os$ *are stable effective*

M.E. Maietti, G. R.
Elementary quotient completions *Theory Appl.Categ.* 2013

A cocompletion for primary doctrines

Theorem

Let $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ *be a primary doctrine*

The elementary doctrine $\mathcal{R}_P^{\mathrm{op}} \xrightarrow{\mathrm{Des}} \mathcal{P}os$ *is cofree on* $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$

Corollary

Given a theory T in first order logic, the theory of the elementary doctrine $\mathcal{R}_{P_T^+}^{\mathrm{op}} \xrightarrow{\mathrm{Des}} \mathcal{P}os$ *has uniform elimination of imaginaries*

J. Emmenegger, F. Pasquali, G. R.
Elementary doctrines as coalgebras *J.Pure Appl.Algebra* 2020

B. Poizat
Une théorie de Galois imaginaire *J.Symb.Log.* 1983

The elementary quotient completion

Theorem

Let $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ *be an elementary doctrine*

The elementary doctrine $(\mathcal{R}_{P/\,\mathrm{ext}})^{\mathrm{op}} \xrightarrow{\mathrm{Des}} \mathcal{P}os$ *has descent quotients of equivalence and comprehensive diagonals*

Moreover quotients in $\mathcal{R}_P{}^{\mathrm{op}} \xrightarrow{\mathrm{Des}} \mathcal{P}os$ *are stable effective*

Examples

For $\mathcal{A}$ with weak pullbacks consider for P the doctrine $\quad \mathrm{Vrn}\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{P}os$
Then $\mathcal{R}_{P/\,\mathrm{ext}} \equiv \mathcal{A}_{\mathrm{ex/wlex}}$ and $\mathrm{Des} = \mathrm{Sub}\colon (\mathcal{A}_{\mathrm{ex/wlex}})^{\mathrm{op}} \longrightarrow \mathcal{P}os$

M.E. Maietti, G. R.
Elementary quotient completions *Theory Appl.Categ.* 2013
A. Carboni, E. Vitale
Regular and exact completions *J.Pure Appl.Algebra* 1998

Logical structure and the elementary quotient completion

$\mathcal{B}^{\text{op}} \xrightarrow{P} \mathcal{P}os$ an elementary doctrine

The doctrines $\mathcal{R}_P{}^{\text{op}} \xrightarrow{\text{Des}} \mathcal{P}os$ and $(\mathcal{R}_{P/\,\text{ext}})^{\text{op}} \xrightarrow{\text{Des}} \mathcal{P}os$ inherit the following

- all fibres $P(b)$ have finite coproducts and each reindexing $P(f)\colon P(b') \longrightarrow P(b)$ preserves them (P has *finite joins*)
- all fibres $P(b)$ are cartesian closed and each reindexing $P(f)\colon P(b') \longrightarrow P(b)$ preserves exponentials (P has *implications*)
- each reindexing along a projection $\pi_1\colon a \times b \longrightarrow a$ has a left adjoint

  $\exists_{a,b}\colon P(a \times b) \longrightarrow P(a)$ and they are natural in a:

$$\begin{array}{ccc} P(a \times b) & \xrightarrow{\exists_{a,b}} & P(a) \\ {\scriptstyle P(g \times \text{id}_b)}\downarrow & & \downarrow{\scriptstyle P(g)} \\ P(a' \times b) & \xrightarrow{\exists_{a',b}} & P(a') \end{array}$$

- each reindexing along a projection $\pi_1\colon a \times b \longrightarrow a$ has a right adjoint $\forall_{a,b}\colon P(a \times b) \longrightarrow P(a)$ and they are natural in a

A.M. Pitts
Categorical logic *Handbook of Logic in Computer Science, vol. 5, 2000*

Weak comprehension

Definition

The primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ has *weak comprehension* if for every b in $\mathcal{C}$ and φ in $P(b)$ there is an arrow $j_\varphi \colon c \longrightarrow b$ in $\mathcal{B}$ such that

- $x : c \vdash_{T_P} \varphi(j_\varphi(x))$
- for every $g \colon a \longrightarrow b$ such that $y : a \vdash_{T_P} \varphi(g(y))$ there is a filling $\begin{array}{ccc} a & & \\ {\scriptstyle k}\downarrow & \searrow^{g} & \\ c & \xrightarrow{j_\varphi} & b \end{array}$

It is *full* when a commutative $\begin{array}{ccc} c & & \\ {\scriptstyle k}\downarrow & \searrow^{j_\varphi} & \\ d & \xrightarrow{j_\psi} & b \end{array}$ implies that $x : b \mid \varphi(x) \vdash \psi(x)$

F.W. Lawvere
Equality in hyperdoctrines and the comprehension schema as an adjoint functor
Proc.AMS Symp.Pure Math. 1970

Examples of weak comprehension

$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathcal{P}os$ has full comprehension for $\mathcal{A}$ with finite limits

$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Vrn}} \mathcal{P}os$ has weak full comprehension for $\mathcal{A}$ with weak pullbacks and finite products

Doctrines obtained by changing the base, for instance:

$\mathcal{T}op_{\mathrm{o}}^{\mathrm{op}} \xrightarrow{U^{\mathrm{op}}} \mathcal{S}et^{\mathrm{op}} \xrightarrow{\wp} \mathcal{P}os \qquad \mathcal{PA}sm^{\mathrm{op}} \xrightarrow{U^{\mathrm{op}}} \mathcal{S}et^{\mathrm{op}} \xrightarrow{\wp} \mathcal{P}os$

Also, comprehension can be freely added to any primary doctrine (and it is full)
For instance, comprehension can be freely added to a tripos

Categorical structure and the elementary quotient completion

$\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ an elementary doctrine with weak comprehension

The doctrine $(\mathcal{R}_{P/\,\mathrm{ext}})^{\mathrm{op}} \xrightarrow{\mathrm{Des}} \mathcal{P}os$ inherits the following properties

- P has finite joins, is existential, and $\mathcal{B}$ has weak disjoint coproducts
- P has implications, is universal, and $\mathcal{B}$ is weakly locally cartesian closed
- P has implications, is universal, and $\mathcal{B}$ has a weak classifier

In each case, the structure is turned into a strong one.

A. Carboni
Some free constructions in realizability and proof theory *J.Pure Appl.Algebra* 1995

C.J. Cioffo
Biased Elementary Doctrines preprint 2021

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