Sheaf Cohomology and Univalent Type Theory

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Sheaf cohomology

Cohomology groups: invariants attached to mathematical structures

Ex.: Brauer group (1927), commutative group associated to *any* field

Extension groups: Schreier (1926), group acting on an abelian group

There, 2nd cohomology group (Teichmüller 1940, 3rd cohomology group)

Leray (1940s): sheaf cohomology over topological spaces

Grothendieck unified these notions with the concept of *site* and *topos*

Sheaf cohomology

This relied on the notion of *injective* resolutions

Grothendieck alluded in an alternative approach using n-stacks

We explain the case $n = 1$, and one difficulty that appears for $n = 2$

We try to show that the language of dependent type theory is (surprisingly) well adapted to describe what is going on, and present new proofs (due to David Wärn, formalised in Agda by Louise Leclerc) of the basic results

Sheaf cohomology

This language deals with the general case of n-stacks in a *uniform* way

Provides several representations of “the” category of abelian groups

All this can be expressed in a constructive and weak metatheory (weaker than second-order arithmetic)

Topos Theory

It [the category of sheaves] functions as a kind of "superstructure of measurement", called the "Category of Sheaves" (over the given space), which henceforth shall be taken to incoorporate all that is most essential about that space. This is in all respects a lawful procedure, (in terms of "mathematical common sense") because it turns out that one can "reconstitute" in all respects, the topological space by means of the associated "category of sheaves" (or "arsenal" of measuring instruments)

Consider the set formed by all sheaves over a (given) topological space or, if you like, the formidable arsenal of all the "rulers" that can be used in taking measurements on it.

Sheaf models

The collection of "all" sheaves over a space has strong similarity with the collection of "all" sets

In general, the law of Excluded Middle may not be valid

The Axiom of Choice may not be valid as well

Sheaf models

We have "partial" or "virtual" elements

For instance G-sets for $G = \mathbb{Z}/2\mathbb{Z}$ we have $T = \{a_0, a_1\}$ with a_0, a_1 swapped by the G-action then $T \to 1$ is epi but T has no global element

Sheaf models

We now give an example how we can use the collection of all sheaves of sets as a way to “measure” a space

How to distinguish the real line and the circle?

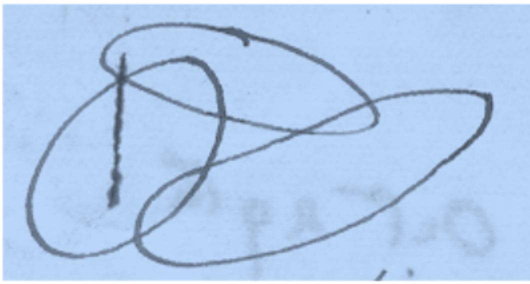
For the real line all $\mathbb{Z}$-torsors are trivial

But for the circle we can find a non trivial $\mathbb{Z}$-torsor

A $\mathbb{Z}$-torsor is a $\mathbb{Z}$-set which is locally isomorphic to $\mathbb{Z}$

Sheaf models

Cover the circle S_1 with 3 open U_0, U_1, U_2



Screenshot of a letter from Ramsey to Russell 1927

Sheaf models

Consider the helix over S_1

Define $H(U_i) = \mathbb{Z}$ with transition maps

$t_{01}(n) = n + 1$ and $t_{12}(n) = t_{20}(n) = n$

H has no global element

But the map $H \to 1$ is epi

H is a non trivial $\mathbb{Z}$-torsor

Sheaf models

This is to be compared with the *real line* $\mathbb{R}$

If we have U_0, U_1, U_2 intervals and $U_0 \cap U_1,\ U_1 \cap U_2$ nonempty and $U_0 \cap U_1 \cap U_2$ empty then $U_0 \cap U_2$ has to be empty

In term of topos over $\mathbb{R}$, this is reflected by the fact that any $\mathbb{Z}$-torsor is *trivial*

This illustrates how the notion of *topos*, as a mathematical universe, can be seen as a way to "measure" a space, here a difference between the real line and the circle

Sheaf models

This example goes back to Russell's point-free presentation of $\mathbb{R}$ (for representing time)

Here we can define a point as a maximal set of open interval meeting two by two

He was trying to generalise this for more complex spaces

Ramsey pointed out to him that we may have in general convex open meeting two by two but not globally containing a point

Torsors

For a given group G in a topos, we can look at the collection of all G-torsors

This is actually one way to define the first cohomology group of a space for a given sheaf of groups G

$H^1(X, G)$ can be defined as the set of all G-torsors, up to isomorphisms

$H^1(\mathbb{R}, \mathbb{Z}) = 0$ and $H^1(S_1, \mathbb{Z}) = \mathbb{Z}$

Torsors

See J. Baez *Torsors Made Easy*

See also A. Blass *Cohomology detects failures of the Axiom of Choice*, 1983

If G is a group a G-torsor T is like the group G that has forgotten its identity

An affine space is like a vector space that has forgotten its origin

In physics, potential energy is only given up to an arbitrary constant

We have access to $W = -\Delta U$ and not to "absolute" energy

A G-torsor is isomorphic to G but not *canonically*

Torsors

Let G be a group, a G-torsor is a set T with a G-action $a, g \mapsto a \cdot g$ such that $T \times G \to T \times T$ is an isomorphism *and* $T \to 1$ is epi

E.g. $T = \{a_0, a_1\}$ as above is a G-torsor, $G = \mathbb{Z}/2\mathbb{Z}$

Torsors

The collection of G-torsors form a groupoid BG

Any map between two G-torsors is an isomorphism!

Locally $T(U_i) \to T'(U_i)$ has some *unique* inverse

Hence we can patch these inverses to build a global inverse $T' \to T$

Torsors

G itself can be seen as a G-torsor t_G the "trivial" G-torsor

The group $Aut(t_G)$ of automorphisms of t_G is exactly G

If we have a morphism $f : G \to H$ we can build a functor $BG \to BH$ sending t_G to t_H and which induces f via $Aut(t_G) \to Aut(t_H)$

If T is a G-torsor how to define $f(T)$??

Torsors

Presentation in a paper of Deligne "Le symbole modéré", 1991, 5.2, 5.3

Locally $T(U_i)$ is isomorphic to $G(U_i)$ and we can take $H(U_i)$

But there might be a patching problem

The key idea is to define $f(T)$ *together with* a map $f : T \to f(T)$ such that $f(t \cdot g) = f(t) \cdot f(g)$

This pair $f(T), f$ is determined up to unique *isomorphism*

Torsors

We can patch together these local data: this is a patching problem where we patch together *sets*

If we have F_i sheaf on U_i, with transition maps $\alpha_{ij} : F_i|U_{ij} \rightarrow F_j|U_{ij}$ on $U_{ij} = U_i \cap U_j$

and we have the cocycle condition $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$

We need this whenever $U_i \cap U_j \cap U_k$ is nonempty

Then we can patch together the F_i to a global sheaf F with isomorphisms $\alpha_i : F|U_i \rightarrow F_i$

Internal language

Can we present these arguments in an internal way?

We are going to compare the proofs of

-any morphism of G-torsors $g : T \to T'$ is an isomorphism

-any group morphism $f : G \to H$ can be lifted to a functor $BG \to BH$

Internal language

If $g : T \to T'$ is a map of G-torsors we prove

$\forall_{t':T'} \exists !_{t:T}\ g(t) = t'$

Indeed, given $t' : T'$ for proving $\exists !_{t:T}\ g(t) = t'$ we can assume T inhabited

This follows from $T \to 1$ is epi

So to prove a given proposition ψ we can always assume that we have some inhabitant of T

We can then apply *unique* choice: if we have $\forall_{t':T'} \exists !_{t:T}\ g(t) = t'$ we have a function that give t' from t

In a topos, choice may fail but unique choice always holds!

Internal language

-any map $f : G \to H$ can be lifted to a map $BG \to BH$

The "usual" way: build $f(T)$ as a quotient $T \times H/ \simeq$

with $(t, h) \simeq (t \cdot g^{-1}, f(g) \cdot h)$

But Deligne's argument *cannot* directly be represented as such internally

It uses the fact that $f(T), f$ is determined up to unique isomorphism

This is a *new* principle of unique choice, which does not hold in higher order logic, but holds in *univalent type theory*

Torsors

As Deligne writes

$H^1(X, G)$ is the set of isomorphisms class of G-torsors

For $H^2(X, G)$ we need to look at sheaves of *groupoids*

The language of torsors and gerbes is essentially equivalent to that of Čech cocycles. It is more convenient and more intrinsic if we agree to speak of an object defined up to unique isomorphism, of category defined up to unique equivalence up to unique isomorphism, and recollement of stacks.

Stacks

On the "non-commutative" side, we have a good foundation work with Giraud's thesis, but this is limited to a formalism of 1-stacks, lending itself to a direct geometric expression of objects of cohomology up to dimension 2 only. The question of developing a cohomological formalism not commutative in terms of n-stacks, imperiously suggested *by numerous examples, encountered serious conceptual difficulties.*

Considering the disaffection or, to put it better, the general contempt, into which have fallen the questions of foundations in a certain world, these difficulties have never been addressed before I started to look at them a little over two years ago

Grothendieck in "Récoltes et Semailles"

Stacks

Already for groupoids, it would be quite difficult to define delooping directly using a quotient operation

We need to follow Deligne's more abstract method and characterise delooping by some universal property

Stacks

For torsors: we need the notion of "up to unique isomorphism"

For gerbes: "up to unique equivalence up to unique isomorphism"

For 2-groupoids, "up to unique 2-equivalence up to unique equivalence up to unique isomorphism"

And so on

2-equivalence: *weak* notion of equivalence

Two questions: how to describe what is going on and how to explain what are such objects (n-stacks)

Higher Topos Theory

Generalize sheaves to stacks: sheaves of groupoids, 2-groupoids, $\ldots$

Use language of *dependent type theory* to describe what is going on

We provide later one explanation of what are such objects

Use universes (type of types) $\mathcal{U}$ and “equality” type $a_0 =_A a_1$

Dependent Type Theory

Stratification of types

proposition $\prod_{a_0:A} \prod_{a_1:A} \; a_0 =_A a_1$

0-type or set $\prod_{a_0:A} \prod_{a_1:A} \; \mathsf{isProp}\ (a_0 =_A a_1)$

1-type of groupoid $\prod_{a_0:A} \prod_{a_1:A} \; \mathsf{isSet}\ (a_0 =_A a_1)$

Contractible type $A \times \mathsf{isProp}\ A$

All these types are themselves propositions!

Dependent Type Theory

We need $\|A\|$ propositional *truncation*

An element of $\|A\|$ intuitively corresponds to local existence

$A \to 1$ epi

Dependent Type Theory

A is $(n+1)$-type if $a_0 = a_1$ n-type for all a_0 and a_1 in A

A is $(n+1)$-connected if $a_0 = a_1$ is n-connected for all a_0 and a_1 in A *and* we have $\|A\|$

If $f : A \rightarrow B$ and $b : B$ then $\sum_{a:A} f(a) = b$ is the *fiber* of f at b

$f : A \rightarrow B$ is an equivalence if all fibers are contractible

Dependent Type Theory

We can define what is a *group* G in a given universe $\mathcal{U}$

It has to be a *set* with a binary operation satisfying the usual laws

We can form the collection of all groups (in a given universe) and *prove* that it is a groupoid

For defining what is a torsor, we need to have an operation $\|A\|$ proposition which expresses that $A \rightarrow 1$ is epi

Torsors in Dependent Type Theory

If G is a group we can then define the type BG of G-torsors

$$BG \ := \ \sum_{X:\mathcal{U}} \|X\| \times \sum_{a:X\times G\to X} \psi(X,a)$$

where $\psi(X,a)$ is a proposition/type expresssing that a is a G-action such that $X \times G \to X \times X$ is a bijection

One can prove in type theory that BG is a *groupoid*

Torsors in Dependent Type Theory

This type is *pointed*: it has an element t_G which is the trivial G-torsor

With univalence, we can show that $t_G =_{BG} t_G$ is equal to G

If A, a is a pointed type then $\Omega(A, a)$ is the pointed type $a =_A a$ inhabited by the constant path, this is the *loop space* of A, a

$G, 1$ is the loop space of $K(G, 1) = (BG, t_G)$

$K(G, 1)$ Eilenberg-MacLane space associated to G

Torsors in Dependent Type Theory

Deligne's argument can then be expressed directly in type theory

Given $f : G \to H$ group morphism and $T : BG$ then we can look at

$$\sum_{T':BH} \sum_{h:T\to T'} \prod_{t:T} \prod_{g:G} h(t \cdot g) = h(t) \cdot f(g)$$

We show that this type is *contractible* by an argument following the type theoretic version of Deligne's argument

We don't need quotient, and we use *unique choice* instead which follows from the fact that to be *contractible* is a *proposition*

Torsors in Dependent Type Theory

Something similar holds for the result stating that for a ring R the groupoid of locally free modules of rank 1 (the *Picard group* of R) and the groupoid of $R^{\times}$-torsors are canonically equivalent

The proof in the Stacks project Lemma 21.6.1 uses an "explicit" construction with a quotient

Torsors in Dependent Type Theory

In type theory, this can be seen as a special case of the following result

$Mod(R)$ is the groupoid of R-modules and M_1 the free R-module of rank 1 so that $M_1 = M_1$ is $R^{\times}$

$T_R = \sum_{M:Mod(R)} \|M = M_1\|$ is the type of locally free modules of rank 1

Lemma: *If A is a groupoid and $a : A$ then the canonical map*

$$T_a \to B(a = a) \qquad x \mapsto (x = a)$$

is an equivalence

Torsors in Dependent Type Theory

The proof of this follows the same kind of reasoning: we have to show that for any $(a = a)$-torsor T the fiber $\sum_{(x,p):T_a}(x = a) = T$ is contractible

It is enough to show it when T of the trivial torsor $a = a$ and this is proved directly

This is because to be contractible is a *proposition* and any torsor is merely equal to the trivial torsor $a = a$.

Torsors in Dependent Type Theory

Once again, Deligne presents such an argument closer to the one of type theory in the 1977 notes

"Étale cohomology: starting points"

(first chapter of SGA4 1/2)

Torsors in Dependent Type Theory

So we can follow Deligne's argument and deloop a map $G \to H$

David Wärn noticed that the same argument proves something stronger: the loop map

$$\Omega : ((BG, t_G) \to_{\bullet} (BH, t_H)) \to \mathsf{Grp}(G, H)$$

is an equivalence/bijection

Indeed, one can prove that this type

$$\sum_{T':BH} \sum_{h:T\to T'} \prod_{t:T} \prod_{g:G} h(t \cdot g) = h(t) \cdot f(g)$$

is equivalent to the *fiber* of the map Ω at $f : G \to H$

Torsors in Dependent Type Theory

Type theoretically, this story has a nice formulation

The groupoids of the form BG, t_G can be characterised as exactly pointed 0-connected 1-types

Let $K_1(\mathcal{U})$ be the type of pointed 0-connected 1-types

Let $\mathsf{Grp}(\mathcal{U})$ the collection of groups, the same argument as before shows that $K_1(\mathcal{U}) \to \mathsf{Grp}(\mathcal{U})$ is fully faithful

And it is essentially surjective since $\Omega(BG, t_G)$ as a group is G

Torsors in Dependent Type Theory

Thus, we can present *group theory* by working with pointed 0-connected 1-types and pointed maps

This is the basis of the *Symmetry* book

https://unimath.github.io/SymmetryBook/book.pdf

M. Bezem, U. Buchholtz, P. Cagne, B. I. Dundas, D. R. Grayson

Connected types

More generally, the *same* argument gives a (new) proof of the following result.

Theorem: *If A is n-connected and B $(2n)$-type then for a in A and b in B*

$$((A,a) \to_{\bullet} (B,b)) \to (\Omega(A,a) \to_{\bullet} \Omega(B,b))$$

is an equivalence

(David Wärn, formalised in Agda by Louise Leclerc)

Application 1

Let $EM_n(\mathcal{U})$ be the type of pointed $(n-1)$-connected n-types with $n > 0$

Let $\mathcal{U}^\bullet$ be the type of pointed types $\sum_{X:\mathcal{U}} X$

We have the loop space function $\Omega : \mathcal{U}^\bullet \to \mathcal{U}^\bullet$

This defines a map $\Omega : EM_n(\mathcal{U}) \to EM_{n-1}(\mathcal{U})$

It follows from the Theorem that this map is fully faithfull

Application 1

We can then show that

$\Omega : EM_n(\mathcal{U}) \rightarrow EM_{n-1}(\mathcal{U})$

is an equivalence for $n > 2$

Furthermore all these types are actually equivalent to the type $\mathsf{Ab}(\mathcal{U})$ of *abelian* groups

Categories in Dependent Type Theory

Each type $EM_m(\mathcal{U})$ is equivalent to the type $\mathsf{Ab}(\mathcal{U})$ (and all are 1-types)

Each category $EM_m(\mathcal{U})$ is equivalent to the category $\mathsf{Ab}(\mathcal{U})$ of abelian groups

(All this is effective if we have an effective model of type theory)

For L abelian group we write $K(L, n)$ the corresponding object in $EM_m(\mathcal{U})$

This is the nth Eilenberg-MacLane space associated to L

Application 2: Universal characterisation of $K(L,n)$

To give a map $K(L,n) \to_\bullet (A,a)$ for any n-type A is equivalent to give a map $\Omega(K(L,n)) \to_\bullet \Omega(A,a)$, i.e. $K(L,n-1) \to \Omega(A,a)$

And then $K(L,n-2) \to_\bullet \Omega^2(A,a)$, and so on, until we get $L \to \Omega^n(A,a)$

We thus get that to give a pointed map $K(L,n) \to_\bullet (A,a)$ is the same as to give a group morphism $L \to \Omega^n(A,a)$

Application 2

The n-sphere (S_n, b) has a similar universal characterisation: to give a map $(S_n, b) \to_\bullet (A, a)$ for *any* pointed n-type is the same as to give a map $\mathbb{Z} \to \Omega^n(A, a)$

$K(\mathbb{Z}, n)$ satisfies the same universal property, but for A n-type

It then follows that $K(\mathbb{Z}, n)$ is the nth truncation of S_n

Hence $\pi_k(S_n, b) = 0$ if $k < n$ and $\pi_n(S_n, b) = \mathbb{Z}$!

Comparison with other approaches

The proof in the HoTT book uses Freudenthal Suspension Theorem (the original proof by G. Brunerie and D. Licata uses the decode-encode method)

Actually, this was used to define $K(\mathbb{Z}, n)$ as the nth truncation of S_n

Short exact sequence

We thus get another presentation of the category of Abelian groups

What happens to the notion of short exact sequence?

How is it expressed in $EM_n(\mathcal{U})$?

A sequence $(A,a) \to_{\bullet} (B,b) \to_{\bullet} (C,c)$ corresponds to a short exact sequence if, and only if, it is a *fibration sequence*

Short exact sequence

If $L \to M \to N$ is a short exact sequence of abelian groups we get the fibration sequence $K(L,n) \to_\bullet K(M,n) \to_\bullet K(N,n)$

We can then obtain the long fibration sequence

$\dots K(L,n-1) \to_\bullet K(M,n-1) \to_\bullet K(N,n-1)$

$\to_\bullet K(L,n) \to_\bullet K(M,n) \to_\bullet K(N,n)$

This can be done for *any* n

Sheaf models

So far, we have shown how to use the language of dependent type theory to express what should happen in a sheaf model

How to model types as stacks over a site?

All $(\infty, 1)$*-toposes have strict univalent universes*
M. Shulman

Constructive sheaf models of type theory
Th. C., F. Ruch and Ch. Sattler

Sheaf models

The second paper is developped in a *constructive* meta theory

It is done in a relatively *weak* meta theory

Proof theoretic strength: dependent type theory with hierarchy of universes

Weaker than second-order arithmetic

The notion of universes in an intuitionistic setting is surprisingly weak (P. Martin-Löf, P. Hancock, P. Aczel, E. Palmgren, E. Griffor, M. Rathjen)

How to recover cohomology groups?

If L is a sheaf of abelian group over a space X

Compute $K(L,n)$ in the sheaf model over X

Apply global section $\Gamma(K(L,n))$: this is a global n-type

$\pi_0(\Gamma(K(L,n))$ is $H^n(X,L)$!

This is the special case $X \to 1$ and can be generalised to $X \to Y$

How to recover cohomology groups

If we apply global section, and then π_0 to the sequence

$$\dots K(L, n-1) \to_\bullet K(M, n-1) \to_\bullet K(N, n-1)$$

$$\to_\bullet K(L, n) \to_\bullet K(M, n) \to_\bullet K(N, n) \dots$$

we get the long exact sequence

$$\dots H^{n-1}(X, L) \to_\bullet H^{n-1}(X, M) \to_\bullet H^{n-1}(X, N)$$

$$\to_\bullet H^n(X, L) \to_\bullet H^n(X, M) \to_\bullet H^n(X, N) \dots$$

Example

Let F be a field and X be the étale site of finite separable extensions of F

This site is quite natural as a constructive representation of the separable closure of F

See *Dynamic Newton-Puiseux Theorem*, Th. C. and B. Mannaa, 2013

We have a sheaf L on X which represents the *separable closure* of F, and the sheaf G_m which represents the invertible elements of L

Example

We can prove constructively that $H^1(X, G_m)$ is trivial, by showing that any global G_m-torsor is trivial

We expect to be able to prove constructively that $H^2(X, G_m)$ is the Brauer group of F

Milnor's conjecture

We can formulate Milnor's conjecture in a weak meta theory: the canonical map $K_n^M(F)/2 \rightarrow H^n(X, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism

Milnor's conjecture is about the Eilenberg-MacLane space of $\mathbb{Z}/2\mathbb{Z}$ in this topos

Summary

We have different presentations of the category of abelian groups (in a given universe)

The elements are pointed $(n-1)$-connected n-types

This is the method that Grothendieck alluded to in a letter to Larry Breen 1975 where we don't need injective resolutions, and which can be represented faithfully in a weak meta theory

The uniformity and elegance of this picture is well expressed using the language of dependent types

Summary

If we apply this to the sheaf/stack model of dependent types, we get a definition of cohomology groups of sheaves of abelian group, with rather direct proofs of the basic properties (e.g. long exact sequence of cohomology groups associated to a short exact sequence)

This can be formally represented in dependent type theory, and the formal proofs are quite close to the informal proofs

Dependent Type Theory

We only make use of the two Lemmas

Lemma 1: *If* A n*-connected and* B n*-type then the canonical map* $B \rightarrow B^A$ *is an equivalence*

Lemma 2: *If* A n*-connected with* $a : A$ *and* $P(x)$ *family of* $n+k+1$*-types over* A *then all fibers of the evaluation map* $(\prod_{x:A} P(x)) \rightarrow P(a)$ *are* k*-types*

Particular case: if $k = -2$ then the fibers are contractible