Univalent Foundations and Applied Mathematics

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I shall try to convince you that the project is promising and important and worth to be pursued further. In this talk I try to interpret and develop some of Vladimir's ideas (admittedly speculative) without claiming any accomplished result.

Vladimir Voevodsky (1966-2017) made important contributions into two different branches of mathematics: the Algebraic Geometry and the Foundations of Mathematics.

He also spent a significant part of his time and effort working in Applied Mathematics and, more specifically, in the Mathematical Biology. This project remained unfinished.

Interview with Roman Mikhailov, 2012

Since Fall 1997 I realised that my main contribution into Motive theory and Motive Cohomology was already accomplished. Since then I was very consciously and actively looking for . . . a theme to work on [. . .]. [C]onsidering tendencies of development of mathematics as a science I realised that [. . .] mathematics is at the edge of crisis, more precisely, two crises.

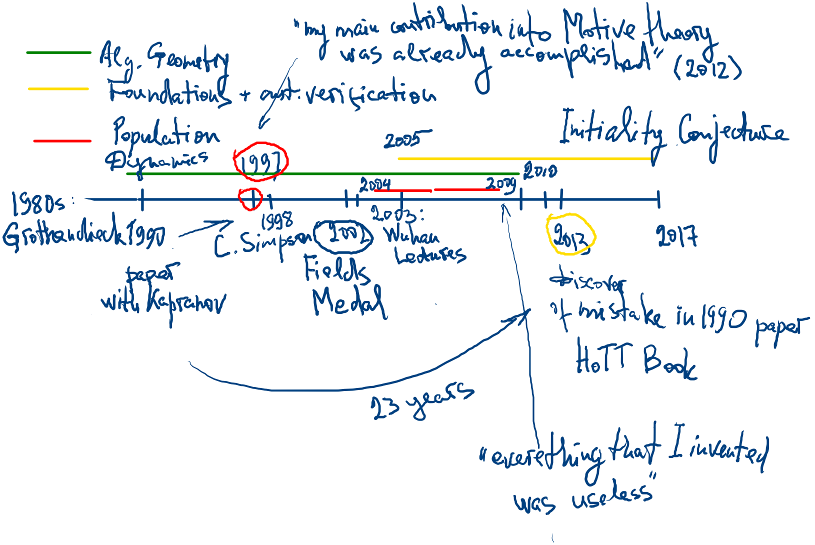
The first crisis concerns the gap between "pure" and applied mathematics. It is clear that sooner or later there will arise the question of why the society should pay money to people, who occupy themselves with things having no practical application.

The second crisis, which is less evident, concerns the fact that mathematics becomes very complex. As a consequence, once again, sooner or later mathematical papers will become too difficult for a detailed checking, and there will begin the process of accumulation of errors.

I decided to do something in order to prevent these crises. In the first case that meant to find an applied task, which would require for its solution methods of pure mathematics developed during the last years or at least during the last decades. [. . .]

[I] was looking for open problems, which would be interesting for me, where I could apply today's mathematics. Finally, I chose — as I now understand wrongly — the problem of reconstruction of history of populations [of living organisms] on the basis of their present genetic constitution. I worked on this problem about two years and finally realised in 2009 that everything that I invented was useless. [. . .]

Timeline


"my main contribution into Motive theory was already accomplished" (2012)
Alg. Geometry
Foundations + aut. verification
Population Dynamics
1997
2005
Initiality Conjecture
2010
2004
2009
1980s:
1998
2003:
Wuhan Lectures
Grothendieck 1990 paper with Kapranov
C. Simpson
2002 Fields Medal
2013
2017
discover of mistake in 1990 paper
HoTT Book
23 years
"everything that I invented was useless"

What is most important for mathematics in the near future? Wuhan U, Dec. 2003

There are two most urgent needs in today's mathematics:

1. (A) To build a computerised library of mathematical knowledge, i.e., a computerised version of Bourbaki's *Elements*;
2. (B) To bridge Pure and Applied Mathematics.

How to do that?

(A) We should gradually move from a hyperlinked mathematical text to a mathematical text verifiable with computer.

(B) We discovered very fundamental classes of new objects including categories, sheaves, cohomology, simplicial sets. They may turn out to be as important in science as algebraic groups. But presently we don't use them for solving problems outside the Pure Mathematics.

Why todays maths is not applied?

One reason can be sociological. Only few people have a profound knowledge both of modern mathematics and of some other research field where an application of modern mathematics can be possible.

Another reason concerns the current scientific policies. In order to apply an abstract mathematical theory to a concrete practical problem one needs, first of all, to generalise this problem and abstract away the intuition associated with this problem. But the current funding policies favour rather fast solutions of concrete practical problems such as, for example, designing "the billion dollar drug".

How to proceed?

In order to apply mathematics to a given problem outside mathematics one should begin with the opposite move. Instead of trying to concentrate on future applications of a mathematical theory to the real life, one should abstract yourself from the real life and look at the given problem as a formal game or puzzle. This is a reason why new mathematics too often strikes one, wrongly, as what moves away from real-world problems.

So the only reasonable policy in mathematical research and in science in general is to support one's curiosity and one's sense of beauty in science.

Four Layers of Today's Mathematics

1. Elementary Mathematics: Pythagoras theorem, Quadratic Equations, etc.; it emerged more than 1000 years ago;
2. "Higher" Mathematics : Integral and Differential Calculus, Differential Equations, Probability theory [17-18 c.];
3. Modern Mathematics: Modern Algebra (Galois theory, Group theory), Basic Topology, Logic (including Gödel Incompleteness theorems) and Set theory [c. 1850-1950];
4. Synthetic Mathematics: Representation theory, Algebraic Geometry, Homotopy theory (in particular the Motivic Homotopy theory), Differential Topology [since 1950].

Four Layers of Today's Mathematics (cntd)

- Elementary Mathematics is integrated into the everyday life;
- "Higher" Mathematics is integrated into most sciences;
- Modern Mathematics is integrated into *some* sciences;
- Synthetic Mathematics is very poorly integrated (if at all).

Comment

This partitioning of mathematics is very similar (save the most recent "synthetic" mathematics) to historical "stages of development" of mathematics earlier proposed by A.N. Kolmogorov and widely accepted by Soviet historians of mathematics.

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It is remarkable, however, that Vladimir thinks about these parts of mathematics as the co-existing “layers” rather than as consequent historical stages that succeed one another. Compare this conception of mathematics with a typical European city that comprises houses and parts of the infrastructure built at different times during the last 3-5 centuries.

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Vladimir presented the same historical view on mathematics once again in 2014 in his Paul Bernays Lectures in ETH Zürich titled "Foundations of Mathematics : their past, present, and future".

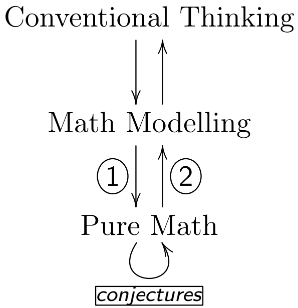
Mathematics and the Outside World (AMS-India meeting Bangalore, Dec. 2003

Mathematics is an integral — albeit very special — part of general problem-solving activity, which by itself is a pre-scientific human condition.

Various practical problems, which are conceptualised and approached with the common aka *conventional* thinking are more effectively solved via the *mathematical modelling* aka applied mathematics, which in its turn gives rise to the pure mathematics.

The pure mathematics grows with solving such *external* problems, which come via the mathematical modelling, and also with formulating and solving its own *internal* problems (in the form of proving mathematical *conjectures*).

Flow of Problems and Solutions



Flow of Problems and Solutions (cntd)

Over the last few decades the situation [as described above] was getting more and more out of balance. Arrows 1 and 2 shown at the diagram, which connect pure and applied mathematics, were weakening.

A weak incoming flow of external problems restrains the internal development of pure mathematics. A weak outgoing flow of useful solution restrains the support of mathematics provided by the Society.

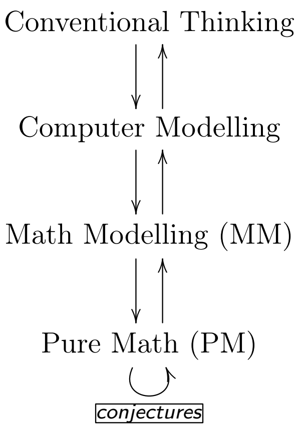
Breakdown of arrow 2 means eventually no salary for mathematicians.

Breakdown of arrow 1 means eventually no new ideas in mathematics.

What we should do to improve the situation

Change the existing pattern of using computer technologies in the general problem-solving!

The existing pattern



The existing pattern (cntd)

The flow of problems down to the "mathematical modelling" layer is filtered through the "computer modelling layer".

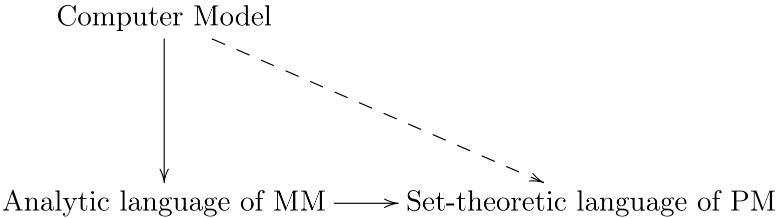
As a result the "mathematical modelling" layer, and as a consequence also the "pure mathematics" layer, receive less problems than they used to receive before the rise of computer technologies.

This particularly affects today's abstract mathematics. Problems, which pass through the filter, are formulated in the old-style language of variables and analytic functions, while the language of today's abstract mathematics is the Set theory.

Comment

In 2003 Vladimir believed that Set theory was the only legitimate foundation for the contemporary "synthetic" mathematics. The idea that Homotopy theory is relevant in Foundations first appears in Vladimir's talk given in IAS in March 2006; the Univalent Foundations make their first public appearance in 2010 in a talk in CMU in February 2010. As late as 2009 Vladimir continues to explore the idea of using Set theory as a link between pure and applied mathematics ("Singletons").

Double Translation of Problems (back to Vladimir's 2003 talk)



The new pattern

Computer Model

↓

Set-theoretic Model

↓ *sometimes*

Analytic Model

Comment

Example:
Computational models in the Climate Research are usually based, theoretically, on the Navier-Stocks equation and apply a plenty of smart algorithms that compute its solutions numerically. New mathematical methods are involved here only on the level of such numerical algorithms. As long as the Navier-Stocks equation remains a theoretical basis for the climate models there is no chance for "synthetic" mathematical theories to be applied at the basic theoretical level.

A possible translation of the Navier-Stocks equation into the language of Set theory illustrates the "double translation" phenomenon. Such a translation, by itself, hardly helps to facilitate the "incoming flow" of new mathematical ideas in this field.

Implementation (back to Vladimir's 2003 talk)

In order to implement this new scheme we need, in particular, to reformulate fundamental and applied scientific theories in the language of today's abstract mathematics, viz., the set-theoretic language.

For this end we need to specify for each theory a notion of basic *unit* and then consider sets of such units.

Sciences and Their [Ontological] Units

Science	Unit
Population Biology and Demography	Individuals (individual
Financial Mathematics	Companies
Political Science	Voters
Particles Physics	Particles
Population Genetics	Genes
Future Theoretical Chemistry, which will be able to account for individual molecules	Molecules

End of Vladimir's 2003 talk

"The most important task for mathematicians is to produce examples that demonstrate the effectiveness of this approach."

Set theory as a mathematical foundations for Science?

Marshall Stone 1961 promoting the (set-based Bourbaki-style) "New Maths" :

"While several important changes have taken place since 1900 in our conception of mathematics or in our points of view concerning it, the one which truly involves a revolution in ideas is the discovery that mathematics is entirely independent of the physical world."

Set theory as a mathematical foundations for Science?

Vladimir Arnold 1998 - a sharp critic of Bourbakism and New Maths, a promoter of the 19th century synthetic style mathematics.

"Mathematics is a part of Physics. Physics is an experimental empirical science, a part of Natural Science. Mathematics is a part of Physics where experiments are cheap."

Cf. Cassirer 1907: "The [Kantian] principle according to which our [mathematical] concepts should be sourced in intuition means that they should be sourced in the mathematical physics and should prove effective in this field."

Vladimir's attempt to stick to the set-theoretic foundations and at the same time promote the unity of mathematics with physics and other sciences is very unique and ... in my understanding, hopeless.

Set theory as a mathematical foundations for Science?

- Georg Cantor considered applications of his Set theory in Biology (following Riemann; letter to Mittag-Lefler Sept. 1884);
- “semantic view of theories”: Bourbaki-style approach in science: Patrick Suppes et al. since 1953
- formal ontologies in Analytic Metaphysics and CS.

Set theory as a mathematical foundation for Science ?

Cf. Lawvere 1970: "a 'set theory' ... should apply not only to *abstract* sets divorced from time, space, ring of definition, etc., but also to more general sets, which do in fact develop along such parameters." (By "concrete" set theory means here the Topos theory.)

Category theory as an alternative foundation? Is CT more science-friendly than ST?

There is a large body of works aiming at applications of Category theory (flat and higher categories) in theoretical Physics: Lawvere, Baez, Schreiber (QFT) et al.

Interestingly, Vladimir did not follow this path but looked for applications of recent "synthetic" mathematics in empirically-oriented (data-driven?) rather than theoretical scientific studies.

Vladimir 2012 on the data-driven research

"Science should collect and comprehend a new knowledge. The collection part is very important. There is a view according to which all important observations are already done, the general world image is clear, so it remains only to arrange this knowledge and pack it into a compact and elegant theory. This view is wrong. It is not only wrong but also supports a very negative tendency to ignore everything that doesn't fit a ready-made theory or hypothesis. This is one of the most important problems of today's science. "

UF for Science and Applied Maths?

It appears plausible that Voevodsky's dismissal of his Population Dynamics research in 2009 is related to the change of his views on Foundations of Mathematics that occurred about the same time. See the timeline.

Vladimir 2012 on Pure and Applied Mathematics

"Concerning pure and applied mathematics, I have the following picture. Pure mathematics is working with models of high abstraction and low complexity (mathematicians like to call this low complexity elegance). Applied mathematics is working with more concrete models but on the higher complexity level (many equations, unknowns, etc). Interesting application of the modern pure mathematics are most likely in the area of high abstraction and high complexity. This area is practically inaccessible today, mostly due to the limitations of the human brain [. . .]."

Vladimir 2012 on Pure and Applied Mathematics cntd.

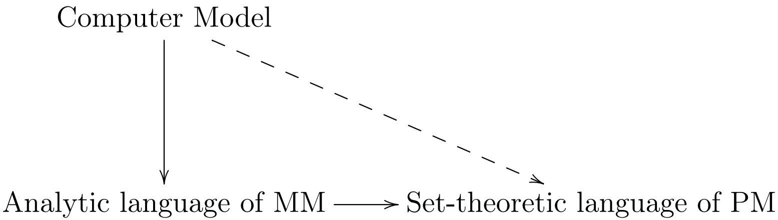
"When we will learn how to use computers for working with abstract mathematical objects this problem will be no longer important and interesting applications of ideas of today's abstract mathematics will be found. "

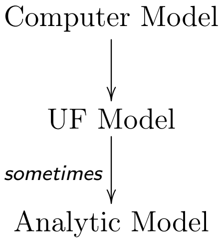
"That's why I think that my present work on computer languages [i.e., on Univalent Foundations - *A.R*] that allow one to work with such objects, will be also helpful for application of ideas of today's pure mathematics in applied problems. "

Comment

This is the only textual evidence I found that supports the claim that Vladimir conceived of a possible use of UF in applied mathematics. In any event this idea needs to be explored.

Solving the double translation problem with UF:





Objection 1:

The same shortcut can be made with any proof-assistant that implements today's mathematical theories on computer. Why UF in particular?

Using type theories with dependent types became a common standard in the computer-verified mathematics (Lean etc.). Using higher identity types, higher inductive types and other special features of UF did not.

The current version of Lean uses the UIP (Uniqueness of Identity Proof) axiom: all proofs of a given proposition are equal. This reduces the *h*-hierarchy of types to trivial and makes HoTT wholly redundant.

Kevin Buzzard 2022 on using UF for formalising schemes:

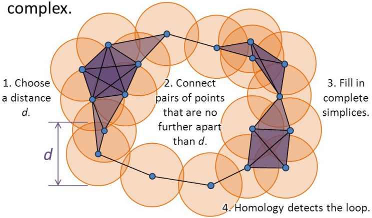
Univalence does not solve the problem which we encountered here. After the substitution in a univalent system, we would have a proof of exactness of one diagram, and we want to prove exactness of another diagram, and the diagrams now have the same objects, however we need to check that they have the same morphisms! Checking this of course boils down to checking that the squares commute, so the lion's share of the work still needs to be done.

Objection 2:

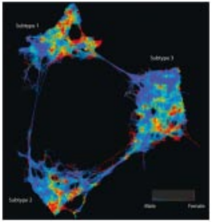
Topological Data Analysis (TDA) appears as a partial realisation of Vladimir's dream : here a relatively recent mathematical theory, viz., Persistent Homology theory, effectively applies directly to raw data on computer.

Topological Data Analysis

Idea: Connect nearby points, build a simplicial complex.



Problem: How do we choose distance *d*?



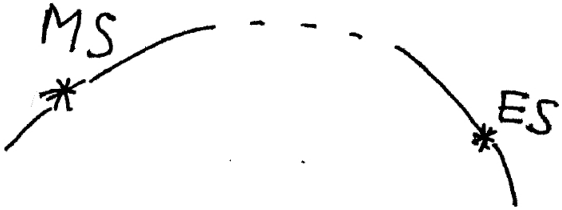
Response in favour of UF (very speculative):

To improve on the relationships between the pure maths and its applications in sciences and technologies it is not sufficient to invent a smart way to implement the existing maths on computer.

One needs to provide maths with new and more application-friendly foundations.

Does UF fit the bill?

In low dimensions UF better reflects basic intuitions about space and time, cf. Cassirer 1907 above.



It appears that we presently don't have any definite concurrent proposal . . . . So it's worth to be further explored in any event.

Voevodsky's unfinished project: Filling the gap between pure and applied mathematics

BioSystems, vol. 204:104391 (June 2021)

https://arxiv.org/abs/2012.01150

THANK YOU!