Nuclear ideal systems in tensor-$*$ categories

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Collaborators

This talk is based on work by Abramsky, Blute and me:
Abramsky, S., Blute, R., and Panangaden, P. (1999). Nuclear and trace ideals in tensored-$*$-categories. Journal of Pure and Applied Algebra, 143(1-3), 3-47.

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Later we formalized conformal field theory:
Blute, R., Panangaden, P., and Pronk, D. (2007). Conformal field theory as a nuclear functor. Electronic Notes in Theoretical Computer Science, 172, 101-132.

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- We can *repartition the interface*: $R(x_1, \ldots, x_m; y_1, \ldots, y_n)$ can be *transposed* to give $R'(x_1, \ldots, x_{m-1}; x_m, y_1, \ldots, y_n)$.

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- We can *repartition the interface*, moving indices around by *transposing* matrices or higher tensors.

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- If there is a natural iso $A \otimes B \cong B \otimes A$ (plus some conditions) we have a **symmetric** monoidal category.

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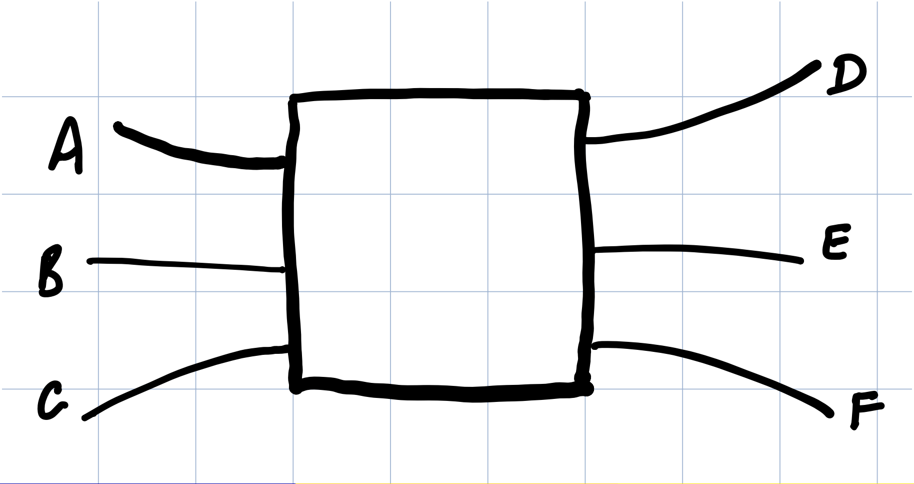
- $\hom(A \otimes X, B) = \hom(A, X \multimap B)$.
- Think of $A \multimap B$ as the space of "linear maps" from A to B.

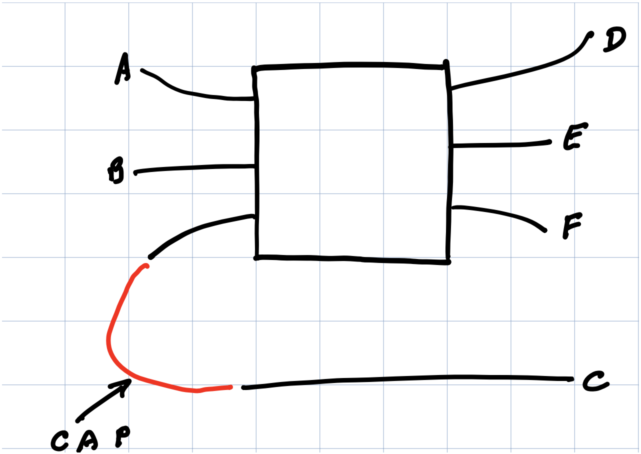
Repartitioning - 1

Figure: A morphism from $A \otimes B \otimes C$ to $D \otimes E \otimes F$



Repartitioning - 2

Figure: A morphism from $A \otimes B$ to $D \otimes E \otimes F \otimes C$



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- Finite-dimensional vector spaces and linear maps are the classic example of a compact closed category.
- The other basic example is sets and binary relations.

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- If we write $I = \{\bullet\}$ then $\nu : I \longrightarrow X \otimes X^*$ is $\bullet\nu(x, x)$ for all x; similarly for ψ.

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- $(g \circ f)(x, z) = \int_Y f(x, y) g(y, z) \mathrm{d}y$.
- If all works well we hope to get a compact closed category.

Small Problem

Schwartz

There is no function that can serve as an identity for this operation. There is no “function” δ such that:

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Schwartz, Gelfand

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- Then we tried using measure theory and thinking of the Dirac delta “function” as a measure. Again we failed to construct a compact closed category.
- Finally Rick Blute realized this was a pattern and formulated the notion of nuclear ideals and realized that there was a well-known example from Hilbert space theory.

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- Nevertheless, the maps of interest can sit as ideals inside a bona-fide monoidal category.
- The maps in the nuclear ideal do behave strikingly like they were part of a compact closed category: one can transpose freely.
- This is what Grothendieck was doing with Banach spaces: when can the maps be thought of as “matrices”?

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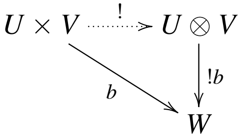
- It also has a $*$ functor like vector spaces.
- For complex Hilbert spaces we also have conjugation or equivalently a “dagger” (more later).

Universal property of tensor products?



There is a unique map, !, from $U \times V$ to $U \otimes V$ such that: given a *bilinear* map from $U \times V$ to W, there is a unique **linear** map from $U \otimes V$ to W making the diagram commute.

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- Alas, the identity is not Hilbert-Schmidt!
- So we cannot have a category of Hilbert spaces and Hilbert-Schmidt maps.

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- Nuclear spaces are typically not describable as normed vector spaces; the only spaces that are nuclear and normed are finite dimensional.

Hilbert-Schmidt and Trace Ideals

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- The composite $g \circ f : \mathcal{H} \longrightarrow \mathcal{H}$ of two nuclear maps $f : \mathcal{H} \longrightarrow \mathcal{K}$ and $g : \mathcal{K} \longrightarrow \mathcal{H}$ is always trace class.

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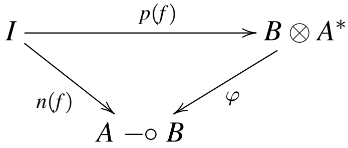

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For any symmetric monoidal closed category, the full subcategory of nuclear objects is compact-closed.

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If $f\colon A \longrightarrow B$ is a nuclear morphism, we can use the bijection θ and the $*$-functor to construct various *transposes*.
The bijection θ must preserve all of the tensored $*$-structure.

Nuclear Ideal - III

Finally, θ has to satisfy a naturality property and a “compactness” property.

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- Blute, P. and Pronk (2007) gave an alternate definition of nuclear ideals in terms of dagger compact categories.

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- These are well known in probability as Markov kernels.

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- But it is not clear what transposition would mean here.

The Radon-Nikodym theorem

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- This is a tool to construct Markov kernels.

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- How do we compose these things?

From joint measures to Markov kernels

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- So we can go back and forth between distributions on the product space $X \times Y$ and a pair consisting of a kernel $h : X \longrightarrow \Sigma_Y$ and a measure on X.
- And, of course we could instead use a kernel $k : Y \longrightarrow \Sigma_X$ and a measure on Y.

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The identity on $(X, \Sigma_X, \mu)$is $\Delta(A \times B) = \mu(A) \cdot \mu(B)$ which can be extended to all the measurable sets of $X \times X$. The associated kernel is the Dirac delta "function".

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- While f itself is only unique almost everywhere, the measure with which f is associated is easily viewed - in a canonical way - both as a member of $Hom(X, Y)$ and as a member of $Hom(I, X \times Y)$.
- Thus every element of the set $Hom(I, X \otimes Y)$ is associated with a measure that has a functional kernel which is in turn one of the members of the set $\mathcal{N}(X, Y)$.

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- The putative identity is too singular to be a function, but we can realize it as a measure.
- The category we get by including such measures is not compact closed.
- But the original functions do form a nuclear ideal.

Simple examples

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- One can construct a nuclear ideal by looking at functions whose domain consists of exactly one element and throw in the everywhere undefined function as well.

Using Schwartz distributions

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- These distributions are perfect for studying differential equations.
- We developed another $*$-tensor category based on a special kind of distribution and showed that the functional versions of these distributions give a nuclear ideal.

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- This involved some interesting mathematics: cobordisms, Riemann surfaces etc.

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- Would they be useful for formalizing infinite-dimensional quantum mechanics?
- We defined trace ideals in terms of nuclear ideals. Is there a more intrinsic way?